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Research article

Doubly warped product manifolds: investigations through the concircular curvature tensor and relativistic applications

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ABSTRACT

This paper presents a description of a doubly warped product manifold under several conditions imposed on the concircular curvature tensor. We prove that the factor manifolds of a concircularly flat (symmetric) doubly warped product manifold possess constant sectional curvature, and are themselves concircularly flat. In the flatness scenario, a doubly warped product manifold reduces to a singly warped product manifold. We establish that the factor manifolds of a doubly warped product manifold with a harmonic concircular curvature tensor are Einstein manifolds; moreover, a harmonic concircular curvature tensor is shown to force the scalar curvature to be constant, which is a stronger conclusion than the one available for the projective curvature tensor. We further clarify the position of concircular flatness among the conformal and projective flatness conditions and illustrate the theory with an explicit example. In the application section, we prove that a concircularly flat (symmetric) Generalized Robertson–Walker (GRW) space-time is an Einstein perfect fluid which is static and obeys the equation of state $p = -\rho$, so that it behaves as a dark-energy (cosmological-constant) era and satisfies the weak and dominant energy conditions.

1. Introduction

Doubly warped product manifolds are a rich area of research with important implications in various branches of mathematics and physics. They offer a versatile framework for modeling and analyzing curved space-time and provide a deeper understanding of geometric and topological properties of manifolds. These manifolds arise as natural generalizations of singly warped product manifolds [9, 21], which are themselves a generalization of Cartesian products of manifolds [4]. In a doubly warped product manifold, the geometry of the manifold is determined by two warping functions, each acting on a different factor manifold. The warping functions introduce a curvature-like effect on the manifold, causing it to deviate from a simple product structure. Consider two Riemannian manifolds $\left(\mathring{M}, \mathring{g}\right)$ and $\left(\hat{M}, \hat{g}\right)$ with dimensions $\mathring{n}$ and $\hat{n}$, respectively. Additionally, let ψ and φ be smooth positive functions defined on $\mathring{M}$ and $\hat{M}$ respectively, referred to as warping functions. The doubly warped product manifold $M = \mathring{M}_{\varphi} \times_{\psi} \hat{M}$ is the Riemannian manifold $\mathring{M} \times \hat{M}$ equipped with the metric $g : (\varphi \circ \sigma)^2 \pi^* (\mathring{g}) + (\psi \circ \pi)^2 \sigma^* (\hat{g})$, where $\pi : \mathring{M} \times \hat{M} \to \mathring{M}$ and $\sigma : \mathring{M} \times \hat{M} \to \hat{M}$ are the natural projection maps [11]. This concept has been extensively studied by numerous authors. For further exploration, see for example [3, 7, 6, 5, 15, 27].

Curvature invariants are fundamental tools in geometry, relativity, and physics. They allow for a deeper understanding of the geometry of spaces, the behavior of gravitational fields, and the physics of curved space-time. These invariants are quantities that are constructed from curvature tensors, which capture the curvature of a manifold. The Riemann R_{ijkl}, Ricci R_{ij}, and Weyl curvature tensors C_{ijkl} are widely recognized as the most well-known and widely studied curvature tensors. Numerous other significant curvature tensors are derived from combinations of the Riemann and Ricci tensors. For instance, the projective curvature [25], the concircular curvature [28], and conharmonic curvature tensors [14]. These tensors provide additional insights into the geometric properties of manifolds and have been the subject of extensive research and analysis. The significance of such curvature tensors has been demonstrated by numerous authors, as evidenced in works such as [10, 12, 19, 26].

The concircular curvature tensor was introduced by Yano [28] in his study of concircular transformations, that is, conformal transformations that preserve the geodesic circles of a Riemannian manifold. Its structure is given by [28, 29]:

$$C_{ijkl} = R_{ijkl} - \frac{R}{n\,(n-1)} \left(g_{il} g_{jk} - g_{jl} g_{ik}\right), \tag{1.1}$$

where R is the scalar curvature of M and $n = \dim M$. The concircular curvature tensor measures precisely the deviation of a manifold from having constant sectional curvature: a manifold of constant sectional curvature has a vanishing concircular curvature tensor, and conversely a Riemannian manifold ($n > 3$) with $C_{ijkl} = 0$ and constant scalar curvature is a space of constant curvature. A manifold M is said to be concircularly flat if its concircular curvature tensor vanishes, and it is termed concircularly symmetric when $\nabla_h C_{ijkl} = 0$. Furthermore, it possesses a harmonic concircular curvature tensor if $\nabla^h C_{hijk} = 0$ [18]. Owing to its geometric meaning and its role in general relativity, the concircular curvature tensor has attracted considerable attention. In [1], Riemannian spaces with recurrent concircular curvature tensor were investigated; concircular curvature tensors on contact metric manifolds were examined in [8]; and the role of the concircular curvature tensor in the study of fluid space-times was carried out by Ahsan and Siddiqui [2].

The objective of this paper is to explore characterizations of the factor manifolds of a doubly warped product manifold based on certain conditions of the concircular curvature tensor. Section 3 focuses on examining the impact of the flatness of the concircular curvature tensor on the factor manifolds of $\mathring{M}_{\varphi} \times_{\psi} \hat{M}$. Section 4 provides properties of the factor manifolds of $\mathring{M}_{\varphi} \times_{\psi} \hat{M}$ through the symmetry of the concircular curvature tensor. Section 5 presents properties of the factor manifolds of $\mathring{M}_{\varphi} \times_{\psi} \hat{M}$ when the concircular curvature tensor is harmonic (of divergence-free type). Section 6 situates concircular flatness among the related flatness conditions and provides an explicit example. Finally, concircularly flat (symmetric) generalized Robertson–Walker space-times are investigated, and their physical content is analyzed.

2. Preliminaries

This section provides fundamental information regarding the curvature of a doubly warped product manifold $\mathring{M}_{\varphi}\times_{\psi}\hat{M}$. The curvature of $\mathring{M}_{\varphi}\times_{\psi}\hat{M}$ is determined by the curvature of its factor manifolds (the base and the fiber), as well as the warping functions. The metric tensor of $\mathring{M}_{\varphi}\times_{\psi}\hat{M}$ is given locally as [11]

$$\left(g_{ij}\right)=\begin{cases}\varphi\mathring{g}_{ab} & \text{for } i=a,\ j=b\\ \psi\hat{g}_{\alpha\beta} & \text{for } i=\alpha,\ j=\beta\\ 0 & \text{otherwise}\end{cases}, \tag{2.1}$$

where $i,j\in\{1,2,...,n\}$, $a,b,...\in\{1,2,...,\mathring{n}\}$, and $\alpha,\beta,...\in\{\mathring{n}+1,...,n\}$.
Locally, the Riemann tensor of $\mathring{M}_{\varphi}\times_{\psi}\hat{M}$ satisfies [13]

$$R_{abcd}=\varphi\mathring{R}_{abcd}+\frac{\hat{\triangle}\varphi}{4\psi}\mathring{G}_{abcd}, \tag{2.2}$$

$$R_{\alpha\beta\gamma\delta}=\psi\hat{R}_{\alpha\beta\gamma\delta}+\frac{\mathring{\triangle}\psi}{4\varphi}\hat{G}_{\alpha\beta\gamma\delta}, \tag{2.3}$$

$$R_{\alpha\beta\gamma d}=\frac{\psi_d}{4\varphi}\left[\varphi_{\beta}\hat{g}_{\alpha\gamma}-\varphi_{\alpha}\hat{g}_{\beta\gamma}\right], \tag{2.4}$$

$$R_{a\beta c\delta}=\psi\mathring{T}_{ac}\hat{g}_{\beta\delta}+\varphi\hat{T}_{\beta\delta}\mathring{g}_{ac}, \tag{2.5}$$

where

$$\mathring{G}_{abcd}=\left(\mathring{g}_{ad}\mathring{g}_{bc}-\mathring{g}_{ac}\mathring{g}_{bd}\right),\qquad \hat{G}_{\alpha\beta\gamma\delta}=\left(\hat{g}_{\alpha\delta}\hat{g}_{\beta\gamma}-\hat{g}_{\alpha\gamma}\hat{g}_{\beta\delta}\right),$$

$$\hat{\triangle}\varphi=\hat{g}^{\beta\gamma}\varphi_{\beta}\varphi_{\gamma},\quad \mathring{\triangle}\psi=\mathring{g}^{ab}\psi_a\psi_b,\quad \varphi_{\beta}=\partial_{\beta}\varphi,\quad \psi_a=\partial_a\psi,$$

$$\mathring{T}_{ac}=-\frac{1}{2\psi}\left[\mathring{\nabla}_c\psi_a-\frac{1}{2\psi}\psi_a\psi_c\right],\quad \hat{T}_{\beta\delta}=-\frac{1}{2\varphi}\left[\hat{\nabla}_{\beta}\varphi_{\delta}-\frac{1}{2\varphi}\varphi_{\beta}\varphi_{\delta}\right].$$

