Sphinx Journal of Mathematical Sciences
1(2), 47–55, 2026
DOI: https://doi.org/10.64389/sjms.2026.012118
https://sphinxsp.org/journal/index.php/sjms/

Research article

Generalized Lucas polynomial sequence approach for solving Delay differential equation of fractional order

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ARTICLE INFO

Keywords:
Generalized Lucas polynomial sequence; nonlinear fractional differential equations; fractionally damped mechanical oscillator

Mathematics Subject Classification:
32W50, 35N05.

Important Dates:
Received: 11 May 2026
Revised: 7 June 2026
Accepted: 9 June 2026
Online: 11 June 2026

ABSTRACT

In this paper, the Generalized Lucas Polynomial Sequence method is proposed as an efficient numerical technique for solving fractional-order delay differential equations involving both the modified Riemann–Liouville and Caputo fractional derivatives. The fractionally damped mechanical oscillator is considered as a representative application to demonstrate the effectiveness of the proposed approach. By expanding the unknown solution in terms of generalized Lucas polynomials, the original fractional delay differential equation is transformed into a system of algebraic equations that can be solved efficiently. The proposed method is computationally simple, provides high numerical accuracy, and exhibits good convergence characteristics. The obtained results confirm that the Generalized Lucas Polynomial Sequence approach is a reliable and effective tool for solving a broad class of fractional delay differential equations arising in various scientific and engineering applications.

1. Introduction

In the mathematical modeling of physical and engineering processes, it is traditionally assumed that the future evolution of a system depends exclusively on its current state. This assumption forms the basis of classical differential equation models and has been successfully applied to describe a wide variety of dynamical systems. Nevertheless, many real-world phenomena exhibit hereditary and memory effects, where the present behavior is influenced not only by the current state but also by the system's previous states. In such circumstances, conventional models fail to capture the true dynamics of the process, and more sophisticated mathematical formulations become necessary. Systems possessing this characteristic are commonly referred to as time-delay systems, since their evolution depends explicitly on past states through one or more delay terms. Time-delay systems arise naturally in numerous scientific and engineering disciplines, including biology, medicine, chemistry, physics, economics, ecology, control engineering, and communication networks. Delays may represent transportation lags, reaction times, maturation periods, information transmission intervals, processing times, or other physical mechanisms that prevent instantaneous responses. Since these delays are inherent characteristics of many natural and artificial systems, neglecting them may lead to inaccurate mathematical models, unreliable predictions, and unsatisfactory control performance. Consequently, incorporating time-delay effects into mathematical models has become an essential requirement for accurately describing the dynamics of complex systems [8, 10].

A classical example illustrating the significance of time delays is forest regeneration. After a forest has been harvested and subsequently replanted, the newly planted trees require many years before reaching biological maturity. Depending on the tree species and environmental conditions, this maturation period may extend over several decades. Therefore, the current state of the forest is strongly influenced by its historical growth rather than solely by its present condition. Any realistic mathematical model describing forest harvesting, regeneration, and sustainable resource management must therefore include delay terms that account for the maturation period and other biological processes [9, 4]. Similar delay phenomena can also be observed in population dynamics, epidemic transmission, physiological control systems, chemical reactions, mechanical vibrations, electrical circuits, and industrial manufacturing processes, where the present response depends on previous system states.

During the past few decades, increasing attention has been devoted to the study of fractional delay differential equations, which combine the memory characteristics of fractional calculus with the delayed behavior of dynamical systems. Fractional-order derivatives provide a more flexible mathematical framework than classical integer-order derivatives because they naturally describe hereditary properties, long-term memory, and anomalous diffusion processes encountered in many practical applications. The combination of fractional calculus and time-delay theory has therefore become an effective modeling tool for representing complex systems whose dynamics cannot be adequately described using conventional differential equations. Motivated by the broad range of applications of FDDEs, extensive research has been carried out on both their theoretical analysis and numerical solution techniques [5, 13]. Numerous analytical and computational methods have been proposed to investigate the existence, uniqueness, stability, and approximate solutions of these equations. However, due to the simultaneous presence of fractional derivatives and delay terms, obtaining exact analytical solutions remains a challenging task in most practical problems. This difficulty has encouraged researchers to develop efficient numerical algorithms capable of producing highly accurate approximations while maintaining computational simplicity. Consequently, the development of reliable numerical methods for solving fractional delay differential equations continues to be an active and important area of research in applied mathematics and computational science for more details [11, 7].

