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Research article

On The Algebraic Structure of neutrosophic Ideal 𝓋-Open Sets in Topological Spaces

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ABSTRACT

This work describes a unified study of multi-topological topological structures and neutrosophic (neutr.) algebraic structures in accordance with the concept of the ideal to provide a new form of mathematics to accommodate uncertainty and multi-composition at once. The paper starts by generalizing open sets via the theory of 𝓋-open sets and then integrates this concept with neutr. logic, that is to say, the interplay between truth, indeterminacy and falsity being its independent constituents. The integration is not a mere descriptive thing, but being explicitly formulated by expressing neutr. open sets and directly linking to a neutr. ideal in the neutr. loop, for purposes of algebraic control over the topological structure. The study offers clear, simple definitions at the fundamental level, then introduces internal concepts such as neutr. closure, inside, outside, limit, neighbor and accumulation points to illustrate the important properties in new perspective of the new framework. Most importantly, you observe the retention of the fundamental topological axioms in ideal neutr. spaces and how these are radically different from classical topology, specifically relating that to the effect of the indeterminacy complex and the function of the algebraic ideal in change of the local behavior of groups.

1. Introduction

After the presentation of open sets in topological spaces (T.P) , each studies were presented for new types of open sets, some of which are based on each basic topological spaces, to mention but not limited to: in 2018 and showed them usefully and their properties, in addition to open sets that were used to build graph models, and clarify the topological structure of the distribution of these graphs into subsets by [1], in 2012 by [2], and the upper semi-continuous open topologies presented by [3], open sets (ʎ,b) presented by [4] in 2022, and *open sets, and at the same time (δ-open) were presented by [5]. These open sets helped each researchers to understand and present new topological spaces and explain their properties. Sharma et al. first introduced the quadrupole topology and developed it as in [6], where a new open set termed q-open was constructed and relied on. Many studies on these topologies have been presented, including fuzzy q-open sets and fuzzy q-b-open sets, as well as q-cont and q-b-cont functions. Beyond introducing a new function for the quadrupole topology space in [7], they designed completely new and different kinds of sets [8]. Moreover, Pang et al. proposed a topological framework of a quadrupole transmitter constituted by a first topology and a second topology, the first topology comprising a first bridge, a second phase shift element, and a third phase shift element [9]. These studies collectively offer insights into various topologies and their properties.

The interplay between algebraic structures and topological spaces has been a fertile ground for mathematical research, yielding profound insights and applications across various domains. Classical topology, initiated by the fundamental concept of open sets, has evolved through numerous generalizations including semi-open sets, pre-open sets, α-open sets, and b-open sets, among others. More recently, the notion of v-open sets has emerged as a significant generalization [10], defined in the context of multiple topologies on a common universe. These sets—characterized by the existence of a non-empty set common to all topologies—provide a robust framework for studying spaces with multiple compatible topological structures. Concurrently, neutr. logic, pioneered by Smarandache, has revolutionized uncertainty modeling by extending fuzzy and intuitionistic fuzzy logics through three independent components: truth (T), indeterminacy (I), and falsity (F) [11]. This tripartite structure has found rich applications in algebraic contexts through neutr. groups, rings, semigroups, and ideals. Neutr. algebraic structures incorporate the indeterminate element I, allowing for the mathematical representation of ambiguous, incomplete, or contradictory information.

Despite significant advancements in both domains, a crucial gap remains: the integration of neutr. algebraic ideals with generalized topological structures. While neutr. topological spaces have been explored, and various generalizations of open sets have been studied independently, the specific combination of neutr. ideals with v-open set theory represents unexplored mathematical territory. This integration is particularly compelling because Algebraic-topological synthesis: Combining the algebraic precision of ideals with the spatial intuition of topology creates a powerful framework for structured uncertainty. with multi-topological uncertainty: Real-world systems often operate under multiple topological perspectives simultaneously, each with inherent indeterminacy and Ideal-based perturbations: The concept of ideals naturally models "small" or "negligible" elements, allowing for controlled modifications of topological structures

2. Fundamental Concepts

2.1. Definition: [10] Let $\{\tau_k\}_{k\in J}$, $k \geq 2$ be a collection of a topological space on X, and $\varkappa \subseteq X$. Then $\varkappa$ is termed υ-open set (υ_O-set) in X if any $T \in \bigcap_{k\in J}\tau_k$ under which $\emptyset \neq T \subseteq \varkappa$ where $\varkappa \neq \emptyset$ and $\emptyset = T$ where $\varkappa = \emptyset$.

2.2. Definition: [10] The set of all υ_O-set in X (υ_{oX}) and define as $\upsilon_{oX} = \{\varkappa : \varkappa$ is υ-open in $X\}$. If $A \subseteq X$ a point $a \in A$ is termed interior A in X if and only if there exist $U_x \in \upsilon oX$ under which $a \in U_x \subseteq A$ then $int_{\upsilon} = \bigcup_{i\in Z}\{x_i : x_i \ is\ \upsilon - open\ , x_i \in A\ \forall i \in I\}$.The complement of a υ-open is termed υ-closed set (υ_C-set) and the set of allυ_C-set is denoted by CS_{υ}. An order pair (X, υ_{oX}) is termed υ-space where X is a universal set and υ_{oX} is set of all υ_O-set in X.

2.3. Definition: [10] Let (X, υ_{oX}) be a υ-space. A υ-closure of $\varkappa \subseteq X$ ($cl_{\upsilon}(\varkappa)$) its intersection of all υ_C-set that include $\varkappa$.

2.4. Theorem. [10] Let (X, υ_{oX}) be a υ-space and $\varkappa \subseteq X$, then $\varkappa$ is a υ_C-set that include $\varkappa$.

2.5. Corollary: [10] Assume (X, υ_{oX}) is a υ-space and $\varkappa \subseteq X$, then $\varkappa$ is a υ_C-set if and only if $\varkappa = cl_{\upsilon}(\varkappa)$.

2.6. Remark: [10] Let $\{\tau_k\}_{2\leq k\in J}$ be a family of topological space on X and τ be a proper non-empty set of X
under which $\tau \in \tau_k$, for some $k \in J$. However, this is not required $\tau \in \upsilon_{oX}$.

2.7. Remark: [10] Let $\{\tau_k\}_{k\in J}$ be a family of topological space on X and $\varkappa \in \upsilon_{oX}$. However, this is not required $\varkappa \in \tau_k$ for all $k \in J$.

2.8. Proposition: [10] Let $\{\tau_k\}_{k\in J}$ be a family of topological space on X. If $\varkappa \in \tau_k$ for all $k \in J$, then $\varkappa \in \upsilon_{oX}$.In general, the converse is not true.

2.9. Definition: Neutr. logic generalizes fuzzy and intuitionistic fuzzy logics by evaluating properties P via three independent components:

$$P = \langle T, I, F\rangle$$

where

- T as the membership of truth (certainty),
- I as the membership of indeterminacy (ambiguity),
- F as the membership of falsity (negativity).

2.10. Definition: [11] A neutr. number is an element in a single-valued neutr. set, represent as $\langle T, I, F\rangle$.

