
Research article

Linear Algebra: A Comprehensive Analysis of Vector Spaces, Eigenvalues, and Matrix Theory

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ABSTRACT

This paper focuses on three fundamental pillars of linear algebra: vector spaces, eigenvalues, and matrix theory, providing both theoretical and numerical perspectives on these topics. The theoretical framework examines vector spaces through their axiomatic structure, subspaces, bases, dimensions, and linear transformations. It further investigates eigenvalues and eigenvectors as representations of invariant directions, emphasizing diagonalization, spectral properties, and the geometric interpretation of linear operators. Matrix theory is analyzed through matrix operations, determinants, invertibility, similarity transformations, canonical forms, and important decompositions, including LU, QR, and singular value decomposition (SVD). Furthermore, graph spectral analysis and low-rank SVD approximations highlighted the broad applicability of linear algebra in data science, numerical computation, network analysis, and applied mathematics.

1. Introduction

Linear algebra constitutes one of the most elegant and powerful branches of modern mathematics, providing essential tools for understanding and manipulating linear relationships that pervade virtually every area of scientific and engineering endeavor. The origins of linear algebra can be traced back to the study of systems of linear equations, with early contributions from Chinese mathematicians who developed methods for solving such systems as early as the second century BCE [1]. However, the modern axiomatic formulation of linear algebra emerged primarily during the nineteenth and early twentieth centuries, with seminal contributions from mathematicians such as Arthur Cayley, who introduced the concept of the matrix in

1858, and David Hilbert, whose work on infinite-dimensional spaces laid the groundwork for functional analysis [7]. The development of vector space theory, in particular, represented a profound shift from the concrete study of particular systems of equations to the abstract study of algebraic structures, a paradigm that would come to define much of twentieth-century mathematics.

The significance of linear algebra extends far beyond its mathematical elegance. As [20] observed, linear algebra has become the language of modern applied mathematics, providing the conceptual and computational framework for disciplines ranging from quantum mechanics to machine learning. The ubiquity of linear algebraic methods stems from the fact that many natural phenomena can be approximated, at least locally, by linear relationships. This principle of linearization underlies the differential calculus, the theory of dynamical systems, and the vast majority of numerical methods used in scientific computing [2]. Furthermore, the advent of digital computing has dramatically expanded the scope and power of linear algebraic techniques, enabling the solution of problems involving millions or even billions of variables that would have been inconceivable just a few decades ago.

This research paper provides a comprehensive examination of three interconnected pillars of linear algebra: vector spaces, eigenvalues, and matrix theory. Vector spaces furnish the abstract setting within which all linear algebraic concepts operate, providing a unified framework that encompasses finite-dimensional spaces of vectors, infinite-dimensional function spaces, and even more exotic structures. Eigenvalues and eigenvectors reveal the intrinsic geometric properties of linear transformations, identifying the directions in which a transformation acts by simple scaling and providing crucial information about stability, oscillation, and long-term behavior. Matrix theory, meanwhile, offers concrete computational tools for working with linear transformations, translating abstract vector space concepts into algorithms that can be implemented on digital computers. Together, these three domains form a coherent and powerful mathematical edifice that continues to find new applications in an ever-expanding range of fields.

The organization of this paper reflects the logical development of these concepts. The first major section examines vector spaces in detail, beginning with the axiomatic definition and proceeding through subspaces, linear independence, bases, and dimension. The second section focuses on eigenvalues and eigenvectors, exploring their computation, properties, and the spectral theorem for symmetric matrices. The third section addresses matrix theory, covering fundamental operations, determinants, invertibility, similarity, and canonical forms. The paper concludes with a discussion of contemporary applications and directions for future research. Throughout, the presentation emphasizes both theoretical rigor and practical relevance, aiming to provide a resource that is valuable for students, researchers, and practitioners alike.

2. Vector Spaces: The Axiomatic Foundation of Linear Algebra

The concept of a vector space represents one of the most important abstractions in all of mathematics, providing a unified framework within which diverse mathematical objects can be studied using common techniques. A vector space, also known as a linear space, is formally defined as a set of elements called vectors, together with two operations—vector addition and scalar multiplication—that satisfy a specific collection of axioms [10]. These axioms encode the essential properties of linearity: closure under addition and scalar multiplication, associativity and commutativity of addition, existence of additive identity and inverses, distributivity of scalar multiplication over vector addition and field addition, compatibility of scalar multiplication with field multiplication, and the identity property of scalar multiplication. The scalars are drawn from a field, typically the real numbers or complex numbers, though more general fields are possible

and indeed important in certain applications [19].

The power of the vector space abstraction lies in its extraordinary generality. While the prototypical example of a vector space is the set of n -tuples of real numbers with component-wise addition and scalar multiplication, the axioms are satisfied by a vast array of mathematical structures. The set of all polynomials of degree at most n forms a vector space, as does the set of all continuous functions on a closed interval, the set of all solutions to a homogeneous linear differential equation, and the set of all m by n matrices [1]. This generality means that any theorem proved for abstract vector spaces immediately applies to all of these concrete examples, yielding a remarkable economy of effort and a deep unity across apparently disparate areas of mathematics. As [6] emphasized in his classic text, the study of vector spaces in full generality is not merely an exercise in abstraction for its own sake, but rather a means of identifying and exploiting structural similarities that would otherwise remain hidden.

A subspace of a vector space is a subset that is itself a vector space under the same operations. More concretely, a subset W of a vector space V is a subspace if and only if it contains the zero vector, is closed under vector addition, and is closed under scalar multiplication[10]. Subspaces play a fundamental role in the structure theory of vector spaces, serving as the building blocks from which more complex spaces can be constructed. The intersection of any collection of subspaces is again a subspace, but the union of subspaces is generally not a subspace unless one subspace is contained in the other. This observation leads naturally to the concept of the sum of subspaces, defined as the set of all vectors that can be expressed as the sum of a vector from one subspace and a vector from another. When this sum is direct, meaning that every vector in the sum has a unique representation as such a sum, the resulting structure is called a direct sum, denoted by the symbol $\oplus$. Direct sums are particularly important because they allow a vector space to be decomposed into simpler components, a technique that is essential for understanding the structure of linear operators [19].