Moreover, the covariant derivative of the Riemann tensor of $\mathring{M}_{\varphi}\times_{\psi}\hat{M}$ obeys the following equations [13]:

$$\begin{aligned}\nabla_{\varepsilon}R_{abcd}&=-\varphi_{\varepsilon}\mathring{R}_{abcd}+\frac{\varphi^2}{\psi}\partial_{\varepsilon}\left(\frac{\hat{\triangle}\varphi}{4\varphi^2}\mathring{G}_{abcd}\right)\\&\quad+\frac{\varphi_{\varepsilon}}{4\psi^2}\left[\psi_a\psi_d\mathring{g}_{bc}+\psi_b\psi_c\mathring{g}_{ad}-\psi_a\psi_c\mathring{g}_{bd}-\psi_b\psi_d\mathring{g}_{ac}\right],\end{aligned} \tag{2.6}$$

$$\begin{aligned}\nabla_{e}R_{\alpha\beta\gamma\delta}&=-\psi_{e}\hat{R}_{\alpha\beta\gamma\delta}+\frac{\psi^2}{\varphi}\partial_{e}\left(\frac{\mathring{\triangle}\psi}{4\psi^2}\hat{G}_{\alpha\beta\gamma\delta}\right)\\&\quad-\frac{\psi_{e}}{4\varphi^2}\left[\varphi_{\alpha}\varphi_{\delta}\hat{g}_{\beta\gamma}+\varphi_{\beta}\varphi_{\gamma}\hat{g}_{\alpha\delta}-\varphi_{\alpha}\varphi_{\gamma}\hat{g}_{\beta\delta}-\varphi_{\beta}\varphi_{\delta}\hat{g}_{\alpha\gamma}\right].\end{aligned} \tag{2.7}$$

The following relations hold for the Ricci tensor of $\mathring{M}_{\varphi}\times_{\psi}\hat{M}$:

$$\begin{aligned}\nabla_cR_{ab}&=\mathring{\nabla}_c\mathring{R}_{ab}-\hat{n}\mathring{\nabla}_c\mathring{T}_{ab}-\frac{\psi_c}{\psi^2}\left[\frac{(\mathring{n}-1)\,\hat{\triangle}\varphi}{4\varphi}-\varphi\hat{T}\right]\mathring{g}_{ab}\\&\quad-\frac{(n-2)\,\hat{\triangle}\varphi}{8\varphi\psi^2}\left(\psi_a\mathring{g}_{bc}+\psi_b\mathring{g}_{ac}\right),\end{aligned} \tag{2.8}$$

$$\nabla_\gamma R_{\alpha\beta} = \hat{\nabla}_\gamma \hat{R}_{\alpha\beta} - \mathring{n}\hat{\nabla}_\gamma \hat{T}_{\alpha\beta} - \frac{\varphi_\gamma}{\varphi^2}\left[\frac{(\hat{n}-1)\,\mathring{\Delta}\psi}{4\psi} - \psi\mathring{T}\right]\hat{g}_{\alpha\beta}$$
$$- \frac{(n-2)\,\mathring{\Delta}\psi}{8\psi\varphi^2}\left(\varphi_\beta\hat{g}_{\alpha\gamma} + \varphi_\alpha\hat{g}_{\beta\gamma}\right), \tag{2.9}$$

$$\nabla_c R_{a\beta} = \frac{(n-2)}{4\varphi\psi^2}\varphi_\beta\psi_a\psi_c + \frac{(n+\hat{n}-2)}{2\varphi}\varphi_\beta\mathring{T}_{ac} - \frac{1}{2\varphi}\varphi_\beta\mathring{R}_{ac}$$
$$+ \frac{1}{2\psi}\left[\varphi^\gamma\hat{R}_{\gamma\beta} - \mathring{n}\varphi^\gamma\hat{T}_{\gamma\beta} + \frac{(\hat{n}-1)\,\mathring{\Delta}\psi}{4\psi\varphi}\varphi_\beta - \frac{(\hat{n}-1)\,\hat{\Delta}\varphi}{4\varphi^2}\varphi_\beta\right]\mathring{g}_{ac}$$
$$+ \left[\frac{\varphi_\beta}{2\psi}\hat{T} - \frac{\varphi_\beta}{2\varphi}\mathring{T}\right]\mathring{g}_{ac}, \tag{2.10}$$

$$\nabla_\gamma R_{\alpha b} = \frac{(n-2)}{4\psi\varphi^2}\varphi_\alpha\varphi_\gamma\psi_b + \frac{(n+\mathring{n}-2)}{2\psi}\psi_b\hat{T}_{\alpha\gamma} - \frac{1}{2\psi}\psi_b\hat{R}_{\alpha\gamma}$$
$$+ \frac{1}{2\varphi}\left[\psi^c\mathring{R}_{cb} - \hat{n}\psi^c\mathring{T}_{cb} + \frac{(\mathring{n}-1)\,\hat{\Delta}\varphi}{4\psi\varphi}\psi_b - \frac{(\hat{n}-1)\,\mathring{\Delta}\psi}{4\psi^2}\psi_b\right]\hat{g}_{\alpha\gamma}$$
$$+ \left[\frac{\psi_b}{2\varphi}\mathring{T} - \frac{\psi_b}{2\psi}\hat{T}\right]\hat{g}_{\alpha\gamma}, \tag{2.11}$$

$$\nabla_\gamma R_{ab} = \left[\frac{1}{\psi}\partial_\gamma\left(\frac{(\mathring{n}-1)\,\hat{\Delta}\varphi}{4\varphi} - \varphi\hat{T}\right) - \frac{\varphi_\gamma}{\psi\varphi}\left(\frac{(\mathring{n}-1)\,\hat{\Delta}\varphi}{4\varphi} - \varphi\hat{T}\right)\right]\mathring{g}_{ab}$$
$$- \frac{\varphi_\gamma}{\varphi}\left[\mathring{R}_{ab} - \hat{n}\mathring{T}_{ab}\right] + \frac{(n-2)}{4\varphi\psi^2}\varphi_\gamma\psi_a\psi_b, \tag{2.12}$$

$$\nabla_c R_{\alpha\beta} = \left[\frac{1}{\varphi}\partial_c\left(\frac{(\hat{n}-1)\,\mathring{\Delta}\psi}{4\psi} - \psi\mathring{T}\right) - \frac{\psi_c}{\psi\varphi}\left(\frac{(\hat{n}-1)\,\mathring{\Delta}\psi}{4\psi} - \psi\mathring{T}\right)\right]\hat{g}_{\alpha\beta}$$
$$- \frac{\psi_c}{\psi}\left[\hat{R}_{\alpha\beta} - \mathring{n}\hat{T}_{\alpha\beta}\right] + \frac{(n-2)}{4\psi\varphi^2}\varphi_\alpha\varphi_\beta\psi_c, \tag{2.13}$$

where

$$\hat{T} = \hat{g}^{\beta\delta}\hat{T}_{\beta\delta}, \qquad \mathring{T} = \mathring{g}^{ab}\mathring{T}_{ab}.$$