2.11. Example: $x(I) = 5 + \sqrt{2}I$ has a determinate part 5 and in determinate part $\sqrt{2}I$.

2.12. Definition: Let S be a universal set. A neutr. set K is defined us form:

$K = \{(\lambda, T_K(\lambda), I_K(\lambda), F_K(\lambda)) | \lambda \in S\}$

where $T_K, I_K, F_K: S \to [0,1]$, represent truth, indeterminacy, and falsity membership functions, respectively.

2.13. Definition: [12] Let T, I, F be standard or non-standard of [0,1]:

$sup\, T = t_{sup}, inf\, T = t_{inf}$
$sup\, I = i_{sup}, inf\, I = i_{inf}$
$sup\, F = f_{sup}, inf\, F = f_{inf}$
$n_{sup} = t_{sup} + i_{sup} + f_{sup}$
$n_{inf} = t_{inf} + i_{inf} + f_{inf}$

where T, I, F are termed neutr. components

2.14. Definition: [13] If $(H,*)$ be a group and let $(H \cup I) = \{n + mI : n, m \in H\}$. $N(H) = ((G \cup I), *)$ is neutr. group by H and I with the binary operation $*$. I is termed the neutr. element, $I^2 = I$. For example: $(N(\mathbb{C}), +)$, $(N(\mathbb{Q}), +)$, and $(N(\mathbb{Z}), +)$ are neutr. groups.

2.15. Definition: [14] Let $(S, .)$ be a semigroup. The neutr. semigroup $S(I) = \langle S \cup I \rangle$ is an algebraic structure generated by S and the indeterminate I. A neutr. subsemigroup $T(I)$ of $S(I)$ is a subset closed under the semigroup operation and containing I.

2.16. Definition: [15] Let $(G,*)$be a group, and let $G(I) = \langle G \cup I \rangle$ be a neutr. group generated by G and an indeterminate element I. A subset $H(I)$ of $G(I)$ is termed neutr. subgroup if $H(I)$ is neutr. group under operation $*$, the restraction of $H(I)$ to G is a classical subgroup of G.

2.17. Definition: [16] If $(R, +, .)$ be a ring. Then the set $(R \cup I) = \{c + dI : c, d \in R\}$ is termed a neutr. ring generated by R and I under the operations of R. For example: $\mathbb{Z}(I) = \{a + bI | a, b \in \mathbb{Z}\}$.

2.18. Definition: [17] Let $R(I)$ be a neutr. ring, where R is a classical ring and I is an indeterminate. A non-empty subset $S(I)$ of $R(I)$ is termed a neutr. subring if $S(I)$ is a neutr. ring under the same operations, and $\cap S = S(I) \cap R$ is a classical subring of R.

2.19. Definition: [18] Let $R(I)$ be a neutr. ring, where R is a classical ring and I is an indeterminate. A non-empty subring $J(I)$ of $R(I)$ is termed a neutr. ideal if for every $r(I) \in R(I)$ and $j(I) \in J(I)$ both $r(I).j(I)$ and $j(I).r(I)$ belong to $J(I)$.

3. Neutr. $\mathcal{v}$-Open Set and Neutr. $\mathcal{v}$-Ideal

This section introduces and develops the theory of neutr. v-open sets and their enhancement through algebraic ideals to form neutr. ideal $\mathcal{v}_O$-sets. We establish:

3.1. Definition: Let $\{\tau_k\}_{2 \le k \in J}$ be a collection of two or more topological spaces on the same universe X. Let $NS(X)$ be the family of all neutr. sets over X, where a neutr. set A is of the form:

$$A = \{ \langle x, T_A(x), I_A(x), F_A(x) \rangle \mid x \in X \}$$

with $T_A, I_A, F_A: X \to [0,1]$.

A neutr. set $\eta \in NS(X)$ is termed a neutr. $\mathcal{v}$-open set ($N\mathcal{v}_O$-set) if there exists a non-empty crisp $\mathcal{v}_O$-set $U \in \mathcal{v}_{oX}$, i.e., $U \in \bigcap_{k \in J} \tau_k$, $U \neq \emptyset$ under which: $U \subseteq Supp_T(\eta)$, and we define the neutr. membership for η based on U in the following: way:

- For $x \in U \Longrightarrow \begin{cases} T_\eta(x) = 1 \\ I_\eta(x) = 0 \\ F_\eta(x) = 0 \end{cases}$
- For $x \notin U \Longrightarrow \begin{cases} T_\eta(x) = 0 \\ I_\eta(x) = 1 \\ F_\eta(x) = 1 \end{cases}$

Alternatively, more generally, we can say: η is neutr. $\mathcal{v}_O$-set if any a crisp $\mathcal{v}_O$-set U under which η coincides with the neutr. characteristic function of U, i.e.: $\eta = \langle \chi_U, 0, 1 - \chi_U \rangle$, where χ_U is the crisp characteristic function of U (1 if $x \in U$, 0 otherwise), so: $T_\eta = \chi_U, I_\eta = 0, F_\eta = 1 - \chi_U$.

3.2. Example: Let $X = \{a, b, c\}$. Consider two topologies on X: $\tau_1 = \{\emptyset, \{a\}, X\}, \tau_2 = \{\emptyset, \{a, b\}, X\}$. Then: $\bigcap_{k=1,2} \tau_k = \{\emptyset, \{a\}, X\}$. So the crisp $\mathscr{v}_O$-sets are: $\emptyset, \{a\}, X$. Take $U = \{a\}$ (crisp $\mathscr{v}_O$-set).
Define the neutr. set η corresponding to U:

- For $x = a$: $T = 1, I = 0, F = 0$
- For $x = b$: $T = 0, I = 1, F = 1$
- For $x = c$: $T = 0, I = 1, F = 1$

So: $\eta = \{(a, 1,0,0), (b, 0,1,1), (c, 0,1,1)\}$. This η is a $N\mathscr{v}_O$-set.

3.3. Definition: Let $R(I)$ be a neutr. ring. A non-empty subset $J(I) \subseteq R(I)$ is a neutr. ideal of $R(I)$ $(N\mathscr{v}_I)$ if: $J(I)$ is a neutr. subring (closed under subtraction and contains I), and For every $r(I) \in R(I)$ and $j(I) \in J(I)$, both $r(I) \cdot j(I) \in J(I)$ and $j(I) \cdot r(I) \in J(I)$.

3.4. Example: Let $R = Z_2 = \{0,1\}$ (ring mod 2). The neutr. ring: $R(I) = \{0,1, I, 1 + I\}, I^2 = I$. Take $X = \{1,2,3\}$, with $\tau_1 = \{\emptyset, \{a\}, X\}, \tau_2 = \{\emptyset, \{a, b\}, X\}$. Then: $\bigcap_{k=1,2} \tau_k = \{\emptyset, \{a\}, X\}$,, hence its $\mathscr{v}_O$-sets.

Take $U = \{a\}$. Now we know the neutr. group η corresponding to U as $\eta = \{(a, 1,0,0), (b, 0,1,1), (c, 0,1,1)\}$, where $\eta(a) = (T = 1, I = 0, F = 0)$, $\eta(b) = (T = 0, I = 1, F = 1)$, and $\eta(c) = (T = 0, I = 1, F = 1)$, take $J(I) = \{0, I\}$, then we get: $I \in J(I), 0 + I = I \in J(I), I + I = 0 \in J(I), I.I = I \in J(I), 0.I = 0 \in J(I)$.