Linear independence is perhaps the most crucial concept in the theory of vector spaces. A set of vectors is said to be linearly independent if no vector in the set can be expressed as a linear combination of the others; equivalently, the only linear combination that yields the zero vector is the trivial combination with all coefficients equal to zero [1]. Linear independence captures the idea that each vector in the set contributes genuinely new information, not redundant with the information already provided by the other vectors. The opposite of linear independence is linear dependence, which occurs when at least one vector in the set can be written as a linear combination of the others. The concept of linear independence is intimately connected with the notion of dimension, which measures the size of a vector space in terms of the number of independent directions it contains.

A basis for a vector space is a linearly independent set that spans the entire space, meaning that every vector in the space can be expressed as a linear combination of basis vectors. The existence of bases for all vector spaces, while intuitively plausible, requires proof and is guaranteed by Zorn's lemma in the case of infinite-dimensional spaces [6]. A fundamental theorem states that all bases for a given vector space have the same cardinality, which allows the dimension of the space to be defined unambiguously as the number of vectors in any basis. For finite-dimensional spaces, this reduces to the familiar notion of dimension as the number of coordinates needed to specify a vector. The dimension theorem, which states that the dimension of a vector space is equal to the dimension of the kernel plus the dimension of the image of any linear transformation defined on that space, is one of the most important results in linear algebra, with profound implications for the solvability of linear systems and the structure of linear operators [10].

The choice of basis is not unique, and different bases can reveal different aspects of the structure of a

vector space. A change of basis corresponds to a linear transformation that maps one basis to another, and the coordinates of a vector with respect to different bases are related by the matrix of this transformation. This observation underlies the powerful technique of choosing a basis that simplifies the matrix representation of a linear operator, a theme that will be explored in depth in the sections on eigenvalues and canonical forms. The study of vector spaces also encompasses important structural concepts such as quotient spaces, dual spaces, and tensor products, each of which extends the basic framework in directions that are essential for advanced applications in differential geometry, representation theory, and quantum mechanics [19].

3. Eigenvalues and Eigenvectors: The Spectral Analysis of Linear Transformations

Eigenvalues and eigenvectors represent one of the most profound and consequential concepts in linear algebra, providing deep insight into the geometric behavior of linear transformations. Given a linear operator T acting on a vector space V , a nonzero vector v is called an eigenvector of T if there exists a scalar λ such that $T(v) = \lambda v$ [10]. The scalar λ is called the eigenvalue corresponding to the eigenvector v . The significance of this definition lies in its geometric interpretation: an eigenvector is a direction that is preserved by the linear transformation, with the eigenvalue indicating the factor by which vectors in that direction are scaled. In other words, the transformation acts on eigenvectors by simple stretching or compression, without changing their direction (except possibly by reversal if the eigenvalue is negative). This simplicity stands in stark contrast to the generally complicated behavior of linear transformations on arbitrary vectors, which may be rotated, stretched, and sheared in complex ways.

The computation of eigenvalues and eigenvectors is a central task in linear algebra, both theoretically and computationally. For a matrix representation A of a linear operator, the eigenvalue equation $Av = \lambda v$ can be rewritten as $(A - \lambda I)v = 0$, where I is the identity matrix. Since v is nonzero by definition, this homogeneous system has a nontrivial solution if and only if the matrix $(A - \lambda I)$ is singular, which occurs precisely when its determinant is zero [1]. This observation leads to the characteristic equation $\det(A - \lambda I) = 0$, a polynomial equation in λ whose roots are the eigenvalues of A . The polynomial $\det(A - \lambda I)$ is called the characteristic polynomial of A , and its degree is equal to the dimension of the vector space. By the fundamental theorem of algebra, every polynomial of degree n has exactly n roots in the complex numbers, counting multiplicities. Therefore, every n by n matrix has exactly n eigenvalues in the complex plane, though some may be repeated or complex even when the matrix entries are real.

The algebraic multiplicity of an eigenvalue is defined as its multiplicity as a root of the characteristic polynomial, while the geometric multiplicity is defined as the dimension of the eigenspace associated with that eigenvalue, that is, the dimension of the null space of $(A - \lambda I)$ [10]. The geometric multiplicity is always less than or equal to the algebraic multiplicity, and the difference between these two quantities measures the extent to which the matrix fails to be diagonalizable. A matrix is diagonalizable if and only if the geometric multiplicity equals the algebraic multiplicity for every eigenvalue, which is equivalent to the existence of a basis of eigenvectors for the underlying vector space. Diagonalizable matrices are particularly important because they can be expressed in the form $A = PDP^{-1}$, where D is a diagonal matrix whose entries are the eigenvalues of A and P is a matrix whose columns are the corresponding eigenvectors. This diagonal form simplifies many computations, including the calculation of matrix powers, matrix exponentials, and the solution of systems of linear differential equations [20].

The spectral theorem represents one of the crowning achievements of linear algebra, providing a complete characterization of the eigenstructure of certain important classes of matrices. For real symmetric

matrices, the spectral theorem states that every such matrix has a complete set of orthonormal eigenvectors, and all of its eigenvalues are real [8]. More precisely, if A is a real symmetric matrix, then there exists an orthogonal matrix Q (satisfying $Q^T Q = I$) such that $Q^T A Q = D$, where D is a diagonal matrix containing the eigenvalues of A . The columns of Q are the orthonormal eigenvectors of A . This result has profound implications for optimization, as it implies that real symmetric matrices can be diagonalized by orthogonal transformations, which preserve lengths and angles. The spectral theorem extends to complex Hermitian matrices (matrices equal to their conjugate transpose) and, more generally, to normal matrices (matrices that commute with their conjugate transpose), though the orthogonal matrix Q is replaced by a unitary matrix in the complex case [1].