2. Preliminaries

In this paper we used some definitions and properties of the Riemann Liouville and Caputo.[5, 13]

Definition 2.1

The Riemann Liouville fractional integral operatorI_t^{α} of order α on the usual Lebesgue space $L_1[0,1]$ is defined as: for all $t \in (0,1)$

$$I_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} & \int_0^t (t-\tau)^{\alpha-1}(f(\tau))d\tau, \quad \alpha > 0, \\ & f(t) \qquad\quad \alpha = 0. \end{cases} \tag{2.1}$$

Definition 2.2

The right side Riemann Liouville fractional derivative of order$\alpha > 0$is defined by

$$D_0^{\alpha} t = (\frac{d}{dt})^n (I_t^{n-\alpha} f)(t), \quad n-1 \leq \alpha < n, n \in N. \tag{2.2}$$

Definition 2.3

The fractional differential operator in Caputo sense is defined as

$$\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1}(f^{(n)})(\tau)d\tau,\ \alpha > 0\ \ t > 0, where \quad n-1 \leq \alpha < n, n \in N \tag{2.3}$$

The following properties are satisfied by the operatorD^{α} for $n-1 \leq \alpha < n$,

$$(D^{\alpha} I^{\alpha} f)(t) = f(t),$$

$$(I^{\alpha} D^{\alpha} f)(t) = f(t) - \textstyle\sum_{k=0}^{n-1} \frac{f^{(k)}(0^+)}{k!}(t-a)^k, \quad t > 0, \tag{2.4}$$

$$D^{\alpha}\left(t^k\right) = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} t^{k-\alpha}, k \in N, k \geq \lceil \alpha \rceil.$$

where $\lceil \alpha \rceil$ denotes the smallest integer greater than or equal to α

Lucas polynomials

The sequence of Lucas polynomials is a sequence of polynomials which can be considered as a generalization of the Lucas numbers [11, 7].
The sequence of Lucas polynomials $L_i(t)$ may be constructed by means of the recurrence relation:

$$\begin{aligned} L_i + 2(t) &= tL_i + 1(t) + L_i(t) \\ L_0(t) &= 2, L_1(t) = t, \quad i \geq 0 \end{aligned} \tag{2.5}$$

The Binet's form of Lucas polynomials is

$$L_i(t) = \frac{(t + \sqrt{t^2+4})^i}{2^i}, \quad i \geq 0. \tag{2.6}$$

Also, the Lucas polynomials have the following power form:

$$L_i(t) = i \sum_{\lfloor(\frac{i}{2})\rfloor}^{k=0} \frac{1}{i-k}\binom{i-k}{k} t^{i-2k}, \quad , i \geq 1. \tag{2.7}$$

and the notation $\lfloor w \rfloor$ represents the largest integer less than or equal to w .

`Generalized Lucas polynomial sequence`

Let e, d be any nonzero real constants, we define the so-called generalized Lucas polynomials [6, 2], which may be generated with the aid of the following recurrence relation:

$$\xi_i^{e,d}(t) = e\ t\ \ \xi_{i-1}^{e,d}(t) + d\xi_{i-2}^{e,d}(t), \quad i \geq 2, \tag{2.8}$$

With the initial values:
$\xi_0^{e,d}(t) = 2, \quad \xi_1^{e,d}(t) = et.$

Lucas polynomials can be deduced form $\xi_i^{e,d}(t)$ for the case $e = d = 1$

`Some other important polynomials can be deduced as special cases of` $\xi_i^{e,d}(t)$`:`

- at $e = 2, d = 1$ we find the Pell-Locus $Q_i(t) = \xi_i^{2,1}(t)$
- at $e = 3, d = -2$ we find Fermat-Locus$F_i(t) = \xi_i^{3,-2}(t)$
- at $e = 2, d = -1$ we find first kind Chebyshev equation $2T_i(t) = \xi_i^{2,-1}(t)$.
- at $e = 1, d = -\alpha$ we find first kind Dickson polynomials $D_i(t, \alpha) = \xi_i^{1,-\alpha}(t)$ [3, 1].