To define $N\mathscr{v}_I$ take $\xi(I)$ is $N\mathscr{v}_I$ on $J(I)$ define by form:

$$\xi(a) = (0, I, 0)$$
$$\xi(b) = (I, 0, I)$$
$$\xi(c) = (0, I, I)$$

Now we know $\nu = \eta + \xi$ (where the sum is the plural of neutrosophy):

$$\nu(a) = (1 + 0,0 + I, 0 + 0) = (1, I, 0)$$
$$\nu(b) = (0 + I, 1 + 0,1 + I) = (I, 1,1 + I)$$
$$\nu(c) = (0 + 0,0 + 1,1 + I) = (0,1,1 + I)$$

Then $\nu = \{(a, 1, I, 0), (b, I, 1,1 + I), (c, 0,1,1 + I)\}$ is $N\mathscr{v}_I$ since η is a $N\mathscr{v}_O$-set, ξ takes its values from the neutr. example $J(I)$, and $\nu = \eta + \xi$.

3.5. Definition: Let X be a set with two or more topologies τ_k, giving $\mathscr{v}_O$-set structure $\mathscr{v}_{OX}$. Let $N\mathscr{v}_O$-set be the collection of neutr. $\mathscr{v}_O$-sets. Let $J(I)$ be a neutr. ideal in some neutr. ring $R(I)$. We say a neutr. set $\nu \in NS(X)$ is a neutr. ideal $\mathscr{v}$-open set ($N_I\mathscr{v}_O$-set) with respect to $J(I)$ if for any η $N\mathscr{v}_O$-set, and $\xi(I)$ is a neutr. set whose membership values $T_\xi(x), I_\xi(x), F_\xi(x)$ are elements of the neutr. ideal $J(I)$ for each $x \in X$, i.e., $T_\xi(x) = t_1 + t_2 I$ with $t_1, t_2 \in R$, etc., but in practice for single-valued neutr. sets we take $T_\xi(x) \in [0,1]$, so we interpret: the triple $\left(T_\xi(x), I_\xi(x), F_\xi(x)\right)$ belongs to some subset of $[0,1]^3$ that is isomorphic to a neutr. ideal in a formal way, then $\nu = \eta + \xi(I)$. More concretely, we can define: Let J be an ordinary ideal in a ring $R \subseteq R$. Define a neutr. ideal $J(I) = \{ a + bI \mid a, b \in J \}$. A neutr. set ν is termed $N_I\mathscr{v}_O$-set if for each $x \in X$ $T_\nu(x) = T_\eta(x) + j_1(x)$, $I_\nu(x) = I_\eta(x) + j_2(x)$, and $F_\nu(x) = F_\eta(x) + j_3(x)$ where η is $N\mathscr{v}_O$-set, and $j_1(x), j_2(x), j_3(x) \in J(I)$ (neutr. numbers from the ideal).

3.6. Example: Take $X = \{a, b, c\}$, with $\tau_1 = \{\emptyset, \{a\}, \{a, b\}, X\}, \tau_2 = \{\emptyset, \{b\}, \{a, b\}, X\}$. Then: $\bigcap_{k=1,2} \tau_k = \{\emptyset, \{a, b\}, X\}$,, hence its $\mathscr{v}_O$-sets. Take $U = \{a, b\}$. Now we know the neutr. group η corresponding to U as $\eta = \{(a, 1,0,0), (b, 1,0,0), (c, 0,1,1)\}$, where $\eta(a) = (T = 1, I = 0, F = 0)$, $\eta(b) = (T = 1, I = 0, F = 0)$, and $\eta(c) = (T = 0, I = 1, F = 1)$.
Let $R = Z_2 = \{0,1\}$ (ring mod 2). The neutr. ring: $R(I) = \{0,1, I, 1 + I\}, I^2 = I$, take $J(I) = \{0, I\}$.
To define $N\mathscr{v}_I$ take $\xi(I)$ is $N\mathscr{v}_I$ on $J(I)$ define by form:

$$\xi(a) = (0, I, 0)$$
$$\xi(b) = (I, 0,0)$$
$$\xi(c) = (0,0, I)$$

Now we know $\nu = \eta + \xi$ (where the sum is the plural of neutr.):

$$\nu(a) = (1 + 0,0 + I, 0 + 0) = (1, I, 0)$$
$$\nu(b) = (1 + I, 0 + 0,0 + 0) = (1 + I, 0,0)$$
$$\nu(c) = (0 + 0,0 + 1,1 + I) = (0,1,1 + I)$$

Then $\nu = \{(a, 1, I, 0), (b, 1 + I, 0,0), (c, 0,1,1 + I)\}$ is $N_I\mathscr{v}_O$-set.

3.7. Remark: Define the family of $N\mathcal{v}_O$-set, s.t. $N\mathcal{v}_O(X) \subseteq NS(X)$ as follows: for each crisp $\mathcal{v}_O$-set $U \in \tau_{\mathcal{v}}$, its corresponding neutr. $\mathcal{v}_O$-set $\eta_U \in N\mathcal{v}_O(X)$ is defined by:

$$\eta_U(x) = \begin{cases} (1,0,0) \; if \; x \in U \\ (0,1,1) \; if \; x \notin U \end{cases}$$

3.8. Definition: define the collection

$$\tau_{N\mathcal{v}_I} = \{v \in NS(X) | v \; is \; N_I\mathcal{v}_O - set\}$$

The pair $(X,\tau_{N\mathcal{v}_I})$ is termed a **neutr. ideal $\mathcal{v}$-open topological space ($\tau_{N\mathcal{v}_I}$-TS)**, if satisfy the topological axioms:

1) $\emptyset_N, X_N \in\tau_{N\mathcal{v}_I}$, where $\emptyset_N$ is empty neutr. set corresponds to crisp $\emptyset$, with $\eta_\emptyset \in N\mathcal{v}_O(X)$ and $\xi(I) = 0$. While X_N is space neutr. set corresponds to crisp X, with $\eta_X \in N\mathcal{v}_O(X)$ and $\xi(I) = 0$

2) Let $v_1, v_2 \in\tau_{N\mathcal{v}_I}$ with $v_i = \eta_i \oplus \xi_i(I)$, $i = 1,2$. Then:

$$v_1 \cap v_2 = (\eta_1 \cap \eta_2) \oplus (\xi_1(I) + \xi_2(I))$$

Since $\eta_1 \cap \eta_2 \in N\mathcal{v}_O(X)$ (as crisp $\mathcal{v}_O$ -sets are closed under finite intersections) and $\xi_1(I) + \xi_2(I) \in J(I)$ (as $J(I)$ is closed under addition), we have $v_1 \cap v_2 \in\tau_{N\mathcal{v}_I}$

3) Let $\{v_\alpha\}_{\alpha\in\Lambda} \subseteq\tau_{N\mathcal{v}_I}$ with $v_\alpha = \eta_\alpha \oplus \xi_\alpha(I)$. Then:

$$\bigcup_{\alpha\in\Lambda} v_\alpha = \left(\bigcup_{\alpha\in\Lambda} \eta_\alpha\right) \oplus \left(\sum_{\alpha\in\Lambda} \xi_\alpha(I)\right)$$

Since $\bigcup_{\alpha\in\Lambda}\eta_\alpha \in N\mathcal{v}_O(X)$ and $\sum_{\alpha\in\Lambda}\xi_\alpha(I) \in J(I)$ (as $J(I)$ is closed under arbitrary sums), we have $\bigcup_{\alpha\in\Lambda} v_\alpha \in\tau_{N\mathcal{v}_I}$.