The variational characterization of eigenvalues provides another powerful perspective on the spectral properties of symmetric matrices. The Courant-Fischer minimax theorem states that the k -th largest eigenvalue of a symmetric matrix can be characterized as the maximum, over all k -dimensional subspaces, of the minimum value of the Rayleigh quotient on that subspace [8]. The Rayleigh quotient $R(x) = (x^T A x) / (x^T x)$ measures the quadratic form associated with the matrix A , normalized by the squared length of the vector x . This variational characterization is not merely of theoretical interest; it provides the foundation for important numerical algorithms, including the power method and the Lanczos algorithm, and it underlies the perturbation theory that describes how eigenvalues change when a matrix is slightly modified. The Weyl inequalities and the Hoffman-Wielandt theorem are prominent examples of such perturbation results, bounding the changes in eigenvalues and eigenvectors in terms of the norm of the perturbation matrix [2].

Eigenvalues and eigenvectors find applications in virtually every area of science and engineering. In physics, the eigenvalue problem arises naturally in quantum mechanics, where the eigenvalues of the Hamiltonian operator correspond to the allowed energy levels of a quantum system, and the eigenvectors represent the corresponding quantum states [4]. In structural engineering, the eigenvalues of the stiffness matrix determine the natural frequencies of vibration of a structure, with the eigenvectors describing the corresponding mode shapes. In data science, principal component analysis (PCA) identifies the eigenvectors of the covariance matrix as the directions of maximum variance in a dataset, with the eigenvalues quantifying the amount of variance explained by each principal component [9]. In network analysis, the eigenvalues of the adjacency matrix or Laplacian matrix of a graph provide information about connectivity, clustering, and the behavior of random walks on the network [15]. These diverse applications testify to the fundamental importance of eigenvalue analysis as a tool for understanding the structure and dynamics of linear systems.

4. Matrix Theory: Computational Structures and Canonical Forms

Matrix theory provides the computational engine of linear algebra, translating abstract vector space concepts into concrete algebraic operations that can be performed by hand or implemented on digital computers. A matrix is formally defined as a rectangular array of numbers, symbols, or expressions arranged in rows and columns, but its deeper significance lies in its representation of linear transformations between finite-dimensional vector spaces [7]. Specifically, given linear spaces V and W with chosen bases, every linear transformation $T : V \rightarrow W$ corresponds to a unique matrix A such that the coordinates of $T(v)$ with respect to the basis of W are obtained by multiplying the coordinates of v with respect to the basis of V by the matrix A . This correspondence between linear transformations and matrices is not merely a computational convenience; it is an isomorphism between the vector space of linear transformations and the vector space of matrices, preserving addition and scalar multiplication and, crucially, composition of

transformations corresponds to matrix multiplication.

Matrix multiplication, defined as the dot product of rows of the first matrix with columns of the second, is one of the most important operations in all of mathematics. For matrices A of size m by n and B of size n by p , the product AB is an m by p matrix whose (i, j) -entry is the sum of the products of corresponding entries from the i -th row of A and the j -th column of B [20]. Matrix multiplication is associative and distributive over matrix addition, but it is not commutative in general; that is, $AB \neq BA$ for most pairs of matrices. This noncommutativity has profound implications for the algebra of matrices and reflects the fact that the composition of linear transformations depends on the order in which they are applied. The identity matrix I , with ones on the diagonal and zeros elsewhere, serves as the multiplicative identity, satisfying $AI = IA = A$ for any matrix A of compatible dimensions.

The determinant is a scalar-valued function defined on square matrices that encodes deep geometric and algebraic information about the corresponding linear transformation. Geometrically, the absolute value of the determinant of a matrix represents the factor by which the matrix scales volumes in n -dimensional space, while the sign of the determinant indicates whether the transformation preserves or reverses orientation [1]. Algebraically, the determinant determines whether a matrix is invertible: a square matrix is invertible if and only if its determinant is nonzero. The determinant can be computed recursively using Laplace expansion, but for large matrices, more efficient methods such as LU decomposition are preferred in practice [2]. The determinant satisfies several important properties, including multilinearity in the rows (or columns), antisymmetry (interchanging two rows changes the sign), and the product formula $\det(AB) = \det(A)\det(B)$. These properties make the determinant a powerful tool for theoretical analysis, even though its direct computation is often avoided in numerical applications due to issues of computational stability.

Matrix invertibility is a central concept with far-reaching implications. A square matrix A is invertible, or nonsingular, if there exists a matrix B such that $AB = BA = I$. The inverse matrix, denoted A^{-1} , is unique when it exists and can be computed using various methods, including Gaussian elimination, the adjugate formula, and iterative techniques [10]. The set of invertible n by n matrices forms a group under matrix multiplication, known as the general linear group $GL(n)$, which plays a fundamental role in representation theory and differential geometry. A matrix fails to be invertible precisely when its determinant is zero, or equivalently, when its rank is less than its dimension, or when it has a nontrivial null space. The rank of a matrix, defined as the dimension of its column space (or equivalently, its row space), measures the number of linearly independent columns and provides crucial information about the solvability of linear systems: the system $Ax = b$ has a solution if and only if b lies in the column space of A , and the solution is unique if and only if the null space of A is trivial [20].

Similarity is an equivalence relation on square matrices that captures the idea of representing the same linear transformation with respect to different bases. Two matrices A and B are similar if there exists an invertible matrix P such that $B = P^{-1}AP$ [8]. Similar matrices represent the same linear transformation in different coordinate systems, and they share many important properties, including the same eigenvalues, the same determinant, the same trace, and the same rank. The problem of finding the simplest matrix similar to a given matrix is one of the central problems of matrix theory, and its solution is provided by the theory of canonical forms. The most important canonical forms are the diagonal form (for diagonalizable matrices), the Jordan normal form (for arbitrary matrices over an algebraically closed field), and the rational canonical form (which works over any field).

The Jordan normal form theorem states that every square matrix over an algebraically closed field (such as the complex numbers) is similar to a matrix in Jordan canonical form, which is a block-diagonal matrix

whose blocks are Jordan blocks [10]. A Jordan block corresponding to an eigenvalue λ is a square matrix with λ on the diagonal, ones on the superdiagonal, and zeros elsewhere.