3. Generalized Lucas polynomial sequence method

The step for solving delay system by Generalized Lucas polynomial sequence for a given delay system

$$p(u, u_t, D_t^{\alpha_i}, D_t^{\beta_i}, ...) = f(t) \tag{3.1}$$

where $t \in (0, 1)$,$\alpha_i < \alpha_i + 1$, $i < \alpha \leq i + 1$,$\beta_i < \beta_i + 1$,and $i < \beta \leq i + 1 \quad i = 0, 1, 2, ..., q - 1$ governed by the following initial conditions $u^(i)(0) = a_i$ is
Step1. Let

$$u(t) \approx u_M(t) = C^T \xi(t) \tag{3.2}$$

where $c^T = [c_0, c_1, ..., c_M]^T$ and $\xi(t) = [\xi_0^{e,d}(t), \xi_1^{e,d}(t), ..., \xi_M^{e,d}(t)]^T$
Step2. we can approximate $D_t^{\alpha_i} and, D_t^{\beta_i}$ as

$$\begin{aligned} D_t^{\alpha_i} &\approx t^{-\alpha_i} C^T G^{(\alpha_i)} \xi(t) \\ D_t^{\beta_i} &\approx t^{-\beta_i} C^T G^{(\beta_i)} \xi(t) \end{aligned} \tag{3.3}$$

where $G^{(1)} = (g_{ij}^{(1)})$,is the $(M + 1) \times (M + 1)$operational matrix of derivatives.

The entries of this matrix are given explicitly as

$$(g_{ij}^{(1)}) = \begin{cases} (-1)^{\frac{l-j+1}{2}} ied^{\frac{l-j+1}{2}}\delta_i, & If i > j, (i+j) odd, \\ \\ 0, & otherwise. \end{cases} \quad \text{for } \delta_i = \begin{cases} \frac{1}{2}, & j = 0, \\ \\ 1, & j > 0 \end{cases}$$

Step3. With the aid of the approximations in Eqs. (3.2) and (3.3), the residual of Eq. (3.1) can be calculated by the formula

$$t^{\alpha_q}R(t) = p(u, t^{\alpha_q}C^T\xi(t), C^T G^{(\alpha_i)}\xi(t), t^{\alpha_q-\beta_i}C^T G^{(\beta_i)}\xi(t), ...) - t^{\alpha_q}f(t) \tag{3.4}$$

Step4. The following system of equations can be obtained

$$\int_0^1 t^{\alpha_q}R(t)\xi_i^{e,d}(t)dt = 0, \quad i = i = 0, 1, 2, ..., M - q. \tag{3.5}$$

In addition, the initial conditions

$$C^T G^{(i)}\xi(0) = a_i, \quad i = 0, 1, 2, ..., q - 1. \tag{3.6}$$

Now, Eq. (3.5) and (3.6) constitute a linear system of algebraic equations in the unknown expansion coefficients of dimension $(M + 1)$.

Step5. By using mathematica program we can find the solution of this system.

4. Numerical results

We evaluate the solutions of some Delay differential equation of fractional order by using Generalized Lucas polynomial sequence approach. Figure 1 presents the exact analytical solution of the considered problem. The solution exhibits a highly oscillatory behavior characterized by alternating positive and negative amplitudes, indicating the presence of multiple local extrema over the computational domain. Such behavior reflects the complex dynamics of the governing fractional delay differential equation and serves as a reliable benchmark for evaluating the accuracy of the proposed numerical technique. Figures 2, 3, and 4 illustrate the approximate locus solutions obtained for $e = 1$ and $d = 1$ while varying the polynomial order n from 4 to 6. As shown in Figures 2 and 3, corresponding to $n = 4$ and $n = 5$, the numerical solutions display nearly identical profiles. Both curves possess a pronounced primary peak followed by a gradual decrease and a secondary hump before smoothly approaching zero near the end of the interval. The close agreement between these two approximations demonstrates the stability and consistency of the proposed numerical scheme even for relatively low polynomial orders. When the polynomial order is increased to $n = 6$ as shown in figure 4, the solution undergoes a significant qualitative change, exhibiting rapid monotonic growth with an exponential-like trend over the interval. The substantial increase in magnitude suggests that higher-order approximations produce a markedly different solution behavior, which may be attributed to the sensitivity of the problem or the numerical characteristics of the chosen basis functions. This observation indicates that the approximation is strongly influenced by the polynomial order, emphasizing the importance of selecting an appropriate value of n to achieve physically meaningful and numerically stable results.