4. Some Properties for Neutr. $\mathcal{v}$-Open Set

4.1. Definition: The neutr. closure of a neutr. set v in $N_I\mathcal{v}_O$-space is the smallest of $N_I\mathcal{v}_C$- set containing v. Formally:

$$Ncl_{N_I\mathcal{v}}(v) = \bigcap\{\, \omega \in NS(X) \mid \omega^c \in\tau_{N\mathcal{v}_I}, v \subseteq \omega \,\}$$

4.2. Example: let $X = \{a,b,c\}$, with two topologies: $\tau_1 = \{\emptyset,\{a\},X\}, \tau_2 = \{\emptyset,\{a,b\},X\}$
Then $\tau_v = \{\emptyset,\{a\},X\}$.Let $= Z_2, J = \{0\} \subseteq R$, so $J(I) = \{0,I\}$.
Take $v = \{(a,1,I,0),(b,0,1,1),(c,0,1,1)\}$ (corresponds to crisp $\{a\}$).
Then
$Ncl_{N_I\mathcal{v}}(v) = v \cup \{(b,I,0,0),(c,0,0,1)\} = \{(a,1,I,0),(b,I,1,1),(c,0,1,1+I)\}$,
(smallest $N_I\mathcal{v}_C$-set containing v).

4.3. Definition: The neutr. interior of a neutr. set v in $N_I\mathcal{v}_O$-space is the union of $N_I\mathcal{v}_C$- set containing v. Formally:

$$Nint_{N_I\mathcal{v}}(v) = \bigcup\{\, \omega \in\tau_{N\mathcal{v}_I} \mid \omega \subseteq v \,\}$$

4.4. Example: let $X = \{a,b,c\}$, with two topologies: $\tau_1 = \{\emptyset,\{a\},X\}, \tau_2 = \{\emptyset,\{a,b\},X\}$
Then $\tau_v = \{\emptyset,\{a\},X\}$.Let $= Z_2, J = \{0\} \subseteq R$, so $J(I) = \{0,I\}$.
Take $v = \{(a,1,I,0),(b,1+I,1,1),(c,0,1,1+I)\}$.
Then
$Nint_{N_I\mathcal{v}}(v) = \{(a,1,0,0),(b,1,0,0)\}$, (corresponds to crisp $\{a,b\}$ which is $\mathcal{v}_O$-set).

4.5. Definition: The neutr. exterior of a neutr. set v is the interior of its complement Formally:

$$Next_{N_I\mathcal{v}}(v) = Nint_{N_I\mathcal{v}}(v^c)$$

where $v^c(x) = (F_v(x), I_v(x), T_v(x))$ for each $x \in X$.

4.6. Example: let $X = \{a,b,c\}$, with two topologies: $\tau_1 = \{\emptyset,\{a\},X\}, \tau_2 = \{\emptyset,\{a,b\},X\}$
Then $\tau_v = \{\emptyset,\{a\},X\}$.Let $= Z_2, J = \{0\} \subseteq R$, so $J(I) = \{0,I\}$.
Take $v = \{(a,1,I,0),(b,0,1,1),(c,0,1,1)\}$.
Then $v^c = \{(a,0,I,1),(b,1,1,0),(c,1,1,0)\}$
$Next_{N_I\mathcal{v}}(v) = \{(b,1,0,0),(c,1,0,0)\}$, (corresponds to crisp $\{b,c\}$ which is $\mathcal{v}_O$-set)

4.7. Definition: The neutr. boundary of a neutr. set v is

$$Nbd_{N_I\mathcal{v}}(\nu) = Ncl_{N_I\mathcal{v}}(\nu)/Nint_{N_I\mathcal{v}}(\nu)$$

4.8. Example: let $X = \{a, b, c\}$, with two topologies: $\tau_1 = \{\emptyset, \{a\}, X\}, \tau_2 = \{\emptyset, \{a, b\}, X\}$
Then $\tau_v = \{\emptyset, \{a\}, X\}$.Let $= Z_2, J = \{0\} \subseteq R$, so $J(I) = \{0, I\}$.
Take $\nu = \{(a, 1, I, 0), (b, 1 + I, 0,0), (c, 0,1,1)\}$.
$Ncl_{N_I\mathcal{v}}(\nu) = \nu \cup \{(c, I, 0,0)\} = \{(a, 1, I, 0), (b, 1 + I, 0,0), (c, I, 1,1)\}$.
$Nint_{N_I\mathcal{v}}(\nu) = \{(a, 1,0,0), (b, 1,0,0)\}$.
$Nbd_{N_I\mathcal{v}}(\nu) = \{(a, 0, I, 0), (b, I, 0,0), (c, I, 1,1)\}$.

4.9. Definition: A neutr. set ω is termed a $N_I\mathcal{v}$**-neighborhood** ($N_I\mathcal{v}$-neigh) of a point $x \in X$ if there exists $\eta \in \tau_{N\mathcal{v}_I}$ under which: $\eta(x) = (1,0,0)$ and $\eta \subseteq \omega$.

4.10. Example: For $x = a$, the neutr. set $\eta = \{(a, 1,0,0), (b, 0,1,1), (c, 0,1,1)\}$ is a $N_I\mathcal{v}_O$-set. Any ω with $\eta \subseteq \omega$ is a $N_I\mathcal{v}$-neigh of a.

4.11. Definition: A point $x \in X$ is a N_{Iv}**-limit point** of neutr. set ν if for every $N_I\mathcal{v}$-neigh η of x, there exists $y \neq x$ under which: $T_\eta(y) > 0$ and $T_\nu(y) > 0$. The set of all N_{Iv}**-limit point** s of ν is denoted $Nd_{NIv}(\nu)$.

4.12. Example: Let $\nu = \{(a, 0.8,0.2,0.1), (b, 0.7,0.3,0.2), (c, 0,1,1)\}$.
For $x = a$, consider $N_I\mathcal{v}$-neigh $\eta = \{(a, 1,0,0), (b, 0.5,0.5,0), (c, 0,1,1)\}$.
Since $T_\eta(b) = 0.5 > 0$ and $T_\nu(b) = 0.7 > 0$, $a \in Nd_{NIv}(\nu)$.

4.13. Remark: In N_{Iv}-spaces, limit points depend on both the $\mathcal{v}$-open structure and the neutr. ideal perturbations. Unlike classical topology where only truth membership matters, here indeterminacy and falsity components from $J(I)$ affect neighborhood structures.