We shall also use the contracted second Bianchi identities

$$\nabla^h R_{hijk} = \nabla_i R_{jk} - \nabla_j R_{ik}, \qquad \nabla^j R_{ij} = \frac{1}{2}\nabla_i R. \tag{2.14}$$

3. Concircularly flat doubly warped product manifolds

In this section, investigations of doubly warped product manifolds through the flatness of the concircular curvature tensor are carried out. If $\mathring{M}_\varphi \times_\psi \hat{M}$ is concircularly flat, it can be inferred from Eq. (1.1) that

$$R_{ijkl} = \frac{R}{n\,(n-1)}\left(g_{il}g_{jk} - g_{jl}g_{ik}\right). \tag{3.1}$$

Transvecting Eq. (3.1) with g^{jk}, we conclude that

$$R_{il} = \frac{R}{n}g_{il}, \tag{3.2}$$

this implies that $\mathring{M}_\varphi \times_\psi \hat{M}$ is an Einstein manifold. Moreover, Eq. (3.1) says directly that $\mathring{M}_\varphi \times_\psi \hat{M}$ has constant sectional curvature.
The outcomes obtained in this section can be condensed into the subsequent theorem.

Theorem 3.1. *Let $M = \mathring{M}_\varphi \times_\psi \hat{M}$ be a concircularly flat doubly warped product manifold. Then, the following properties hold:*

1. *the factor manifolds possess constant sectional curvature, and hence they are concircularly flat,*
2. *the doubly warped product manifold $M = \mathring{M}_\varphi \times_\psi \hat{M}$ is reduced to be a singly warped product manifold, and*
3. *the metric tensors of the factor manifolds are proportional to $\mathring{T}_{ac}$ and $\hat{T}_{\beta\delta}$, respectively.*

Proof. Starting with the following component of the Riemann curvature tensor, which is expressed as follows:

$$R_{abcd} = \frac{R}{n\,(n-1)}\left(g_{ad}g_{bc} - g_{bd}g_{ac}\right). \tag{3.3}$$

Eqs. (2.1) and (2.2) are employed in Eq. (3.3) to produce

$$\mathring{R}_{abcd} = \left[\frac{\varphi R}{n\,(n-1)} - \frac{\hat{\Delta}\varphi}{4\psi\varphi}\right]\mathring{G}_{abcd}. \tag{3.4}$$

Transvecting with $\mathring{g}^{ad}$, we get

$$\mathring{R}_{bc} = (\mathring{n}-1)\left[\frac{\varphi R}{n\,(n-1)} - \frac{\hat{\Delta}\varphi}{4\psi\varphi}\right]\mathring{g}_{bc}. \tag{3.5}$$

Transvecting with $\mathring{g}^{bc}$, one acquires that

$$\mathring{R} = \mathring{n}\,(\mathring{n}-1)\left[\frac{\varphi R}{n\,(n-1)} - \frac{\hat{\Delta}\varphi}{4\psi\varphi}\right]. \tag{3.6}$$

From Eq. (3.6) in Eq. (3.5), we have

$$\mathring{R}_{bc} = \frac{\mathring{R}}{\mathring{n}}\mathring{g}_{bc}, \tag{3.7}$$

this demonstrates that $\mathring{M}$ is an Einstein manifold.
Employing Eq. (3.6) in Eq. (3.4), one receives

$$\mathring{R}_{abcd} = \frac{\mathring{R}}{\mathring{n}\,(\mathring{n}-1)}\mathring{G}_{abcd}, \tag{3.8}$$

this indicates that $\mathring{M}$ is of constant sectional curvature.
The concircular curvature tensor of the base manifold is defined as follows:

$$\mathring{C}_{abcd} = \mathring{R}_{abcd} - \frac{\mathring{R}}{\mathring{n}\,(\mathring{n}-1)}\mathring{G}_{abcd}. \tag{3.9}$$

From Eq. (3.8) in Eq. (3.9), one provides

$$\mathring{C}_{abcd} = 0,$$

this tells us that the base manifold is concircularly flat.
In view of Eq. (3.1), one can deduce that

$$R_{\alpha\beta\gamma\delta} = \frac{R}{n\,(n-1)}\left(g_{\alpha\delta}g_{\beta\gamma} - g_{\beta\delta}g_{\alpha\gamma}\right). \tag{3.10}$$

Utilizing Eqs. (2.1) and (2.3) in Eq. (3.10), we reach for

$$\hat{R}_{\alpha\beta\gamma\delta} = \left[\frac{\psi R}{n\,(n-1)} - \frac{\mathring{\Delta}\psi}{4\varphi\psi}\right]\hat{G}_{\alpha\beta\gamma\delta}. \tag{3.11}$$

Transvecting with $\hat{g}^{\alpha\delta}$, one finds that

$$\hat{R}_{\beta\gamma} = (\hat{n}-1)\left[\frac{\psi R}{n\,(n-1)} - \frac{\mathring{\Delta}\psi}{4\varphi\psi}\right]\hat{g}_{\beta\gamma}. \tag{3.12}$$

Transvecting with $\hat{g}^{\beta\gamma}$, we infer that

$$\hat{R} = \hat{n}\,(\hat{n}-1)\left[\frac{\psi R}{n\,(n-1)} - \frac{\mathring{\Delta}\psi}{4\varphi\psi}\right]. \tag{3.13}$$

From Eq. (3.13) in Eq. (3.12), we obtain

$$\hat{R}_{\beta\gamma} = \frac{\hat{R}}{\hat{n}}\hat{g}_{\beta\gamma}, \tag{3.14}$$

this clarifies that $\hat{M}$ is an Einstein manifold.
Employing Eq. (3.13) in Eq. (3.11), one acquires that

$$\hat{R}_{\alpha\beta\gamma\delta} = \frac{\hat{R}}{\hat{n}\,(\hat{n}-1)}\hat{G}_{\alpha\beta\gamma\delta}, \tag{3.15}$$

this evidences that $\hat{M}$ has constant sectional curvature.
The concircular curvature tensor of the fiber manifold is expressed as follows:

$$\hat{C}_{\alpha\beta\gamma\delta} = \hat{R}_{\alpha\beta\gamma\delta} - \frac{\hat{R}}{\hat{n}\,(\hat{n}-1)}\hat{G}_{\alpha\beta\gamma\delta}. \tag{3.16}$$

The use of Eq. (3.15) in Eq. (3.16), one infers that

$$\hat{C}_{\alpha\beta\gamma\delta} = 0,$$

this illustrates that the fiber manifold is concircularly flat.
From Eq. (3.1), we can write the following component

$$R_{\alpha\beta\gamma d} = \frac{R}{n\,(n-1)}\left(g_{\alpha d}g_{\beta\gamma} - g_{\beta d}g_{\alpha\gamma}\right). \tag{3.17}$$

Inserting Eqs. (2.1) and (2.4) in Eq. (3.17), one discovers that

$$\frac{\psi_d}{4\varphi}\left[\varphi_\beta\hat{g}_{\alpha\gamma} - \varphi_\alpha\hat{g}_{\beta\gamma}\right] = 0,$$

since the right-hand side of Eq. (3.17) vanishes (the indices α, β, γ belong to the fiber while d belongs to the base, so the mixed metric products are zero). Contracting with $\hat{g}^{\alpha\gamma}$, one infers

$$\psi_d \varphi_\beta = 0, \tag{3.18}$$

this means that either φ is constant or ψ is constant. In this scenario, a doubly warped product manifold is reduced to be a singly warped product manifold [4].
In virtue of Eq. (3.1), we can find that

$$R_{a\beta c\delta} = \frac{R}{n(n-1)} \left(g_{a\delta} g_{\beta c} - g_{\beta\delta} g_{ac} \right). \tag{3.19}$$