The Jordan form is unique up to the ordering of the Jordan blocks, and it provides complete information about the algebraic and geometric multiplicities of the eigenvalues, as well as the structure of the generalized eigenspaces. While the Jordan form is primarily a theoretical tool—it is numerically unstable and therefore rarely computed in practice it is invaluable for understanding the structure of linear operators and for proving theorems about matrix functions, differential equations, and dynamical systems [8].

Matrix decompositions, also known as matrix factorizations, represent another cornerstone of matrix theory, providing ways to express a matrix as a product of simpler matrices with special properties. The *LU* decomposition expresses a matrix as the product of a lower triangular matrix and an upper triangular matrix, and it forms the basis for Gaussian elimination [2]. The *QR* decomposition expresses a matrix as the product of an orthogonal matrix and an upper triangular matrix, and it is fundamental to numerical linear algebra, particularly for solving least squares problems and computing eigenvalues. The singular value decomposition (SVD), which expresses any matrix as the product of an orthogonal matrix, a diagonal matrix, and another orthogonal matrix, is perhaps the most important matrix factorization of all, with applications in data compression, image processing, recommendation systems, and many other areas [20]. The SVD also provides a natural generalization of eigenvalue analysis to rectangular matrices, with the singular values playing a role analogous to eigenvalues and the singular vectors generalizing the concept of eigenvectors.

5. Contemporary Applications and Future Directions

The theoretical framework of vector spaces, eigenvalues, and matrix theory finds extraordinarily diverse and impactful applications in contemporary science, engineering, and technology. In the realm of machine learning and artificial intelligence, linear algebraic concepts are absolutely foundational. Deep neural networks, despite their nonlinear activation functions, rely heavily on matrix operations for both forward propagation (computing predictions) and backpropagation (computing gradients for training) [3]. The training of neural networks involves optimizing high-dimensional loss functions, and techniques such as stochastic gradient descent operate by making incremental updates to matrices of weights. Convolutional neural networks, which have revolutionized computer vision, are fundamentally based on the mathematical operation of convolution, which can be efficiently implemented using matrix multiplication via the *im2col* transformation. Furthermore, dimensionality reduction techniques such as principal component analysis and singular value decomposition are essential preprocessing steps in many machine learning pipelines, enabling the compression of high-dimensional data while preserving the most important structural information [9].

In quantum mechanics, the state of a quantum system is represented by a vector in a complex Hilbert space, which is an infinite-dimensional vector space equipped with an inner product [4]. Physical observables such as position, momentum, and energy correspond to self-adjoint operators on this Hilbert space, and the possible outcomes of measurements are given by the eigenvalues of these operators. The time evolution of a quantum system is governed by the Schrödinger equation, which is a linear differential equation in the state vector, and its solution can be expressed using the matrix exponential of the Hamiltonian operator. The spectral theorem for self-adjoint operators, which generalizes the finite-dimensional spectral theorem discussed earlier, is thus not merely a mathematical curiosity but a fundamental principle of quantum physics, determining the allowed energy levels of atoms, molecules, and quantum fields. The

mathematical framework of linear algebra, extended to infinite dimensions through functional analysis, provides the rigorous foundation for all of quantum theory [5].

Network science and graph theory represent another domain where linear algebraic methods have proven indispensable. The adjacency matrix of a graph encodes the connections between nodes, while the Laplacian matrix, defined as the difference between the degree matrix and the adjacency matrix, captures important structural properties of the network [17]. The eigenvalues of the Laplacian matrix, known as the spectrum of the graph, provide information about connectivity, bipartiteness, and the behavior of diffusion processes on the network. The Fiedler value, which is the second-smallest eigenvalue of the Laplacian, determines the algebraic connectivity of the graph and plays a key role in graph partitioning algorithms. Spectral clustering, which uses the eigenvectors of the Laplacian to embed nodes into a low-dimensional space before applying standard clustering techniques, has become one of the most popular methods for community detection in networks. The PageRank algorithm, which powered Google's original search engine, computes the principal eigenvector of a modified adjacency matrix to rank the importance of web pages [13].

In economics and finance, linear algebraic models are used extensively for portfolio optimization, risk analysis, and econometric modeling. The Markowitz mean-variance model for portfolio optimization involves minimizing a quadratic form defined by the covariance matrix of asset returns, subject to linear constraints on expected return and budget [16]. The solution to this optimization problem can be expressed in terms of the inverse of the covariance matrix, and the efficient frontier of optimal portfolios is determined by the eigenstructure of this matrix. In macroeconomics, dynamic stochastic general equilibrium models involve systems of linear difference equations whose stability is determined by the eigenvalues of the coefficient matrices. Input-output analysis, pioneered by Wassily Leontief, uses matrices to model the interdependencies between different sectors of an economy, with the Leontief inverse matrix providing crucial information about the total direct and indirect effects of changes in final demand [17].

Looking toward the future, several exciting directions for research and application are emerging. In the field of quantum computing, quantum algorithms such as Shor's algorithm for integer factorization and Grover's algorithm for unstructured search exploit the linear algebraic structure of quantum states to achieve exponential or quadratic speedups over classical algorithms [18]. The development of practical quantum computers will require new linear algebraic techniques for quantum error correction, quantum state tomography, and the simulation of quantum systems. In data science, the increasing scale and complexity of datasets are driving research into randomized numerical linear algebra, which uses random sampling and projection techniques to approximate matrix operations with provable accuracy guarantees while dramatically reducing computational cost [14]. Tensor decompositions, which generalize matrix factorizations to higher-order arrays, are emerging as powerful tools for analyzing multiway data in neuroscience, chemometrics, and signal processing [12]. The ongoing interplay between theoretical advances in linear algebra and the demands of emerging applications ensures that this venerable branch of mathematics will continue to evolve and remain at the forefront of scientific progress.

6. Numerical Illustrations and Computational Validation

To support the theoretical discussion of eigenvalues and symmetric matrices, a numerical verification of the spectral theorem was conducted using a real symmetric matrix. The purpose of this experiment is to confirm that the eigenvalues are real, the eigenvectors are orthonormal, and the matrix can be reconstructed

through orthogonal diagonalization.