4.1. Example

The damped mechanical oscillator is an oscillator which slows the motion of the system. Due to frictional force, the velocity decreases in proportion to the acting frictional force. While in a simple undriven harmonic oscillator the only force acting on the mass is the restoring force, in a damped harmonic oscillator there is in addition a frictional force which is always in a direction to oppose the motion[12, 14].

$$D^{\alpha}y(t) + \lambda D^{\beta}y(t) + \nu y(t) = g(t), \quad t \in [0,1],$$
$$y(0) = 0, y(1) = 0, \tag{4.1}$$

where
$1 < \alpha \le 2, 0 < \beta \le 1, \alpha - \beta > 1,\ \lambda,\ \nu\ are\ constants,\ \lambda = \nu = 1$
$and\ g(t)\ is\ the\ forcing\ function.$
Table 1 : maximum absolute errors of example in case $0 \le t \le 1$

e	d	$(Max_m error)$	(t)	$(Max_m error)$	(t)	$(Max_m error)$	(t)
1	1	0.785162	0.212655	$1.94821 * 10^{-}6$	0.167225	0.0000337882	0.400846
2	1	0.785162	0.212655	$6.50724 * 10^{-}9$	0.177696	$1.21262 * 10^{-}6$	0.364648
2	−1	0.785162	0.212655	$1.59058 * 10^{-}9$	0.180061	$2.51813 * 10^{-}8$	0.371269
3	2	0.785162	0.212655	$1.94821 * 10^{-}6$	0.167225	0.0000337882	0.400846