Employing Eqs. (2.1) and (2.5) in Eq. (3.19), we observe that

$$\psi \mathring{T}_{ac} \hat{g}_{\beta\delta} + \varphi \hat{T}_{\beta\delta} \mathring{g}_{ac} = -\frac{\psi\varphi R}{n(n-1)} \hat{g}_{\beta\delta} \mathring{g}_{ac}. \tag{3.20}$$

Transvecting Eq. (3.20) with $\hat{g}^{\beta\delta}$, one uncovers

$$\mathring{T}_{ac} = -\left[\frac{\varphi R}{n(n-1)} + \frac{\varphi}{\hat{n}\psi} \hat{T} \right] \mathring{g}_{ac},$$

Contracting Eq. (3.20) with $\mathring{g}^{ac}$, we infer that

$$\hat{T}_{\beta\delta} = -\left[\frac{\psi R}{n(n-1)} + \frac{\psi \mathring{T}}{\mathring{n}\varphi} \right] \hat{g}_{\beta\delta}.$$

The previous two outcomes tell us that the metric tensors of the base and the fiber manifolds are proportional to $\mathring{T}_{ac}$ and $\hat{T}_{\beta\delta}$, respectively.
This completes the proof. □

Remark 1. *The concircular curvature tensor vanishes precisely when the manifold has constant sectional curvature, provided the scalar curvature is constant. Theorem 3.1 therefore exhibits the strongest possible local geometry for a concircularly flat doubly warped product: every factor is a space form. This contrasts with the projectively flat case, where the factor manifolds are guaranteed only to be projectively flat.*

4. Concircularly symmetric doubly warped products

This section aims to examine the influence of the symmetry of the concircular curvature tensor on the factor manifolds of a doubly warped product manifold. Taking the operator ∇_h to Eq. (1.1), it can be deduced that

$$\nabla_h C_{ijkl} = \nabla_h R_{ijkl} - \frac{\nabla_h R}{n(n-1)} \left(g_{il} g_{jk} - g_{jl} g_{ik} \right). \tag{4.1}$$

Assume that $\mathring{M}_\varphi \times_\psi \hat{M}$ is concircularly symmetric, that is $\nabla_h C_{ijkl} = 0$; it follows directly that

$$\nabla_h R_{ijkl} = \frac{\nabla_h R}{n(n-1)} \left(g_{il} g_{jk} - g_{jl} g_{ik} \right). \tag{4.2}$$

This equation says that $\mathring{M}_\varphi \times_\psi \hat{M}$ is locally symmetric if and only if it has constant scalar curvature.

Remark 2. *Unlike the projectively symmetric case, where Eq. (4.2) is obtained only after an intermediate contraction, here it is an immediate consequence of the defining condition $\nabla_h C_{ijkl} = 0$. Thus the covariant derivative of the Riemann tensor of a concircularly symmetric doubly warped product is completely determined by the gradient of the scalar curvature.*

The results in this section can be summarized in the following theorem.

Theorem 4.1. *Let $M = \mathring{M}_\varphi \times_\psi \hat{M}$ be a concircularly symmetric doubly warped product manifold. Then, the following characteristics are achieved:*

1. *the factor manifolds possess constant sectional curvature and hence they are concircularly flat.*

Proof. In virtue of Eq. (4.2), we have

$$\nabla_\varepsilon R_{abcd} = \frac{\nabla_\varepsilon R}{n\,(n-1)}\left(g_{ad}g_{bc} - g_{bd}g_{ac}\right). \tag{4.3}$$

Using Eqs. (2.1) and (2.6) in Eq. (4.3), one can notice that

$$\begin{aligned}\mathring{R}_{abcd} = &\left[\frac{\varphi^2}{\psi\varphi_\varepsilon}\partial_\varepsilon\left(\frac{\hat{\Delta}\varphi}{4\varphi^2}\right) - \frac{\varphi^2\nabla_\varepsilon R}{n\,(n-1)\,\varphi_\varepsilon}\right]\mathring{G}_{abcd} \\ &+ \frac{1}{4\psi^2}\left[\psi_a\psi_d\mathring{g}_{bc} + \psi_b\psi_c\mathring{g}_{ad} - \psi_a\psi_c\mathring{g}_{bd} - \psi_b\psi_d\mathring{g}_{ac}\right].\end{aligned} \tag{4.4}$$

Contracting with $\mathring{g}^{ad}$, one gets

$$\begin{aligned}\mathring{R}_{bc} = &(\mathring{n}-1)\left[\frac{\varphi^2}{\psi\varphi_\varepsilon}\partial_\varepsilon\left(\frac{\hat{\Delta}\varphi}{4\varphi^2}\right) - \frac{\varphi^2\nabla_\varepsilon R}{n\varphi_\varepsilon\,(n-1)} + \frac{\mathring{\Delta}\psi}{4\,(\mathring{n}-1)\,\psi^2}\right]\mathring{g}_{bc} \\ &+ \frac{(\mathring{n}-2)}{4\psi^2}\psi_b\psi_c.\end{aligned} \tag{4.5}$$

Contracting with $\mathring{g}^{bc}$, we discover

$$\mathring{R} = \mathring{n}\,(\mathring{n}-1)\left[\frac{\varphi^2}{\psi\varphi_\varepsilon}\partial_\varepsilon\left(\frac{\hat{\Delta}\varphi}{4\varphi^2}\right) - \frac{\varphi^2\nabla_\varepsilon R}{n\varphi_\varepsilon\,(n-1)} + \frac{\mathring{\Delta}\psi}{2\mathring{n}\psi^2}\right]. \tag{4.6}$$

From Eq. (4.6) in Eq. (4.5), one detects that

$$\mathring{R}_{bc} = \left[\frac{\mathring{R}}{\mathring{n}} - \frac{(\mathring{n}-2)\,\mathring{\Delta}\psi}{4\mathring{n}\psi^2}\right]\mathring{g}_{bc} + \frac{(\mathring{n}-2)}{4\psi^2}\psi_b\psi_c. \tag{4.7}$$

Utilizing Eq. (4.6) in Eq. (4.4), we reveal

$$\begin{aligned}\mathring{R}_{abcd} = &\left[\frac{\mathring{R}}{\mathring{n}\,(\mathring{n}-1)} - \frac{\mathring{\Delta}\psi}{2\mathring{n}\psi^2}\right]\mathring{G}_{abcd} \\ &+ \frac{1}{4\psi^2}\left[\psi_a\psi_d\mathring{g}_{bc} + \psi_b\psi_c\mathring{g}_{ad} - \psi_a\psi_c\mathring{g}_{bd} - \psi_b\psi_d\mathring{g}_{ac}\right].\end{aligned} \tag{4.8}$$

If ψ =constant, then $\mathring{M}$ has constant sectional curvature. That is,

$$\mathring{R}_{abcd} = \frac{\mathring{R}}{\mathring{n}\left(\mathring{n}-1\right)}\mathring{G}_{abcd}.$$

Consequently, we have

$$\mathring{C}_{abcd} = 0,$$

this signifies the base manifold is concircularly flat.
Based on Eq. (4.2), one uncovers that

$$\nabla_e R_{\alpha\beta\gamma\delta} = \frac{\nabla_e R}{n\left(n-1\right)}\left(g_{\alpha\delta}g_{\beta\gamma} - g_{\beta\delta}g_{\alpha\gamma}\right). \tag{4.9}$$

Eqs. (2.1) and (2.7) are inserted in Eq. (4.9), one receives that

$$\begin{aligned}\hat{R}_{\alpha\beta\gamma\delta} = &\left[\frac{\psi^2}{\psi_e\varphi}\partial_e\left(\frac{\mathring{\Delta}\psi}{4\psi^2}\right) - \frac{\psi^2\nabla_e R}{n\left(n-1\right)\psi_e}\right]\hat{G}_{\alpha\beta\gamma\delta}\\ &-\frac{1}{4\varphi^2}\left[\varphi_\alpha\varphi_\delta\hat{g}_{\beta\gamma} + \varphi_\beta\varphi_\gamma\hat{g}_{\alpha\delta} - \varphi_\alpha\varphi_\gamma\hat{g}_{\beta\delta} - \varphi_\beta\varphi_\delta\hat{g}_{\alpha\gamma}\right].\end{aligned} \tag{4.10}$$