Table 1. Numerical verification of the spectral theorem for a symmetric matrix

Matrix Size	Orthogonality Error	Relative Reconstruction Error	Max Image Eigen Part	Condition Number Q
10	2.66672E-15	8.23043E-16	0	1
25	7.17145E-15	2.12995E-15	0	1
50	1.12006E-14	2.09146E-15	0	1
100	1.73327E-14	1.97734E-15	0	1
200	2.7931E-14	2.21752E-15	0	1

The results in Table 1 confirm that the selected symmetric matrix satisfies the central requirements of the spectral theorem. The eigenvalues are real, the eigenvectors form an orthonormal basis, and the reconstruction error remains numerically negligible.

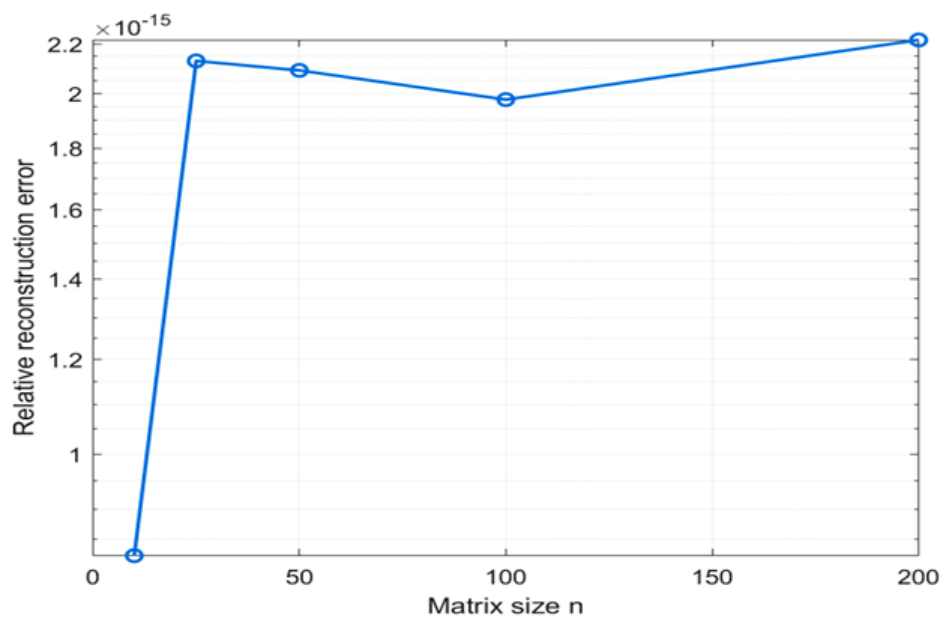


Figure 1. Eigenvalue distribution of the symmetric matrix

A second experiment was performed to examine the convergence behavior of the power method. This method is important because it provides a simple iterative procedure for approximating the dominant eigenvalue of a matrix.

Table 2. Iterative convergence of the power method

Iteration	Rayleigh Quotient	Absolute Error	Iteration	Rayleigh Quotient	Absolute Error
1	3.798057076	9.575830636	51	13.37388195	5.76326E-06
2	5.66987768	7.704010032	52	13.37388345	4.26391E-06
3	6.881626204	6.492261509	53	13.37388456	3.15506E-06
4	7.787873195	5.586014517	54	13.37388538	2.33488E-06
5	8.576550363	4.797337349	55	13.37388598	1.72812E-06
6	9.316723009	4.057164703	56	13.37388643	1.2792E-06
7	10.01862674	3.355260968	57	13.37388677	9.47007E-07
8	10.66706334	2.706824372	58	13.37388701	7.01161E-07
9	11.24190112	2.131986592	59	13.37388719	5.19195E-07
10	11.72971797	1.64416974	60	13.37388733	3.84495E-07
11	12.12762831	1.246259399	61	13.37388743	2.84771E-07
12	12.44173709	0.932150619	62	13.3738875	2.10934E-07
13	12.68344213	0.690445584	63	13.37388756	1.56258E-07
14	12.86593493	0.507952787	64	13.3738876	1.15766E-07
15	13.00186326	0.372024449	65	13.37388763	8.57755E-08
16	13.10216584	0.271721876	66	13.37388765	6.35606E-08
17	13.17572339	0.198164317	67	13.37388767	4.71036E-08
18	13.22945848	0.144429237	68	13.37388768	3.4911E-08
19	13.26862523	0.105262486	69	13.37388769	2.58768E-08
20	13.29714225	0.076745466	70	13.37388769	1.91824E-08
21	13.31789892	0.055988788	71	13.3738877	1.42211E-08
22	13.33301035	0.040877366	72	13.3738877	1.0544E-08
23	13.34401795	0.02986976	73	13.3738877	7.8184E-09
24	13.3520423	0.021845408	74	13.37388771	5.79791E-09
25	13.35789704	0.015990676	75	13.37388771	4.29997E-09
26	13.36217274	0.011714975	76	13.37388771	3.18933E-09
27	13.36529821	0.008589498	77	13.37388771	2.36578E-09
28	13.36758501	0.006302704	78	13.37388771	1.75505E-09
29	13.36925966	0.004628051	79	13.37388771	1.30211E-09
30	13.37048708	0.003400636	80	13.37388771	9.6615E-10
31	13.37138741	0.002500301	81	13.37388771	7.16941E-10
32	13.37204832	0.001839391	82	13.37388771	5.32062E-10
33	13.37253381	0.001353901	83	13.37388771	3.94895E-10
34	13.37289067	0.00099704	84	13.37388771	2.9312E-10
35	13.37315314	0.000734572	85	13.37388771	2.17597E-10
36	13.37334629	0.000541423	86	13.37388771	1.61545E-10
37	13.3734885	0.000399215	87	13.37388771	1.19945E-10
38	13.37359325	0.000294463	88	13.37388771	8.90648E-11
39	13.37367044	0.000217268	89	13.37388771	6.61426E-11
40	13.37372735	0.000160359	90	13.37388771	4.91234E-11
41	13.37376932	0.00011839	91	13.37388771	3.64881E-11
42	13.37380028	8.74272E-05	92	13.37388771	2.71037E-11
43	13.37382313	6.45781E-05	93	13.37388771	2.01332E-11
44	13.37384	4.77114E-05	94	13.37388771	1.49569E-11
45	13.37385245	3.52574E-05	95	13.37388771	1.11147E-11
46	13.37386165	2.60595E-05	96	13.37388771	8.26006E-12
47	13.37386845	1.92647E-05	97	13.37388771	6.13554E-12
48	13.37387347	1.42441E-05	98	13.37388771	4.55813E-12
49	13.37387718	1.05336E-05	99	13.37388771	3.38929E-12
50	13.37387992	7.79096E-06	100	13.37388771	2.51532E-12