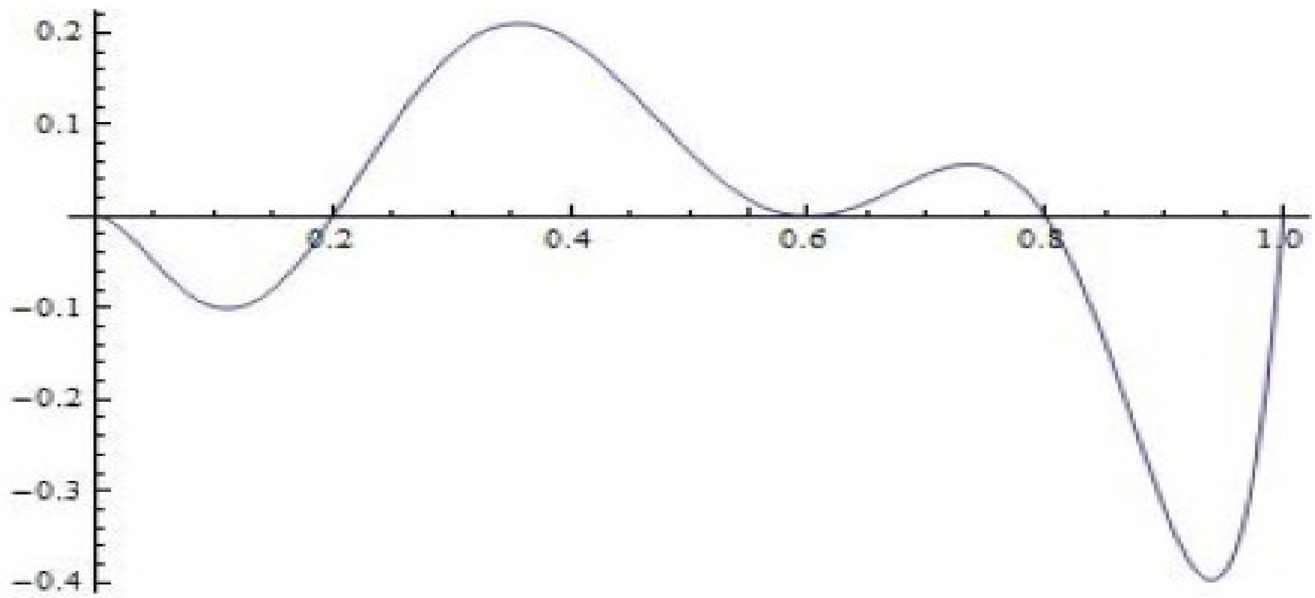

Figure 1. Exact solution

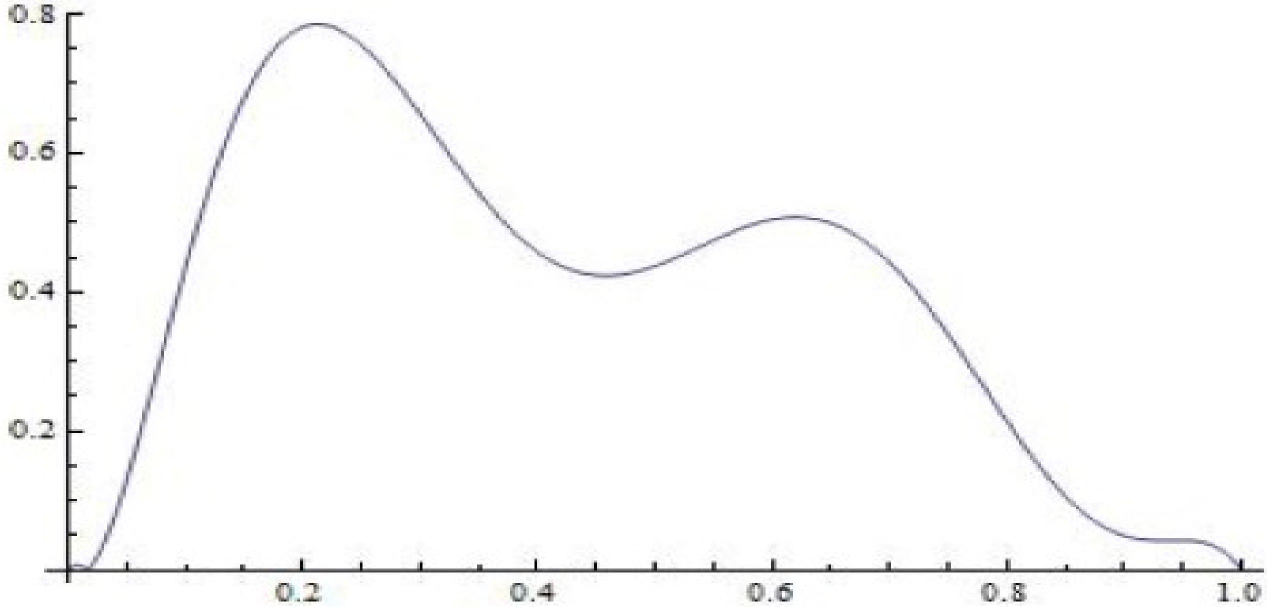

Figure 2. locus solution at n=4,e=1,d=1

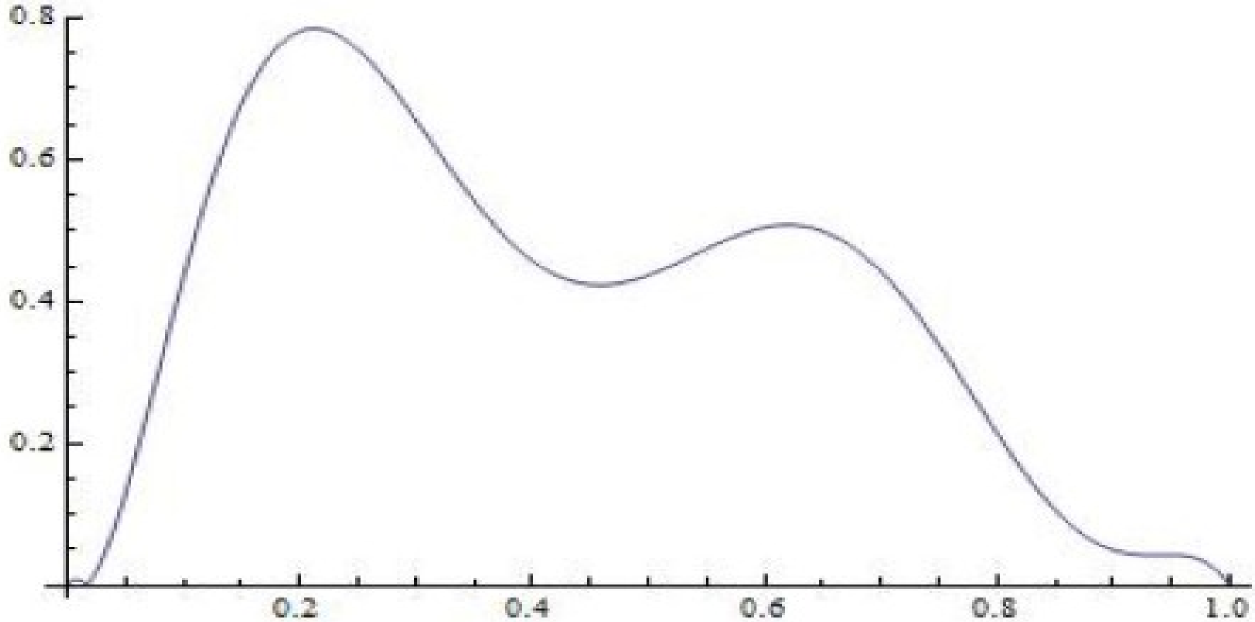

Figure 3. locus solution at n=5,e=1,d=1

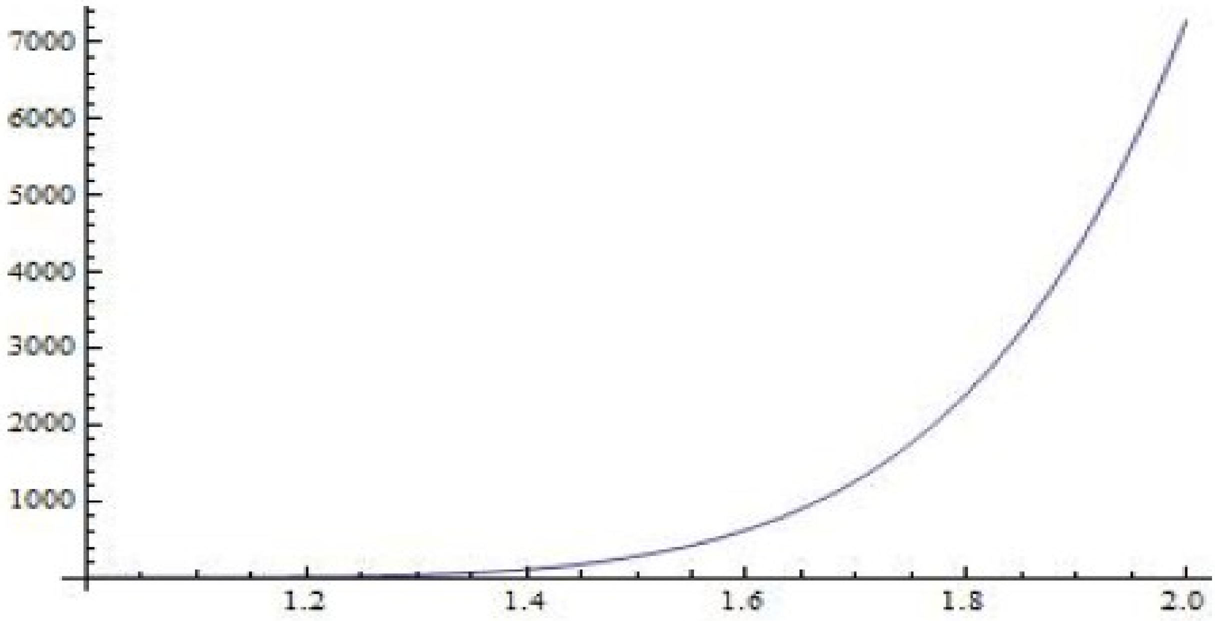

Figure 4. locus solution at n=6,e=1,d=1

5. Conclusions

The primary objective of this article is to develop an efficient numerical technique for solving fractional-order delay differential equations by employing the Tau method based on the Generalized Lucas polynomial sequence approach. Fractional delay differential equations have attracted considerable attention due to their ability to accurately describe dynamic systems possessing memory and hereditary characteristics, which cannot be adequately represented by classical integer-order models. In the proposed formulation, the fractional derivatives are evaluated in both the Riemann–Liouville and Caputo senses, allowing the method to be applied to a broad class of fractional models with different initial condition formulations. The Generalized Lucas polynomial basis provides an accurate and computationally efficient approximation of the unknown solution, while the Tau method transforms the original fractional delay differential equation into a system of algebraic equations that can be solved with relative ease. The proposed approach is straightforward to implement, computationally stable, and