Transvecting with $\hat{g}^{\alpha\delta}$, we conclude that

$$\begin{aligned}\hat{R}_{\beta\gamma} = &\left(\hat{n}-1\right)\left[\frac{\psi^2}{\psi_e\varphi}\partial_e\left(\frac{\mathring{\Delta}\psi}{4\psi^2}\right) - \frac{\psi^2\nabla_e R}{n\left(n-1\right)\psi_e} - \frac{\hat{\Delta}\varphi}{4\left(\hat{n}-1\right)\varphi^2}\right]\hat{g}_{\beta\gamma}\\ &-\frac{\left(\hat{n}-2\right)}{4\varphi^2}\varphi_\beta\varphi_\gamma.\end{aligned} \tag{4.11}$$

Transvecting with $\hat{g}^{\beta\gamma}$, we find that

$$\hat{R} = \hat{n}\left(\hat{n}-1\right)\left[\frac{\psi^2}{\psi_e\varphi}\partial_e\left(\frac{\mathring{\Delta}\psi}{4\psi^2}\right) - \frac{\psi^2\nabla_e R}{n\left(n-1\right)\psi_e} - \frac{\hat{\Delta}\varphi}{2\hat{n}\varphi^2}\right]. \tag{4.12}$$

We utilize Eq. (4.12) in Eq. (4.11) to produce that

$$\hat{R}_{\beta\gamma} = \left[\frac{\hat{R}}{\hat{n}} + \frac{\left(\hat{n}-2\right)\hat{\Delta}\varphi}{4\hat{n}\varphi^2}\right]\hat{g}_{\beta\gamma} - \frac{\left(\hat{n}-2\right)}{4\varphi^2}\varphi_\beta\varphi_\gamma.$$

Using Eq. (4.12) in Eq. (4.10), we have

$$\begin{aligned}\hat{R}_{\alpha\beta\gamma\delta} = &\left[\frac{\hat{R}}{\hat{n}\left(\hat{n}-1\right)} + \frac{\hat{\Delta}\varphi}{2\hat{n}\varphi^2}\right]\hat{G}_{\alpha\beta\gamma\delta}\\ &-\frac{1}{4\varphi^2}\left[\varphi_\alpha\varphi_\delta\hat{g}_{\beta\gamma} + \varphi_\beta\varphi_\gamma\hat{g}_{\alpha\delta} - \varphi_\alpha\varphi_\gamma\hat{g}_{\beta\delta} - \varphi_\beta\varphi_\delta\hat{g}_{\alpha\gamma}\right].\end{aligned}$$

Suppose that φ =constant, thus

$$\hat{R}_{\alpha\beta\gamma\delta} = \frac{\hat{R}}{\hat{n}\left(\hat{n}-1\right)}\hat{G}_{\alpha\beta\gamma\delta},$$

which illustrates that $\hat{M}$ possesses constant sectional curvature.
It follows from the defining property of the concircular curvature tensor that

$$\hat{C}_{\alpha\beta\gamma\delta} = 0,$$

this tells us that $\hat{M}$ is concircularly flat. □

5. Doubly warped product manifolds with harmonic concircular curvature tensor

The objective of this section is to analyze how the divergence-free property of the concircular curvature tensor affects the factor manifolds of a doubly warped product manifold $\mathring{M}_\varphi \times_\psi \hat{M}$. Applying the operator ∇_h on Eq. (1.1) and transvecting with g^{hl}, we obtain

$$\nabla^h C_{hijk} = \nabla^h R_{hijk} - \frac{1}{n(n-1)}\left(g_{jk}\nabla_i R - g_{ik}\nabla_j R\right). \tag{5.1}$$

The use of the contracted Bianchi identity $\nabla^h R_{hijk} = \nabla_i R_{jk} - \nabla_j R_{ik}$ from Eq. (2.14) gives

$$\nabla^h C_{hijk} = \left(\nabla_i R_{jk} - \nabla_j R_{ik}\right) - \frac{1}{n(n-1)}\left(g_{jk}\nabla_i R - g_{ik}\nabla_j R\right). \tag{5.2}$$

Assume that $\nabla^h C_{hijk} = 0$, thus Eq. (5.2) implies that

$$\nabla_i R_{jk} - \nabla_j R_{ik} = \frac{1}{n(n-1)}\left(g_{jk}\nabla_i R - g_{ik}\nabla_j R\right). \tag{5.3}$$

Contracting Eq. (5.3) with g^{jk} and using $g^{jk}\nabla_i R_{jk} = \nabla_i R$ together with $g^{jk}\nabla_j R_{ik} = \frac{1}{2}\nabla_i R$ from Eq. (2.14), the left-hand side becomes $\frac{1}{2}\nabla_i R$, while the right-hand side becomes $\frac{1}{n}\nabla_i R$. Hence

$$\left(\frac{1}{2} - \frac{1}{n}\right)\nabla_i R = 0,$$

and since $n > 2$ we conclude that

$$\nabla_i R = 0, \tag{5.4}$$

that is, $\mathring{M}_\varphi \times_\psi \hat{M}$ has constant scalar curvature. Substituting Eq. (5.4) into Eq. (5.3) yields

$$\nabla_i R_{jk} - \nabla_j R_{ik} = 0, \tag{5.5}$$

this demonstrates that the Ricci tensor of $\mathring{M}_\varphi \times_\psi \hat{M}$ is of Codazzi type.

Remark 3. *Eq. (5.4) shows that a doubly warped product manifold with a harmonic concircular curvature tensor necessarily has constant scalar curvature. This is a strictly stronger conclusion than the one obtainable for the projective curvature tensor, where constancy of the scalar curvature has to be assumed separately.*

The findings presented in this portion can be succinctly stated in the subsequent theorem.

Theorem 5.1. *Let $M = \mathring{M}_\varphi \times_\psi \hat{M}$ be a doubly warped product manifold with $\nabla^h C_{hijk} = 0$. Then M has constant scalar curvature, and*

1. *the concircular curvature tensors of the factor manifolds are divergence-free, and*
2. *the factor manifolds are Einstein.*

Proof. In view of Eq. (5.5), we have

$$\nabla_c R_{ab} - \nabla_a R_{bc} = 0. \tag{5.6}$$

The use of Eq. (2.8) in Eq. (5.6), one acquires that

$$\mathring{\nabla}_c \mathring{R}_{ab} - \mathring{\nabla}_a \mathring{R}_{cb} = \hat{n}\left(\mathring{\nabla}_c \mathring{T}_{ab} - \mathring{\nabla}_a \mathring{T}_{cb}\right) + \frac{1}{\psi^2}\left(\frac{\hat{n} - \mathring{n}}{8\varphi}\hat{\Delta}\varphi + \varphi\hat{T}\right)\left[\psi_a \mathring{g}_{cb} - \psi_c \mathring{g}_{ab}\right]. \tag{5.7}$$

If ψ =constant, then the foregoing equation becomes

$$\mathring{\nabla}_c \mathring{R}_{ab} - \mathring{\nabla}_a \mathring{R}_{cb} = 0, \tag{5.8}$$

this means that the Ricci tensor of $\mathring{M}$ is of Codazzi type.
In light of Eq. (5.5), we derive that

$$\nabla_\gamma R_{\alpha\beta} - \nabla_\alpha R_{\gamma\beta} = 0. \tag{5.9}$$

Eq. (2.9) is inserted in Eq. (5.9) to give

$$\hat{\nabla}_\gamma \hat{R}_{\alpha\beta} - \hat{\nabla}_\alpha \hat{R}_{\gamma\beta} = \mathring{n}\left(\hat{\nabla}_\gamma \hat{T}_{\alpha\beta} - \hat{\nabla}_\alpha \hat{T}_{\gamma\beta}\right) + \frac{1}{\varphi^2}\left(\frac{\mathring{n} - \hat{n}}{4\psi}\mathring{\Delta}\psi + \psi\mathring{T}\right)\left[\varphi_\alpha \hat{g}_{\gamma\beta} - \varphi_\gamma \hat{g}_{\alpha\beta}\right].$$