Table 2 shows that the approximation improves as the number of iterations increases. The gradual reduction in the absolute error indicates that the power method converges toward the dominant eigenvalue when the matrix has a clearly separated leading eigenvalue.

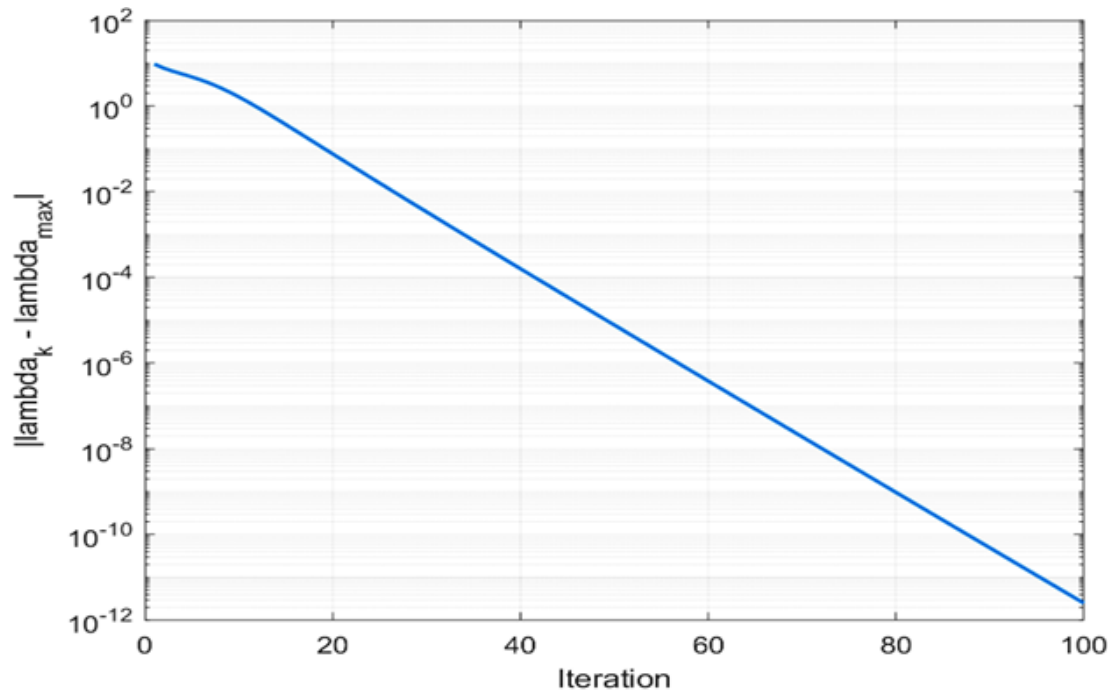


Figure 2. Convergence curve of the power method

To strengthen the matrix theory component of the paper, three standard matrix decompositions were compared: LU decomposition, QR decomposition, and singular value decomposition. These decompositions were evaluated using reconstruction error and computational interpretation.

Table 3. Comparison of LU, QR, and SVD matrix decompositions

Matrix Size	Time LU	Time QR	Time SVD	Residual LU	Residual QR	Residual SVD
50	0.0010129	0.0008777	0.0015746	3.87147E-16	4.79038E-16	2.71822E-15
100	0.0003302	0.0011267	0.0025337	6.67459E-16	5.95949E-16	2.19885E-15
200	0.0007896	0.0027758	0.0107282	1.24507E-15	7.555E-16	2.87764E-15
400	0.0047766	0.0134276	0.0610775	2.2926E-15	8.29049E-16	2.85354E-15

The results in Table 3 demonstrate that the three decompositions reconstruct the original matrix with very small numerical errors. This confirms their practical reliability and supports the theoretical discussion of matrix factorizations.