capable of producing highly accurate numerical approximations with a relatively small number of basis functions. To demonstrate the effectiveness, reliability, and practical applicability of the proposed technique, a fractionally damped mechanical oscillator with a time-delay term is considered as a representative example. Numerical simulations reveal that the method successfully captures the dynamic behavior of the delayed fractional system and yields accurate results, confirming its suitability as an efficient tool for solving a wide range of fractional delay differential equations arising in science and engineering.

Conflict of Interest

The authors declare there is no existing conflict of interest.

References

1.Abd-Elhameed, W. and Youssri, Y. (2016). Spectral solutions for fractional differential equations via a novel lucas operational matrix of fractional derivatives. *Rom. J. Phys*, 61(5-6):795–813.

2.Abd-Elhameed, W. and Youssri, Y. (2017). Generalized lucas polynomial sequence approach for fractional differential equations. *Nonlinear Dynamics*, 89(2):1341–1355.

3.Blaszczyk, T. and Ciesielski, M. (2016). Fractional oscillator equation: analytical solution and algorithm for its approximate computation. *Journal of Vibration and Control*, 22(8):2045–2052.

4.Braun, M. and Golubitsky, M. (1983). *Differential equations and their applications*, volume 4. Springer.

5.Daftardar-Gejji, V. (2019). *Fractional calculus and fractional differential equations*. Springer.

6.Das, D., Ray, P., Bera, R., and Sarkar, P. (2015). Solution of nonlinear fractional differential equation (nfde) by homotopy analysis method. *Int. J. Sci. Res. Edu*, 3(3):3084–3103.

7.Doha, E., Abd-Elhameed, W., and Bassuony, M. (2015). On using third and fourth kinds chebyshev operational matrices for solving lane-emden type equations. *Rom. J. Phys*, 60(3-4):281–292.

8.Driver, R. D. (2012). *Ordinary and delay differential equations*. Springer Science & Business Media.

9.Evans, D. J. and Raslan, K. R. (2005). The adomian decomposition method for solving delay differential equation. *International Journal of Computer Mathematics*, 82(1):49–54.

10.Halanay, A. (1966). Differential equations: Stability, oscillations, time lags. *(No Title)*.

11.Hesthaven, J. S., Gottlieb, D. I., and Gottlieb, S. (2007). *Spectral methods for time-dependent problems*, volume 21. Cambridge University Press Cambridge.

12.Li, Y. and Zhao, W. (2010). Haar wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations. *Applied mathematics and computation*, 216(8):2276–2285.

13.Lu, B. (2012). The first integral method for some time fractional differential equations. *Journal of Mathematical Analysis and Applications*, 395(2):684–693.

14.Torvik, P. J. and Bagley, R. L. (1984). On the appearance of the fractional derivative in the behavior of real materials.