If φ = constant, then the preceding equation implies that

$$\hat{\nabla}_\gamma \hat{R}_{\alpha\beta} - \hat{\nabla}_\alpha \hat{R}_{\gamma\beta} = 0, \tag{5.10}$$

this signifies that the Ricci tensor of $\hat{M}$ is of Codazzi type.
The divergence of the concircular curvature tensors of the factor manifolds obeys, by the same computation as Eqs. (5.1)–(5.2),

$$\begin{aligned}
\mathring{\nabla}^d \mathring{C}_{dabc} &= \left(\mathring{\nabla}_a \mathring{R}_{bc} - \mathring{\nabla}_b \mathring{R}_{ac}\right) - \frac{1}{\mathring{n}\left(\mathring{n}-1\right)}\left(\mathring{g}_{bc}\mathring{\nabla}_a \mathring{R} - \mathring{g}_{ac}\mathring{\nabla}_b \mathring{R}\right),\\
\hat{\nabla}^\delta \hat{C}_{\delta\alpha\beta\gamma} &= \left(\hat{\nabla}_\alpha \hat{R}_{\beta\gamma} - \hat{\nabla}_\beta \hat{R}_{\alpha\gamma}\right) - \frac{1}{\hat{n}\left(\hat{n}-1\right)}\left(\hat{g}_{\beta\gamma}\hat{\nabla}_\alpha \hat{R} - \hat{g}_{\alpha\gamma}\hat{\nabla}_\beta \hat{R}\right).
\end{aligned}$$

Since the Ricci tensors of the factor manifolds are of Codazzi type (Eqs. (5.8) and (5.10)), the contraction g^{jk} of Eq. (5.5) applied to each factor yields $\mathring{\nabla}_a \mathring{R} = 0$ and $\hat{\nabla}_\alpha \hat{R} = 0$; hence the bracketed terms vanish and

$$\begin{aligned}
\mathring{\nabla}^d \mathring{C}_{dabc} &= 0,\\
\hat{\nabla}^\delta \hat{C}_{\delta\alpha\beta\gamma} &= 0,
\end{aligned}$$

these mean that the concircular curvature tensors of the factor manifolds are divergence-free.
With the aid of Eq. (5.5), we can conclude that

$$\nabla_\gamma R_{ab} - \nabla_a R_{\gamma b} = 0. \tag{5.11}$$

From Eqs. (2.10) and (2.12) in Eq. (5.11), one finds that

$$\mathring{R}_{ab} = \left\{\frac{2\varphi}{\psi\varphi_\gamma}\partial_\gamma\left(\frac{\left(\mathring{n}-1\right)\hat{\Delta}\varphi}{4\varphi} - \varphi\hat{T}\right) - \frac{\left(\mathring{n}-1\right)\hat{\Delta}\varphi}{4\psi\varphi} + \frac{\varphi}{\psi}\hat{T} + \mathring{T}\right.$$

$$+\frac{\mathring{n}\varphi\varphi^{\beta}}{\psi\varphi_{\gamma}}\hat{T}_{\gamma\beta}-\frac{\varphi\varphi^{\beta}}{\psi\varphi_{\gamma}}\hat{R}_{\gamma\beta}-\frac{(\hat{n}-1)\,\mathring{\Delta}\psi}{4\psi^{2}}\Bigg\}\mathring{g}_{ab}-(\mathring{n}-2)\,\mathring{T}_{ab}.$$

Assume that $\mathring{T}_{ab}$ is proportional to the metric tensor $\mathring{g}_{ab}$, that is $\mathring{T}_{ab}=\omega\mathring{g}_{ab}$, thus the previous equation becomes

$$\mathring{R}_{ab}=\Bigg\{\frac{2\varphi}{\psi\varphi_{\gamma}}\partial_{\gamma}\left(\frac{(\mathring{n}-1)\,\hat{\Delta}\varphi}{4\varphi}-\varphi\hat{T}\right)-\frac{(\mathring{n}-1)\,\hat{\Delta}\varphi}{4\psi\varphi}+\frac{\varphi}{\psi}\hat{T}+\mathring{T}$$
$$+\frac{\mathring{n}\varphi\varphi^{\beta}}{\psi\varphi_{\gamma}}\hat{T}_{\gamma\beta}-\frac{\varphi\varphi^{\beta}}{\psi\varphi_{\gamma}}\hat{R}_{\gamma\beta}-\frac{(\hat{n}-1)\,\mathring{\Delta}\psi}{4\psi^{2}}-(\mathring{n}-2)\,\omega\Bigg\}\mathring{g}_{ab}.$$

Transvecting the foregoing equation with $\mathring{g}^{ab}$ and combining the result with the previous identity, we obtain

$$\mathring{R}_{ab}=\frac{\mathring{R}}{\mathring{n}}\mathring{g}_{ab}, \tag{5.12}$$

this indicates that $\mathring{M}$ is an Einstein manifold.
According to Eq. (5.5), we express that

$$\nabla_{c}R_{\alpha\beta}-\nabla_{\alpha}R_{c\beta}=0. \tag{5.13}$$

Employing Eqs. (2.11) and (2.13) in Eq. (5.13), one can find that

$$\hat{R}_{\alpha\beta}=\Bigg\{\frac{2\psi}{\varphi\psi_{c}}\partial_{c}\left(\frac{(\hat{n}-1)\,\mathring{\Delta}\psi}{4\psi}-\psi\mathring{T}\right)-\frac{(\hat{n}-1)\,\mathring{\Delta}\psi}{4\psi\varphi}+\frac{\psi}{\varphi}\mathring{T}+\hat{T}$$
$$+\frac{\hat{n}\psi\psi^{b}}{\varphi\psi_{c}}\mathring{T}_{cb}-\frac{\psi\psi^{b}}{\varphi\psi_{c}}\mathring{R}_{cb}-\frac{(\mathring{n}-1)\,\hat{\Delta}\varphi}{4\varphi^{2}}\Bigg\}\hat{g}_{\alpha\beta}-(\hat{n}-2)\,\hat{T}_{\alpha\beta}.$$

Suppose that $\hat{T}_{\alpha\beta}$ is proportional to $\hat{g}_{\alpha\beta}$, that is, $\hat{T}_{\alpha\beta}=\lambda\hat{g}_{\alpha\beta}$; then proceeding as before and transvecting with $\hat{g}^{\alpha\beta}$, we get

$$\hat{R}_{\alpha\beta}=\frac{\hat{R}}{\hat{n}}\hat{g}_{\alpha\beta}, \tag{5.14}$$

which implies that $\hat{M}$ is an Einstein manifold.
This completes the proof. □

6. Concircular flatness among related curvature conditions and an example

The concircular curvature tensor occupies a distinguished position among the classical curvature tensors of a Riemannian manifold. In this section we record the relationship between concircular flatness and the projective and conformal (Weyl) flatness conditions for a doubly warped product manifold, and we provide an explicit model.
Recall that for a Riemannian manifold of dimension $n>3$ the Weyl conformal curvature tensor and the projective curvature tensor are

$$C_{ijkl}=R_{ijkl}-\frac{1}{n-2}\left(g_{il}R_{jk}-g_{jl}R_{ik}+g_{jk}R_{il}-g_{ik}R_{jl}\right)+\frac{R}{(n-1)\,(n-2)}\left(g_{il}g_{jk}-g_{jl}g_{ik}\right),$$

$$P_{ijkl}=R_{ijkl}-\frac{1}{n-1}\left(g_{il}R_{jk}-g_{jl}R_{ik}\right).$$

Proposition 1. *Let $M = \mathring{M}_{\varphi} \times_{\psi} \hat{M}$ be a doubly warped product manifold of dimension $n > 3$ with constant scalar curvature. Then the following are equivalent:*

1. *M is concircularly flat;*
2. *M is of constant sectional curvature;*
3. *M is both conformally flat and projectively flat.*

Proof. (1) $\Rightarrow$ (2) follows from Eq. (3.1) together with the constancy of R. If M is of constant sectional curvature, then $R_{ijkl} = \frac{R}{n(n-1)}\left(g_{il}g_{jk} - g_{jl}g_{ik}\right)$ and $R_{ij} = \frac{R}{n}g_{ij}$. Substituting these into the expressions for C_{ijkl} and P_{ijkl} shows that both vanish, giving (2) $\Rightarrow$ (3). Finally, a manifold that is both conformally flat and projectively flat is of constant curvature; substituting the constant-curvature form of R_{ijkl} into Eq. (1.1) gives $C_{ijkl} = 0$, which is (3) $\Rightarrow$ (1). $\square$

Proposition 1 makes explicit that, among the three classical flatness conditions, concircular flatness is the most restrictive in the presence of constant scalar curvature: it is equivalent to the manifold being a space form. We now exhibit a concrete realization.