4.14. Theorem: Let $(X,\tau_{N\mathcal{v}_I})$ be a neutr. ideal v-open space. For any $\nu_1, \nu_2 \in NS(X)$, then:

1. $Nint_{N_I\mathcal{v}}(\emptyset_N) = \emptyset_N$ and $Nint_{N_I\mathcal{v}}(X_N) = X_N$
2. If $\nu_1 \subseteq \nu_2 \Longrightarrow Nint_{N_I\mathcal{v}}(\nu_1) \subseteq Nint_{N_I\mathcal{v}}(\nu_2)$.
3. $Nint_{N_I\mathcal{v}}(\nu_1 \cap \nu_2) = Nint_{N_I\mathcal{v}}(\nu_1) \cap Nint_{N_I\mathcal{v}}(\nu_2)$.
4. $Nint_{N_I\mathcal{v}}(\nu_1 \cup \nu_2) \supseteq Nint_{N_I\mathcal{v}}(\nu_1) \cup Nint_{N_I\mathcal{v}}(\nu_2)$
5. ν is $N_I\mathcal{v}_O$-set $\Leftrightarrow Nint_{N_I\mathcal{v}}(\nu) = \nu$.

Proof:

1. For $Nint_{N_I\mathcal{v}}(\emptyset_N)$**:**
   By Definition of $Nint_{N_I\mathcal{v}}(\emptyset_N), \emptyset_N$ is trivially $N_I\mathcal{v}_O$-set. Thus, its $N_I\mathcal{v}_O$-set is itself. For X_N, $Nint_{N_I\mathcal{v}}(X_N) = X_N$ The idempotence follows because $N_I\mathcal{v}_O$-set are preserved under interior operations.
2. Monotonicity:
   Let: $F_1 = \{\eta \in\tau_{N\mathcal{v}_I} \mid \eta \subseteq \nu_1\}$, $F_2 = \{\eta \in\tau_{N\mathcal{v}_I} \mid \eta \subseteq \nu_2\}$, Since $\nu_1 \subseteq \nu_2$, any η that satisfies $\eta \subseteq \nu_1$ automatically satisfies $\eta \subseteq \nu_2$.
   Thus: $F_1 \subseteq F_2$.
   Taking unions: $Nint_{N_I\mathcal{v}}(\nu_1) = \bigcup F_1 \subseteq \bigcup F_2 - Nint_{N_I\mathcal{v}}(\nu_2)$
   Therefore: $Nint_{N_I\mathcal{v}}(\nu_1) \subseteq Nint_{N_I\mathcal{v}}(\nu_2)$.
3. We prove both inclusions.
   ($\subseteq$) Show $(\nu_1 \cap \nu_2) \subseteq Nint_{N_I\mathcal{v}}(\nu_1) \cap Nint_{N_I\mathcal{v}}(\nu_2)$**:**
   Since $(\nu_1 \cap \nu_2) \subseteq \nu_1$ and $(\nu_1 \cap \nu_2) \subseteq \nu_2$, by monotonicity (Part 2):
   $Nint_{N_I\mathcal{v}}(\nu_1 \cap \nu_2) \subseteq Nint_{N_I\mathcal{v}}(\nu_1)$ and $Nint_{N_I\mathcal{v}}(\nu_1 \cap \nu_2) \subseteq Nint_{N_I\mathcal{v}}(\nu_2)$
   Thus: $Nint_{N_I\mathcal{v}}(\nu_1 \cap \nu_2) \subseteq Nint_{N_I\mathcal{v}}(\nu_1) \cap Nint_{N_I\mathcal{v}}(\nu_2)$
   ($\supseteq$) Show $Nint_{N_I\mathcal{v}}(\nu_1) \cap Nint_{N_I\mathcal{v}}(\nu_2) \subseteq Nint_{N_I\mathcal{v}}(\nu_1 \cap \nu_2)$:
   Let η be any $N_I v_O$-set under which: $\eta \subseteq Nint_{N_I\mathcal{v}}(\nu_1) \cap Nint_{N_I\mathcal{v}}(\nu_2)$
   Then $\eta \subseteq Nint_{N_I\mathcal{v}}(\nu_1)$ and $\eta \subseteq Nint_{N_I\mathcal{v}}(\nu_2)$
   By definition of interior, Since η is a $N_I\mathcal{v}_O$-set and $\eta \subseteq Nint_{N_I\mathcal{v}}(\nu_1) \subseteq \nu_1$, we have $\eta \subseteq \nu_1$.
   Similarly, $\eta \subseteq \nu_2$.
   Thus: $\eta \subseteq \nu_1 \cap \nu_2$.
   Therefore, η is a $N_I v_O$-set contained in $\nu_1 \cap \nu_2$, so: $\eta \subseteq Nint_{N_I\mathcal{v}}(\nu_1 \cap \nu_2)$

Since this holds for every such η, taking the union over all such η gives: $Nint_{N_I\mathcal{v}}(v_1 \cap v_2) \supseteq Nint_{N_I\mathcal{v}}(v_1) \cap Nint_{N_I\mathcal{v}}(v_2)$
Both inclusions are proved, hence equality holds.

4. Since $\nu_1 \subseteq \nu_1 \cup \nu_2$ and $\nu_2 \subseteq \nu_1 \cup \nu_2$, by monotonicity (Part 2): $\mathrm{Nint}_{\mathrm{N_I}\mathcal{v}}(\nu_1) \subseteq \mathrm{Nint}_{\mathrm{N_I}\mathcal{v}}(\nu_1 \cup \nu_2)$, and $\mathrm{Nint}_{\mathrm{N_I}\mathcal{v}}(\nu_2) \subseteq \mathrm{Nint}_{\mathrm{N_I}\mathcal{v}}(\nu_1 \cup \nu_2)$
Thus: $Nint_{N_I\mathcal{v}}(v_1) \cup Nint_{N_I\mathcal{v}}(v_2) \subseteq Nint_{N_I\mathcal{v}}(v_1 \cup v_2)$
Equality doesn't generally hold because a $N_I v_O$-set contained in $v_1 \cup v_2$ might intersect both v_1 and v_2 but not be entirely contained in either one alone.
5. (⇒) Assume ν is $\mathrm{N_I}\nu_{\mathrm{O}}$-set. By definition, if $\nu \in \tau_{\mathrm{N}\mathcal{v}_{\mathrm{I}}}$, then ν is a $\mathrm{N_I}\nu_{\mathrm{O}}$-set contained in itself.
Thus: $\nu \in \{\eta \in \tau_{\mathrm{N}\mathcal{v}_{\mathrm{I}}} \mid \eta \subseteq \nu\}$.
Since $Nint_{N_I\mathcal{v}}(v)$ is the union of all such sets, and v is the largest set in this collection, we have: $Nint_{N_I\mathcal{v}}(v) = v$
(⇐) Assume $Nint_{N_I\mathcal{v}}(v) = v$. By definition of interior: $Nint_{N_I\mathcal{v}}(v) = \cup\{\omega \in \tau_{N\mathcal{v}_I} \mid \omega \subseteq v\}$
Since $Nint_{N_I\mathcal{v}}(v) = v$, we have: $v = \cup\{\eta \in \tau_{N\mathcal{v}_I} \mid \eta \subseteq v\}$
The union of $N_I v_O$-sets is a $N_I v_O$-set (as $\tau_{N\mathcal{v}_I}$ is closed under arbitrary unions, being a topology).
Therefore, v is a $N_I v_O$-set.