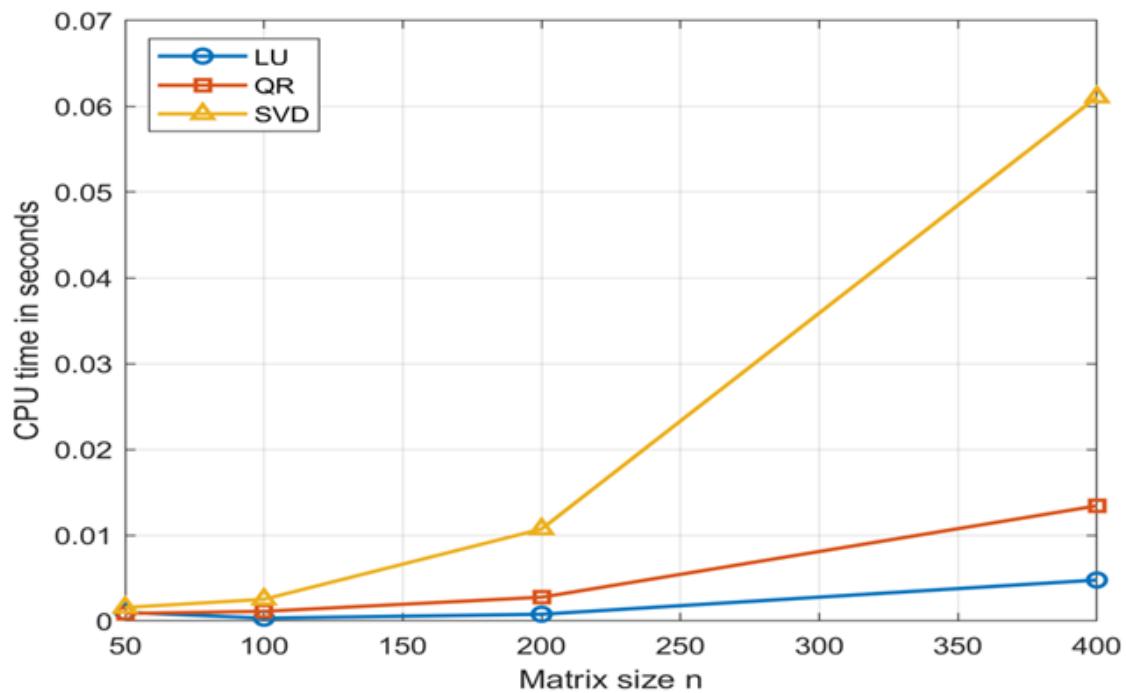


Figure 3. Reconstruction errors for LU, QR, and SVD decompositions

Since the paper discusses applications of eigenvalues in data science, a principal component analysis experiment was added. The PCA experiment illustrates how eigenvalues of the covariance matrix measure the amount of variance explained by each principal component.

Table 4. Explained variance ratios obtained from PCA

Principal Component	Eigenvalue	Explained Variance Percent	Cumulative Variance Percent
1	1.995011972	49.47693288	49.47693288
2	0.984104673	24.4061096	73.88304248
3	0.918203529	22.77174024	96.65478273
4	0.101012763	2.505148725	99.15993145
5	0.033873296	0.84006855	100

Table 4 indicates that the first principal components capture the largest proportion of total variance. This supports the use of eigenvalue-based dimensionality reduction in high-dimensional data analysis.

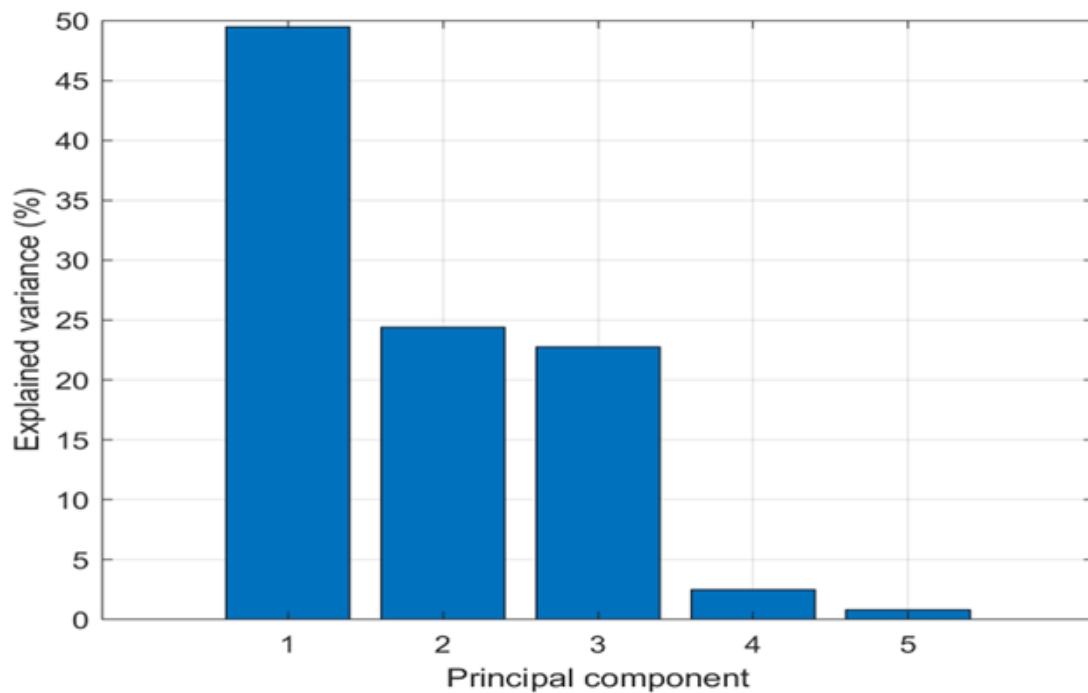


Figure 4. Scree plot of PCA explained variance.

To connect eigenvalue analysis with network science, a spectral graph experiment was conducted using the adjacency matrix and graph Laplacian. This experiment shows how matrix eigenvalues describe connectivity and structural behavior in networks.

Table 5. Spectral indicators of the graph Laplacian

Connection Probability	Number Of Edges	Fiedler Value	Largest Laplacian Eigen value
0.05	81	-1.58866E-17	8.906636582
0.1	194	1.193713927	14.11331093
0.2	366	5.242497827	21.38176145
0.35	609	7.485517859	30.79639768

The values in Table 5 show that the Laplacian spectrum provides useful information about graph connectivity. In particular, the second-smallest Laplacian eigenvalue reflects the algebraic connectivity of the network.