Example 1. *Let $\mathring{M} = \mathbb{S}^{\mathring{n}}(c)$ be the $\mathring{n}$-sphere of constant curvature $c > 0$ and let $\hat{M} = \mathbb{S}^{\hat{n}}(c)$ be the $\hat{n}$-sphere of the same constant curvature c. Take the warping functions to be the positive constants $\varphi \equiv \varphi_0$ and $\psi \equiv \psi_0$ with $\varphi_0 = \psi_0$. Then, by Eqs. (2.2)–(2.3), the warping contributions $\hat{\Delta}\varphi$ and $\mathring{\Delta}\psi$ vanish, and the doubly warped product reduces to the scaled Riemannian product of two space forms of equal curvature. Hence M is of constant sectional curvature and, by Proposition 1, concircularly flat. This shows that the class described in Theorem 3.1 is non-empty. Conversely, whenever one of the warping functions is non-constant the mixed component (3.18) is non-zero, so by Theorem 3.1 a genuinely (non-trivially) doubly warped product cannot be concircularly flat; the flatness forces a reduction to a singly warped product.*

7. Applications

A GRW space-time can be described as a singly warped product manifold with one-dimensional base and its fiber is an $(n-1)$ –Riemannian manifold. Mathematically, this can be represented as $M = \mathring{M} \times_{\Psi^2} \hat{M}$. The structure of the metric tensor for a GRW space-time is expressed as [22, 23]:

$$\left(g_{ij}\right) = \begin{cases} g_{11} = \mathring{g}_{11} = -1 \\ g_{\alpha\beta} = \Psi^2 \hat{g}_{\alpha\beta} \\ g_{1\alpha} = 0 \end{cases}, \tag{7.1}$$

where $\alpha, \beta, ... \in \{2, ..., n\}$.
For a GRW space-time, the Riemann tensor satisfies the following two relations:

$$R_{\alpha\beta\gamma\delta} = \Psi^2 \left[\hat{R}_{\alpha\beta\gamma\delta} + \dot{\Psi}^2 \hat{G}_{\alpha\beta\gamma\delta}\right]. \tag{7.2}$$

$$\nabla_1 R_{\alpha\beta\gamma\delta} = -2\Psi\dot{\Psi}\hat{R}_{\alpha\beta\gamma\delta} + \left[2\ddot{\Psi}\dot{\Psi}^2\Psi - 2\dot{\Psi}^3\Psi\right]\hat{G}_{\alpha\beta\gamma\delta}. \tag{7.3}$$

Theorem 7.1. *A concircularly flat (symmetric) GRW space-time exhibits constant sectional curvature.*

Proof. Eq. (3.1) implies that

$$R_{\alpha\beta\gamma\delta} = \frac{R}{n\,(n-1)}\left[g_{\alpha\delta}g_{\beta\gamma} - g_{\beta\delta}g_{\alpha\gamma}\right]. \tag{7.4}$$

Utilizing Eqs. (7.1) and (7.2) in Eq. (7.4), we get

$$\hat{R}_{\alpha\beta\gamma\delta} = \left[\frac{\Psi^2 R}{n\,(n-1)} - \dot{\Psi}^2\right]\hat{G}_{\alpha\beta\gamma\delta}. \tag{7.5}$$

Transvecting with $\hat{g}^{\alpha\delta}$, the result is

$$\hat{R}_{\beta\gamma} = (n-2)\left[\frac{\Psi^2 R}{n\,(n-1)} - \dot{\Psi}^2\right]\hat{g}_{\beta\gamma}. \tag{7.6}$$

Transvecting with $\hat{g}^{\beta\gamma}$, we realize that

$$\hat{R} = (n-1)\,(n-2)\left[\frac{\Psi^2 R}{n\,(n-1)} - \dot{\Psi}^2\right]. \tag{7.7}$$

Using Eq. (7.7) in Eq. (7.5), it can be concluded that

$$\hat{R}_{\alpha\beta\gamma\delta} = \frac{\hat{R}}{(n-1)\,(n-2)}\hat{G}_{\alpha\beta\gamma\delta}. \tag{7.8}$$

Eq. (4.2) leads to

$$\nabla_1 R_{\alpha\beta\gamma\delta} = \frac{\nabla_1 R}{n\,(n-1)}\left(g_{\alpha\delta}g_{\beta\gamma} - g_{\beta\delta}g_{\alpha\gamma}\right). \tag{7.9}$$

Making use of Eqs. (7.1) and (7.3) in Eq. (7.9), one can find that

$$\hat{R}_{\alpha\beta\gamma\delta} = \left[\ddot{\Psi}\Psi - \dot{\Psi}^2 - \frac{\Psi^3 \nabla_1 R}{2n\,(n-1)\,\dot{\Psi}}\right]\hat{G}_{\alpha\beta\gamma\delta}. \tag{7.10}$$

Transvecting with $\hat{g}^{\alpha\delta}$, we get

$$\hat{R}_{\beta\gamma} = (n-2)\left[\ddot{\Psi}\Psi - \dot{\Psi}^2 - \frac{\Psi^3 \nabla_1 R}{2n\,(n-1)\,\dot{\Psi}}\right]\hat{g}_{\beta\gamma}. \tag{7.11}$$

Transvecting with $\hat{g}^{\beta\gamma}$, one obtains

$$\hat{R} = (n-1)\,(n-2)\left[\ddot{\Psi}\Psi - \dot{\Psi}^2 - \frac{\Psi^3 \nabla_1 R}{2n\,(n-1)\,\dot{\Psi}}\right]. \tag{7.12}$$

Employing Eq. (7.12) in Eq. (7.10), we find

$$\hat{R}_{\alpha\beta\gamma\delta} = \frac{\hat{R}}{(n-1)\,(n-2)}\hat{G}_{\alpha\beta\gamma\delta}. \tag{7.13}$$

Eqs. (7.8) and (7.13) imply that the fiber manifold has constant sectional curvature. Consequently, it is Einstein. □

According to Mantica and Molinari [17], the following two corollaries can be stated:

Corollary 1. *A concircularly flat (symmetric) GRW space-time is a perfect fluid.*

Corollary 2. *For a concircularly flat (symmetric) GRW space-time, the divergence of the Weyl tensor vanishes.*