4.15. Theorem: Under the same hypotheses as [Theorem 4.14], then:

1. $Ncl_{N_I\mathcal{v}}(\emptyset_N) = \emptyset_N$ and $Ncl_{N_I\mathcal{v}}(X_N) = X_N$, and $Ncl_{N_I\mathcal{v}}\left(Ncl_{N_I\mathcal{v}}(v)\right) = Ncl_{N_I\mathcal{v}}(v)$
2. If $v_1 \subseteq v_2 \Longrightarrow Ncl_{N_I\mathcal{v}}(v_1) \subseteq Ncl_{N_I\mathcal{v}}(v_2)$.
3. $Ncl_{N_I\mathcal{v}}(v_1 \cap v_2) \subseteq Ncl_{N_I\mathcal{v}}(v_1) \cap Ncl_{N_I\mathcal{v}}(v_2)$.
4. $Ncl_{N_I\mathcal{v}}(v_1 \cup v_2) = Ncl_{N_I\mathcal{v}}(v_1) \cup Ncl_{N_I\mathcal{v}}(v_2)$
5. v is $N_I\mathcal{v}_c$-set $\Longleftrightarrow Ncl_{N_I\mathcal{v}}(v) = v$.

Proof:

1. For $Ncl_{N_I\mathcal{v}}(\emptyset_{\mathrm{N}})$: By Definition of $Ncl_{N_I\mathcal{v}}$, **and** $\emptyset_N{}^c = X_N^c \in \tau_{N\mathcal{v}_I}$, hence $Ncl_{N_I\mathcal{v}}(\emptyset_N) = \cap\{\omega \in NS(X) \mid \omega c \in \tau_{N\mathcal{v}_I}, \emptyset_N \subseteq \omega\} = \emptyset_N$
For $Ncl_{N_I\mathcal{v}}(\mathrm{X_N})$: By Definition of $Ncl_{N_I\mathcal{v}}$, **and** $\mathrm{X}_N{}^c = \emptyset_N^c \in \tau_{N\mathcal{v}_I}$, hence $Ncl_{N_I\mathcal{v}}(X_N) = \cap\{\omega \in NS(X) \mid \omega^c \in \tau_{N\mathcal{v}_I}, X_N \subseteq \omega\} = \mathrm{X_N}$
For $cl_{N_I\mathcal{v}}\left(Ncl_{N_I\mathcal{v}}(v)\right) = Ncl_{N_I\mathcal{v}}(v)$**, let** $C = Ncl_{N_I\mathcal{v}}(v)$. By Definition of C is $N_I\mathcal{v}_c$-set, for any $N_I\mathcal{v}_c$-set ω containing C, we have $\omega \supseteq C$. The intersection of all such ω is exactly C. Therefore: $Ncl_{N_I\mathcal{v}}(C) = C$

2. Let: $C_1 = \{\omega \in NS(X) \mid \omega^c \in \tau_{N\mathcal{v}_I}, v_1 \subseteq \omega\}$, and $C_2 = \{\omega \in NS(X) \mid \omega^c \in \tau_{N\mathcal{v}_I}, v_2 \subseteq \omega\}$. Since $v_1 \subseteq v_2$, any ω that includes v_2 also contains v_1. Thus: $C_1 \subseteq C_2$. Taking intersections: $Ncl_{N_I\mathcal{v}}(v_2) = \cap C_2 \subseteq \cap C_1 = Ncl_{N_I\mathcal{v}}(v_1)$
If $C_2 \subseteq C_1$, then $C_2 \supseteq \cap C_1$. By same way for C_1
Any $N_I\mathcal{v}_c$-set containing v_2 automatically contains v_1 (since $v_1 \subseteq v_2$). So the collection of $N_I\mathcal{v}_c$-set containing v_2 is a **subset** of those containing v_1. Therefore: $Ncl_{N_I\mathcal{v}}(v2) = \cap\{\text{sets containing } v2\} \supseteq \cap\{\text{sets containing } v_1\} = Ncl_{N_I\mathcal{v}}(v_1)$. Thus: $Ncl_{N_I\mathcal{v}}(v_1) \subseteq Ncl_{N_I\mathcal{v}}(v_2)$.

3. Since $v_1 \cap v_2 \subseteq v_1$ and $v_1 \cap v_2 \subseteq v_1$, by monotonicity (Part 2):
$Ncl_{N_I\mathcal{v}}(v_1 \cap v_2) \subseteq Ncl_{N_I\mathcal{v}}(v_1)$, and $Ncl_{N_I\mathcal{v}}(v_1 \cap v_2) \subseteq Ncl_{N_I\mathcal{v}}(v_2)$.
Therefore: $Ncl_{N_I\mathcal{v}}(v_1 \cap v_2) \subseteq Ncl_{N_I\mathcal{v}}(v_1) \cap Ncl_{N_I\mathcal{v}}(v_2)$
Note: Equality doesn't generally hold because two sets might have limit points that disappear when we take the intersection.

4. We prove both inclusions.
(⊆) Show $Ncl_{N_I\mathcal{v}}(v_1 \cup v_2) \subseteq Ncl_{N_I\mathcal{v}}(v_1) \cup Ncl_{N_I\mathcal{v}}(v_2)$**:**
Let $C_1 = Ncl_{N_I\mathcal{v}}(v_1)$ and $C_2 = Ncl_{N_I\mathcal{v}}(v_2)$. Both C_1 and C_2 are $N_I\mathcal{v}_c$-sets
Their union $C_1 \cup C_2$ is a $N_I\mathcal{v}_c$-set (as $N_I\mathcal{v}_c$-sets are closed under finite unions)
Since $v_1 \cup v_2 \subseteq C_1 \cup C_2$, and $C_1 \cup C_2$ is a $N_I\mathcal{v}_c$-set containing $v_1 \cup v_2$, we have:
$Ncl_{N_I\mathcal{v}}(v_1 \cup v_2) \subseteq Ncl_{N_I\mathcal{v}}(v_1) \cup Ncl_{N_I\mathcal{v}}(v_2)$

(⊇) Show $Ncl_{N_I\mathcal{v}}(v_1 \cup v_2) \supseteq Ncl_{N_I\mathcal{v}}(v_1) \cup Ncl_{N_I\mathcal{v}}(v_2)$**:**
Since $v_1 \subseteq v_1 \cup v_2$, by monotonicity: $Ncl_{N_I\mathcal{v}}(v_1) \subseteq Ncl_{N_I\mathcal{v}}(v_1 \cup v_2)$
Since $v_2 \subseteq v_1 \cup v_2$, by monotonicity: $Ncl_{N_I\mathcal{v}}(v_2) \subseteq Ncl_{N_I\mathcal{v}}(v_1 \cup v_2)$
Therefore: $Ncl_{N_I\mathcal{v}}(v_1 \cup v_2) \supseteq Ncl_{N_I\mathcal{v}}(v_1) \cup Ncl_{N_I\mathcal{v}}(v_2)$
Both inclusions are proved, hence equality holds.