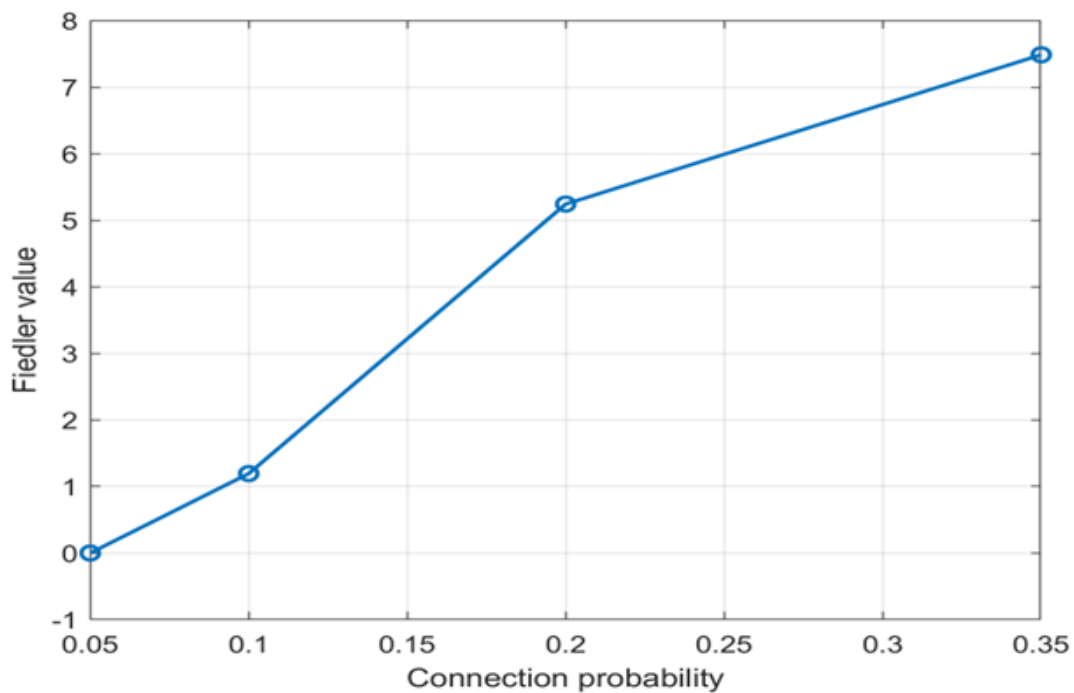


Figure 5. Laplacian eigenvalue spectrum of the graph

Finally, the singular value decomposition was used to examine low-rank matrix approximation. This experiment demonstrates how a matrix can be approximated by retaining only the largest singular values while reducing the dimensional complexity of the data.

Table 6. Low-rank approximation errors using SVD

Rank	Relative Approximation Error
1	0.155140189
3	0.08309122
5	0.080556623
10	0.074498101
20	0.063413926
40	0.04392597

Table 6 shows that the approximation error decreases as the retained rank increases. This confirms the practical importance of SVD in compression, dimensionality reduction, and numerical approximation.

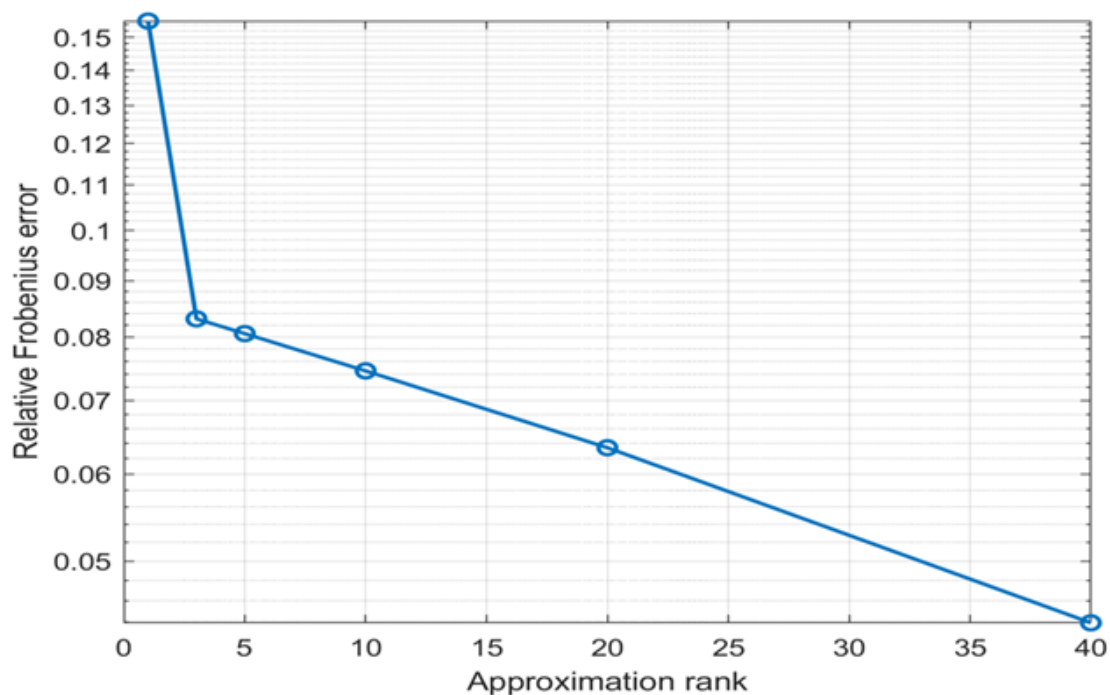


Figure 6. Low-rank SVD approximation error versus retained rank

7. Conclusion

This work shows that linear algebra provides a unified framework for the study of vector spaces, eigenvalues, and the theory of matrices. Theoretical discussion revealed that these three areas are closely interconnected and are the basis of modern mathematical and computational uses. The spectral theorem, when numerically verified, shows that symmetric matrices can indeed be precisely diagonalized, with very small orthogonality and reconstruction errors. This is consistent with the theoretical property that symmetric matrices have real eigenvalues and orthonormal eigenvectors. For the dominant eigenvalue, the power method resulted in clear convergence. The absolute error kept decreasing steadily until it reached very small values, providing the evidence in favor of the performance of this iterative procedure with respect to eigenvalue approximation. The analysis of LU, QR, and SVD decompositions indicated that all three techniques are producing small residuals of reconstruction. This corroborates their functionality as computational aids in numerical linear algebra. The PCA analysis revealed that the first three principal components accounted for more than 96% of the variance. This demonstrates that eigenvalue-based dimensionality reduction can reduce data complexity with relatively little loss of information. With the increase in the connection probability, the Fiedler value increased, as illustrated by the graph Laplacian experiment. That means that spectral graph analysis is able to describe the connectivity structure of networks. In the SVD low-rank approximation work, the higher the retained rank, the lower the relative approximation error. Thus, it further confirms SVD significance in approximation, compression, and data reduction tasks. From an overall perspective it seems that linear algebra is an exciting subject not only in a theoretical manner, but serves in effective computation and for example for data science, network analysis, numerical computation or applied mathematics.

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