Theorem 7.2. *A concircularly flat (symmetric) GRW space-time is static.*

Proof. In a perfect fluid space-time, the expression for the Ricci tensor is as follows [16]:

$$R_{kl} = du_k u_l + sg_{kl} - (n-2)\, u^i u^j C_{iklj}, \tag{7.14}$$

where $d = \frac{R-n\zeta}{n-1}$, $s = \frac{R-\zeta}{n-1}$.
Mantica and Molinari demonstrated that the following two conditions hold in a GRW space-time and are equivalent to each other [17], that is,

$$\nabla^m C_{mikl} = 0 \Leftrightarrow u^m C_{mikl} = 0. \tag{7.15}$$

As mentioned in Corollary 2, $\nabla^m C_{mikl} = 0$, thus Eq. (7.14) becomes

$$R_{kl} = du_k u_l + sg_{kl}. \tag{7.16}$$

By taking the covariant derivative of the foregoing equation, we get

$$\nabla_h R_{kl} = d_h u_k u_l + d\,[u_k \nabla_h u_l + u_l \nabla_h u_k] + g_{kl} s_h,$$

where $d_h = \nabla_h d$, $s_h = \nabla_h s$.
For a concircularly flat (symmetric) GRW space-time, we have $\nabla_h R_{kl} = 0$. Consequently,

$$d_h u_k u_l + d\,[u_k \nabla_h u_l + u_l \nabla_h u_k] + g_{kl} s_h = 0. \tag{7.17}$$

Transvecting Eq. (7.16) with g^{kl}, we reveal that

$$R = -d + ns.$$

By performing the covariant derivative to the two sides, we can conclude that

$$\nabla_h R = -\nabla_h d + n \nabla_h s.$$

As indicated previously, R =constant, therefore

$$d_h = ns_h. \tag{7.18}$$

From Eq. (7.18) in Eq. (7.17), we observe that

$$d\,[u_k \nabla_h u_l + u_l \nabla_h u_k] + [g_{kl} + nu_k u_l]\, s_h = 0.$$

Contracting with u^l, one acquires that

$$d \nabla_h u_k = (1-n)\, u_k s_h.$$

Transvecting with u^k, we detect that

$$s_h = 0.$$

From the last two equations, we can deduce that

$$\nabla_h u_k = 0, \tag{7.19}$$

this implies that u_k is parallel.
Transvecting Eq. (7.19) with u^h, one sees that

$$\dot{u}_h = u^h \nabla_h u_k = 0, \tag{7.20}$$

this demonstrates that u_h has zero acceleration.
Eqs. (7.19) and (7.20) clarify that u_k is vorticity-free [20]:

$$\mu_{kl} = \frac{1}{2}(\nabla_l u_k - \nabla_k u_l) + \frac{1}{2}\dot{u}_k u_l + u_k \dot{u}_l = 0.$$

In a space-time M, $\nabla_l u_k$ possesses a standard decomposition with geometric interpretation:

$$\nabla_l u_k = \frac{\nabla_i u^i}{n-1}(u_k u_l + g_{kl}) - \dot{u}_k u_l + \mu_{kl} + \sigma_{kl},$$

with σ_{kl} being the shear tensor.
It is easily apparent that

$$\sigma_{kl} = 0,$$

this signifies that u_h is shear free.
A space-time M becomes static if it has a unit vector field u_k that is time-like and achieves the following both conditions [24]:

$$\nabla_l u_k = -u_l \dot{u}_k,$$
$$\ddot{u}_l u_k - \ddot{u}_k u_l = 0.$$

Eqs. (7.19) and (7.20) illustrate that the preceding two equations are fulfilled. This means that the considered space-time is static. □

The constancy of the curvature also fixes the physical content of the fluid. Modeling gravity by the Einstein field equations (with cosmological constant Λ)

$$R_{kl} - \frac{1}{2}R g_{kl} + \Lambda g_{kl} = \kappa T_{kl}, \tag{7.21}$$

and writing the energy–momentum tensor of a perfect fluid as

$$T_{kl} = (\rho + p)\, u_k u_l + p g_{kl}, \tag{7.22}$$

with ρ the energy density, p the isotropic pressure and $u_k u^k = -1$, we may extract the equation of state. By Theorem 7.1 the space-time is of constant sectional curvature; hence it is Einstein,

$$R_{kl} = \frac{R}{n} g_{kl}. \tag{7.23}$$

Substituting Eq. (7.23) into Eq. (7.21) gives

$$\kappa T_{kl} = \left[\frac{R}{n} - \frac{R}{2} + \Lambda\right] g_{kl},$$

which is proportional to g_{kl}. Comparing this with Eq. (7.22) forces the coefficient of $u_k u_l$ to vanish, that is

$$\rho + p = 0, \qquad \text{equivalently} \qquad p = -\rho. \tag{7.24}$$

We summarize this in the following statement.

Theorem 7.3. *A concircularly flat (symmetric) GWR space-time ($n \geq 4$) is an Einstein perfect fluid obeying the equation of state $p = -\rho$. Consequently, it represents a dark-energy (cosmological-constant) era: it is a space of constant curvature behaving as a de Sitter (or anti–de Sitter) model, and it satisfies the weak and dominant energy conditions whenever $\rho \geq 0$.*

Proof. The first assertion is Eq. (7.24). The constant value $\rho = -p = \frac{1}{\kappa}\left[\frac{R}{2} - \frac{R}{n} - \Lambda\right]$ is constant because R is constant (Theorem 7.1). An equation of state $p = -\rho$ with constant ρ is exactly that of a cosmological constant; the corresponding constant-curvature solution is de Sitter for $R > 0$ and anti–de Sitter for $R < 0$. The weak energy condition $\rho \geq 0$, $\rho + p \geq 0$ and the dominant energy condition $\rho \geq |p|$ reduce, under $p = -\rho$, to the single requirement $\rho \geq 0$. □

Remark 4. *The chain of conclusions concircular flatness $\Rightarrow$ constant curvature $\Rightarrow$ Einstein perfect fluid $\Rightarrow$ $p = -\rho$ is specific to the concircular curvature tensor: because C_{ijkl} is built directly from the deviation of R_{ijkl} from the constant-curvature model, its vanishing pins down the equation of state completely, whereas projective or conharmonic flatness leave the proportionality factor between R_{kl} and g_{kl} undetermined in general.*

8. Conclusion

The concircular curvature tensor is the curvature invariant that measures the deviation of a Riemannian (or Lorentzian) manifold from a space of constant sectional curvature, since it is built directly from the difference between the Riemann tensor and its constant-curvature model. This geometric meaning makes it a natural object in three connected settings. In differential geometry it characterizes space forms and the concircular transformations that preserve geodesic circles; in general relativity it controls the algebraic type of the Ricci tensor and hence the matter content of a space-time; and in cosmology, where the spatial sections of standard models are spaces of constant curvature, the vanishing of the concircular tensor singles out precisely the maximally symmetric (de Sitter, anti–de Sitter and Minkowski) backgrounds. The concircular tensor therefore provides a single language linking the geometry of warped products to the physics of the universe.

The principal results of this paper sharpen this link in the setting of doubly warped product manifolds. We proved that a concircularly flat (symmetric) doubly warped product has factor manifolds of constant sectional curvature that are themselves concircularly flat, and that genuine (non-trivial) flatness forces the structure to collapse to a singly warped product. We further showed that a harmonic concircular curvature tensor automatically implies constant scalar curvature and Codazzi-type, Einstein factor manifolds—a conclusion strictly stronger than the corresponding one for the projective tensor. Finally, on the relativistic side, a concircularly flat (symmetric) GRW space-time was shown to be an Einstein perfect fluid that is static and obeys the equation of state $p = -\rho$, so that it models a dark-energy (cosmological-constant) era and satisfies the weak and dominant energy conditions.

Several directions remain open for future investigation. The flatness, symmetry and harmonicity conditions studied here could be relaxed to the weaker recurrent, pseudo-symmetric, C-semisymmetric or Z-symmetric settings, and extended from doubly warped products to multiply, sequential and twisted warped products. It would also be valuable to combine the concircular condition with soliton structures (Ricci, Yamabe and Ricci–harmonic solitons) on these manifolds, and to carry the analysis into modified theories of gravity such as $f(R)$, $f(R, G)$ and $f(Q)$ gravity, where the matter content and energy conditions implied by concircular flatness may yield new cosmological models. We hope that the present results will motivate further study of the concircular curvature tensor as a bridge between warped product geometry and relativistic physics.

Conflict of Interest

The authors declare there is no existing conflict of interest.

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