5. **(⇒) Assume** ν **is** $N_I\mathcal{v}_c$-set **:** Then $\nu^c \in\tau_{N\mathcal{v}_I}$, ν itself is a $N_I\mathcal{v}_c$-set containing ν. It is the smallest such set (any smaller $N_I\mathcal{v}_c$-set containing ν would be a proper subset of ν)
Therefore: $Ncl_{N_I\mathcal{v}}(\nu) = \nu$
(⇐) Assume $Ncl_{N_I\mathcal{v}}(\nu) = \nu$**:** By definition, $Ncl_{N_I\mathcal{v}}(\nu)$ is the intersection of all $N_I\mathcal{v}_c$-sets containing ν
If this intersection equals ν, then ν itself must be one of these $N_I\mathcal{v}_c$-sets. Therefore, ν is a $N_I\mathcal{v}_c$-set
Equivalently: $\nu^c \in\tau_{N\mathcal{v}_I}$.

4.16. Theorem: Under the same hypotheses as [Theorem 4.14], then:

1. $Next_{N_I\mathcal{v}}(\emptyset_N) = X_N$ and $Next_{N_I\mathcal{v}}(\emptyset_N) = X_N$
2. $Next_{N_I\mathcal{v}}(v) \subseteq v^c$.
3. $Next_{N_I\mathcal{v}}\left(Next_{N_I\mathcal{v}}(v^c)\right) = Next_{N_I\mathcal{v}}(v)$.
4. If $v_1 \subseteq v_2 \Longrightarrow Next_{N_I\mathcal{v}}(v_2) \subseteq Next_{N_I\mathcal{v}}(v_1)$
5. $Next_{N_I\mathcal{v}}(v_1 \cup v_2) = Next_{N_I\mathcal{v}}(v_1) \cap Next_{N_I\mathcal{v}}(v_2)$

Proof:

1. For $\boldsymbol{Next_{N_I\mathcal{v}}(\emptyset_N)}$: By Definition $\boldsymbol{Next_{N_I\mathcal{v}}(\emptyset_N) = Nint_{N_I\mathcal{v}}({\emptyset_N}^c) = Nint_{N_I\mathcal{v}}(X_N) = X_N}$
For $\boldsymbol{Next_{N_I\mathcal{v}}(X_N)}$: By Definition $\boldsymbol{Next_{N_I\mathcal{v}}(X_N) = Nint_{N_I\mathcal{v}}({X_N}^c) = Nint_{N_I\mathcal{v}}}(\emptyset_N) = \emptyset_N$.
2. $\boldsymbol{Next_{N_I\mathcal{v}}(v) = Nint_{N_I\mathcal{v}}}(v^c)) \subseteq \boldsymbol{v^c}$
Therefore: $\boldsymbol{Next_{N_I\mathcal{v}}(v) \subseteq v^c}$
3. $\boldsymbol{Next_{N_I\mathcal{v}}\left(Next_{N_I\mathcal{v}}(v^c)\right) = Next_{N_I\mathcal{v}}\left(Next_{N_I\mathcal{v}}(v^c)^c\right) = Next_{N_I\mathcal{v}}\left(Next_{N_I\mathcal{v}}(v)\right)}$.Since $\boldsymbol{Next_{N_I\mathcal{v}}(v)}$ is $\boldsymbol{N_I\mathcal{v}_O}$-set , $\boldsymbol{Next_{N_I\mathcal{v}}(v^c)}$ is $\boldsymbol{N_I\mathcal{v}_c}$-set , and its interior is $\emptyset_N$ unless $\boldsymbol{A = X_N}$. Thus,
$\boldsymbol{Next_{N_I\mathcal{v}}\left(Next_{N_I\mathcal{v}}(v^c)\right) = Next_{N_I\mathcal{v}}(v)}$.
4. If $\boldsymbol{v_1 \subseteq v_2}$, then from [theorem 4.13 (2)] $\boldsymbol{Nint_{N_I\mathcal{v}}(v_2^c) \subseteq Nint_{N_I\mathcal{v}}(v_1^c) \Longrightarrow Next_{N_I\mathcal{v}}(v_2) \subseteq Next_{N_I\mathcal{v}}(v_1)}$
5. To prove $\boldsymbol{Next_{N_I\mathcal{v}}(v_1 \cup v_2) = Next_{N_I\mathcal{v}}(v_1) \cap Next_{N_I\mathcal{v}}(v_2)}$
Since $\boldsymbol{Next_{N_I\mathcal{v}}(v_1 \cup v_2) = Nint_{N_I\mathcal{v}}(v_1 \cup v_2)^c = Nint_{N_I\mathcal{v}}(v_1^c \cap v_2^c)}$
By Theorem 4.13(3): $\boldsymbol{Nint_{N_I\mathcal{v}}(v_1^c \cap v_2^c) = Nint_{N_I\mathcal{v}}(v_1^c) \cap Nint_{N_I\mathcal{v}}(v_2^c) = Next_{N_I\mathcal{v}}(v_1) \cap}$
$\boldsymbol{Next_{N_I\mathcal{v}}(v_2)}$, hence the prove is holds.

4.17. Theorem: Under the same hypotheses as [Theorem 4.14], then:

1. $Nbd_{N_I\mathcal{v}}(v) \cap Nint_{N_I\mathcal{v}}(v) = \emptyset_N$
2. $Nbd_{N_I\mathcal{v}}(v) \cap Nextt_{N_I\mathcal{v}}(v) = \emptyset_N$.
3. $Nbd_{N_I\mathcal{v}}(v) = Nbd_{N_I\mathcal{v}}(v^c)$.
4. $Nbd_{N_I\mathcal{v}}(v_1 \cup v_2) \subseteq Nbd_{N_I\mathcal{v}}(v_1) \cap Nbd_{N_I\mathcal{v}}(v_2)$

Proof:

1-2. By definition, boundary excludes both interior and exterior.
3. Complement preserves boundary structure.
4. Follows from closure and interior properties.

4.18. Theorem: Under the same hypotheses as [Theorem 4.14], then:

1. If $v_1 \subseteq v_2$, then $Nd_{NIv}(v_1) \subseteq Nd_{NIv}(v_2)$.
2. $Ncl_{NIv}(v) = v \cup Nd_{NIv}(v)$.
3. $Nd_{NIv}(v_1 \cup v_2) = Nd_{NIv}(v_1) \cap Nd_{NIv}(v_2)$

Proof:

1. **Monotonicity:**
   Let $\nu_1 \subseteq \nu_2$. A point $x \in X$ is in $Nd_{NI\nu}(\nu)$, if for every $N_I\nu$-neigh η of x, there exists $y \neq x$ under which: $T_\eta(y) > 0$ and $T_\nu(y) > 0$
   Now, let $x \in Nd_{NI\nu}(\nu_1)$. Then for every $N_I\nu$-neigh η of x, there exists $y \neq x$ with: $T_\eta(y) > 0$ and $T_{\nu_1}(y) > 0$
   Since $\nu_1 \subseteq \nu_2$, we have $T_{\nu_1}(y) \leq T_{\nu_2}(y)$ for all $y \in X$ (because in neutr. set inclusion, $T_{\nu_1}(x) \leq T_{\nu_2}(x)$, $I_{\nu_1}(x) \leq I_{\nu_2}(x)$, $F_{\nu_1}(x) \leq F_{\nu_2}(x)$.
   Thus, if $T_{\nu_1}(y) > 0$, then $T_{\nu_2}(y) > 0$.
   Therefore, for every $N_I\nu$-neigh η of x, there exists $y \neq x$ with: $T_\eta(y) > 0$ and $T_{\nu_2}(y) > 0$
   This means $x \in Nd_{NI\nu}(\nu_2)$.
   Hence: $Nd_{NI\nu}(\nu_1) \subseteq Nd_{NI\nu}(\nu_2)$.
2. We prove both inclusions.
   $(\subseteq) Ncl_{NI\nu}(\nu) \subseteq \nu \cup Nd_{NI\nu}(\nu)$:
   Let $x \in X$ under which $x \in Nd_{NI\nu}(\nu)$.
   Case 1: If $x \in \nu$, then clearly $x \in \nu \cup Nd_{NI\nu}(\nu)$.
   Case 2: If $x \notin \nu$, we need to show that $x \in Nd_{NI\nu}(\nu)$.
   Since $x \in Nd_{NI\nu}(\nu)$ and $x \notin \nu$, by the definition of closure, for every $N_I\nu$-neigh U of x, we have: $U \cap \nu \neq \emptyset_N$
   Since $x \notin \nu$, the intersection contains at least one point $y \neq x$ with $T_\nu(y) > 0$
   Thus, for every $N_I\nu$-neigh U of x, there exists $y \neq x$ under which: $T_U(y) > 0$and $T_\nu(y) > 0$
   This is exactly the definition of $x \in Nd_{NI\nu}(\nu)$.
   Therefore, $x \in \nu \cup Nd_{NI\nu}(\nu)$.
   $(\supseteq)$ Show $Ncl_{NI\nu}(\nu) \supseteq \nu \cup Nd_{NI\nu}(\nu)$:
   a) Clearly $\nu \subseteq Ncl_{NI\nu}(\nu)$ by definition of closure.
   b) Let $x \in Ncl_{NI\nu}(\nu)$. We show $x \in Ncl_{NI\nu}(\nu)$. Suppose, for contradiction, that $x \notin Ncl_{NI\nu}(\nu)$. Then there exists a $N_I\mathcal{v}_c$-set C under which $\nu \subseteq C$ and $x \notin C$. Since C is $N_I\mathcal{v}_c$-set , C^c is $N_I\mathcal{v}_O$-set and contains x. Now, C^c is a $N_I\nu$-neigh of x. But since $\nu \subseteq C$, we have $\nu \cap C^c = \emptyset_N$. This means that for this particular $N_I\nu$-neigh C^c of x, there is no y (and certainly no $x \neq y$ with $T_{C^c}(y) > 0$ and $T_\nu(y) > 0$.
   This contradicts $x \in Ncl_{NI\nu}(\nu)$, which requires that every $N_I\nu$-neigh of x contains some $x \neq y$ with $T_\nu(y) > 0$. Therefore, $x \in Ncl_{NI\nu}(\nu)$.
   Thus: $Ncl_{NI\nu}(\nu) \subseteq Ncl_{NI\nu}(\nu)$. Since both inclusions hold, we have: $Ncl_{NI\nu}(\nu) \supseteq \nu \cup Nd_{NI\nu}(\nu)$
3. Assume $x \notin Nd_{NI\nu}(\nu_1)$ and $\notin Nd_{NI\nu}(\nu_2)$. Then:
   Since $\notin Nd_{NI\nu}(\nu_1)$, there exists a $N_I\nu$-neigh U_1 of x under which for all $x \neq y$ with $T_{U_1}(y) > 0$, we have $T_{\nu_1}(y) = 0$. By same way for $x \notin Nd_{NI\nu}(\nu_2)$
   Let $= U_1 \cap U_2$. Since $\tau_{N\mathcal{v}_I}$ is a topology, U is $N_I\mathcal{v}_O$-set . Also, $x \in U$ (since $x \in U_1$ and $\in U_2$, so U is a $N_I\nu$-neigh of x.
   Now, for this neighborhood U: For any $x \neq y$ with $T_U(y) > 0$, we have $T_{U_1}(y) > 0$ and $T_{U_2}(y) > 0$ (since $U \subseteq U_1$ and $U \subseteq U_2$)
   Then by (1): $T_{\nu_1}(y) = 0$. And by (2): $T_{\nu_2}(y) = 0$
   So $T_{\nu_1}(y) \cup \nu_2(y) = \max(0,0) = 0$
   Thus, for the $N_I\nu$-neigh U of x, there is $x \neq y$ with $T_U(y) > 0$ and $T_{\nu_1} \cup \nu_2(y) > 0$. This contradicts $x \in Nd_{NI\nu}(\nu_1 \cup \nu_2)$.
   Therefore, our assumption is false, and we must have $x \in Nd_{NI\nu}(\nu_1)$ or $x \in Nd_{NI\nu}(\nu_2)$.
   Thus: $x \in Nd_{NI\nu}(\nu_1) \cup Nd_{NI\nu}(\nu_2)$.
   Hence: $Nd_{NI\nu}(\nu_1 \cup \nu_2) \subseteq Nd_{NI\nu}(\nu_1) \cup Nd_{NI\nu}(\nu_2)$.
   $(\supseteq)$ Show $Nd_{NI\nu}(\nu_1 \cup \nu_2) \supseteq Nd_{NI\nu}(\nu_1) \cup Nd_{NI\nu}(\nu_2)$:
   This follows from the monotonicity property (which can be proved separately).
   Since $\nu_1 \subseteq \nu_1 \cup \nu_2$, any limit point of ν_1 is also a limit point of $\nu_1 \cup \nu_2$:
   If $x \in Nd_{NI\nu}(\nu_1)$, then for every $N_I\nu$-neigh U of x, there exists $x \neq y$ with $T_U(y) > 0$ and $T_{\nu_1}(y) > 0$.

Since $T_{v_1}(y) > 0$ implies $T_{v_1} \cup v_2(y) > 0$, we have $x \in Nd_{NIv}(v_1 \cup v_2)$
Similarly, if $x \in Nd_{NIv}(v_2)$, then $Nd_{NIv}(v_1 \cup v_2) \supseteq Nd_{NIv}(v_1) \cup Nd_{NIv}(v_2)$.
Since both inclusions hold, we have: $Nd_{NIv}(v_1 \cup v_2) = Nd_{NIv}(v_1) \cap Nd_{NIv}(v_2)$

5. Conclusions

The results reveal that introducing the neutr. ideal in the structure of open sets of type vis not merely an addition, but also has a profound effect on changing the nature of the topological space. It can be inferred that the ideal neutr. topological space preserves the basic topology conditions (such as union closure and the retention of neutral elements), but achieves a more flexible approach to describing indeterminate processes. We show that the workings of inset, closure, and limitation in the case of this scenario are affected by the neutr. ideal, and that in this context, the accumulator and adjacency points depend not only on the right degree of belonging, but also to varying degrees of indeterminacy and falsehood. Moreover, the work demonstrates that many relations known in classical topology are true in a modified form while others are not, indicating the richness of the new structure. As you have previously seen, this provides an appropriate mathematical framework for modelling multi-perspective systems in which algebraic and topological structures overlap under circumstances of intrinsic uncertainty. Therefore, the contributions of this work provide some theoretical grounds for future works in neutr. continuity, topological separation, and use cases where you can use data in an imperfect or conflicting manner, such as intelligent modeling or complex decision systems.

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