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Research article

Analyzing Real Data by a New Heavy-Tailed Statistical Model

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ABSTRACT

This study presents the power Mira distribution, an innovative three-parameter probability model that improves baseline distributions by including an extra shaping parameter. The suggested distribution has exceptional adaptability in representing various data characteristics, such as left and right skewness, declining trends, and unimodal patterns. These characteristics render it exceptionally appropriate for modeling risk-related data, an essential component of actuarial science and insurance analytics. We do an extensive theoretical study, delineating essential statistical features and offering a robust framework for parameter estimation. Critical risk metrics, including value-at-risk and tail value-at-risk, are calculated and assessed using comprehensive numerical simulations, validating the precision and efficacy of the suggested estimators. We illustrate the practical value of the power Mira distribution by applying it to a real-world insurance loss dataset and comparing its performance with established models. The findings underscore its exceptional goodness-of-fit and flexibility, affirming its capability as an effective instrument for risk assessment and financial modeling.

1. Introduction

In the last decades, many distributions have been generated using different methods to improve the flexibility. This can be achieved by adding one or more parameters to the parent models. Parameter(s) to the parent models. One of these methods is the power transformation method, a powerful tool for creating new models to better capture complex data sets' underlying patterns. This technique is beneficial when dealing with non-linear relationships between variables, which can be difficult to model using traditional statistical methods. By transforming the data to a new distribution, it is possible to create a more flexible model to fit the data better and make more accurate predictions. The power transformation method can also be used with a wide range of distributions, making it a versatile tool for data analysts and statisticians in various fields. Overall, the power transformation method is an important technique for creating more flexible and accurate statistical models and is an essential tool in the modern data analysis toolkit.

Some of the recent generated distributions include, power-modified Kies-exponential distribution by [3], modified Lindley distribution by[15], the power piecewise exponential model by [22], exponentiated odd Lomax exponential distribution by [16], alpha power transformed extended exponential distribution by [25], unit-Lindley distribution by [27], the arcsine-X family of distributions by [40], generalized odd linear exponential family of distribution by [26], alpha power inverted exponential distribution by [13], type II Quasi Lambert family [24], Gompertz-alpha power inverted exponential distribution by [17], extended half-power exponential model by [34], alpha power Kumaraswamy inverted exponential model by [38], to mention a few.

In disciplines such as insurance, actuarial science, and economics, datasets often exhibit observations that significantly diverge from the central trend, a fact generally ascribed to the existence of outliers or heavy tails. Conventional probability distributions often fail to effectively represent these attributes, resulting in suboptimal model performance and questionable statistical judgments. Thus, the need for heavy-tailed distributions emerges since they provide a more suitable framework for depicting data with significant fluctuations. These distributions provide more flexibility in modeling skewness and kurtosis, rendering them especially beneficial in risk assessment, financial modeling, and dependability analysis. A comprehensive study has been undertaken on the formulation and use of heavy-tailed distributions, focusing on their theoretical characteristics and practical consequences. Heavy-tailed distributions are available in the literature; for more details, see [37, 41, 31, 11, 4, 14]

The main aim of this paper is divided into two parts: the first objective is to introduce a new flexible distribution based on the power method. It can be used in different phenomena, particularly in modeling the insurance loss data set, and can also be used to estimate the risk exposure. This new proposed model is referred to as the power Mira distribution (PMD). The proposed model has three parameters, and it can be unimodal and right-skewed or left-skewed. Furthermore, the two-parameter Mira (TPM) distribution is a special case of our proposed model. The second objective is to derive well-known risk measures for the PMD model.

The following factors have been involved in the design of this work. Section 2 defines the novel power Mira model and its corresponding reliability statistics. Several statistical properties, such as the quantile function, moments, moment generating function, moments of residual life, Rényi entropy, and the distribution of order statistics, are presented in Section 3. Section 4 contains the estimation of the unknown parameters using several techniques. A Monte Carlo simulation study to examine the consistency property of the maximum likelihood estimators is given in Section 5. Section 6 presents some actuarial measures from the power Mira distribution. One data set to examine the potential of the proposed distribution is given in Section 7, followed by some concluding remarks in Section 8.

2. Model formulation

Sen et al. [35] defined the TPM distribution where its cumulative distribution function (CDF) (for $t > 0$) is given as follows

$$G(t) = 1 - \frac{\delta^2\left(\frac{\alpha}{\delta^2} + \frac{\alpha t^2}{2} + \frac{\alpha t}{\delta} + 1\right)e^{-\delta t}}{\alpha + \delta^2}; t, \alpha, \delta > 0, \tag{2.1}$$

and its probability density function (PDF) is defined as follows

$$g(t) = \frac{\delta^3\left(\alpha t^2 + 2\right)e^{-\delta t}}{2\left(\alpha + \delta^2\right)}; t, \alpha, \delta > 0. \tag{2.2}$$

By using the following transformation $X = T^{\frac{1}{\beta}}$ to (2.1), the CDF of the PMD is

$$F(x) = 1 - \frac{e^{-\delta x^\beta}\left(\delta^2\left(\frac{\alpha}{\delta^2} + \frac{1}{2}\alpha x^{2\beta} + \frac{\alpha x^\beta}{\delta} + 1\right)\right)}{\alpha + \delta^2}; x, \alpha, \delta, \beta > 0. \tag{2.3}$$

The associated PDF is defined by

$$f(x) = \frac{\beta\delta^3 x^{\beta-1}\left(\alpha x^{2\beta} + 2\right)e^{-\delta x^\beta}}{2\left(\alpha + \delta^2\right)}; x, \alpha, \delta, \beta > 0.. \tag{2.4}$$

From (2.4), clearly the random variable X is a TPM distribution if $\beta = 1$. Figure 1 shows the PDF plots of the PMD. It is visible that the PDF can take several shapes, including increasing, right-skewed, left-skewed, unimodal, and reverse-J.

2.1. Reliability measures

The PMD's hazard rate function (hrf) and survival function (SF) are provided, respectively, by

$$S(x) = 1 - F(x) = -\frac{e^{-\delta x^\beta}\left(\delta^2\left(\frac{\alpha}{\delta^2} + \frac{1}{2}\alpha x^{2\beta} + \frac{\alpha x^\beta}{\delta} + 1\right)\right)}{\alpha + \delta^2}, \tag{2.5}$$

and

$$h(x) = \frac{f(x)}{S(x)} = \frac{\beta\delta^3\left(\alpha x^{2\beta} + 2\right)}{x\left(2\alpha\delta + 2\left(\alpha + \delta^2\right)x^{-\beta} + \alpha\delta^2 x^\beta\right)}. \tag{2.6}$$

The PMD's cumulative hazard rate function is expressed simply as :

$$H(x) = -\ln[S(x)] = \ln\left\{\frac{e^{-\delta x^\beta}\left(\delta^2\left(\frac{\alpha}{\delta^2} + \frac{1}{2}\alpha x^{2\beta} + \frac{\alpha x^\beta}{\delta} + 1\right)\right)}{\alpha + \delta^2}\right\}. \tag{2.7}$$

Figure 2 shows the possible plots of the hrf of the PMD. The hrf of the PMD can be increasing, reverse-J, bathtub, upside-down bathtub, and bathtub followed by upside-down bathtub shapes.

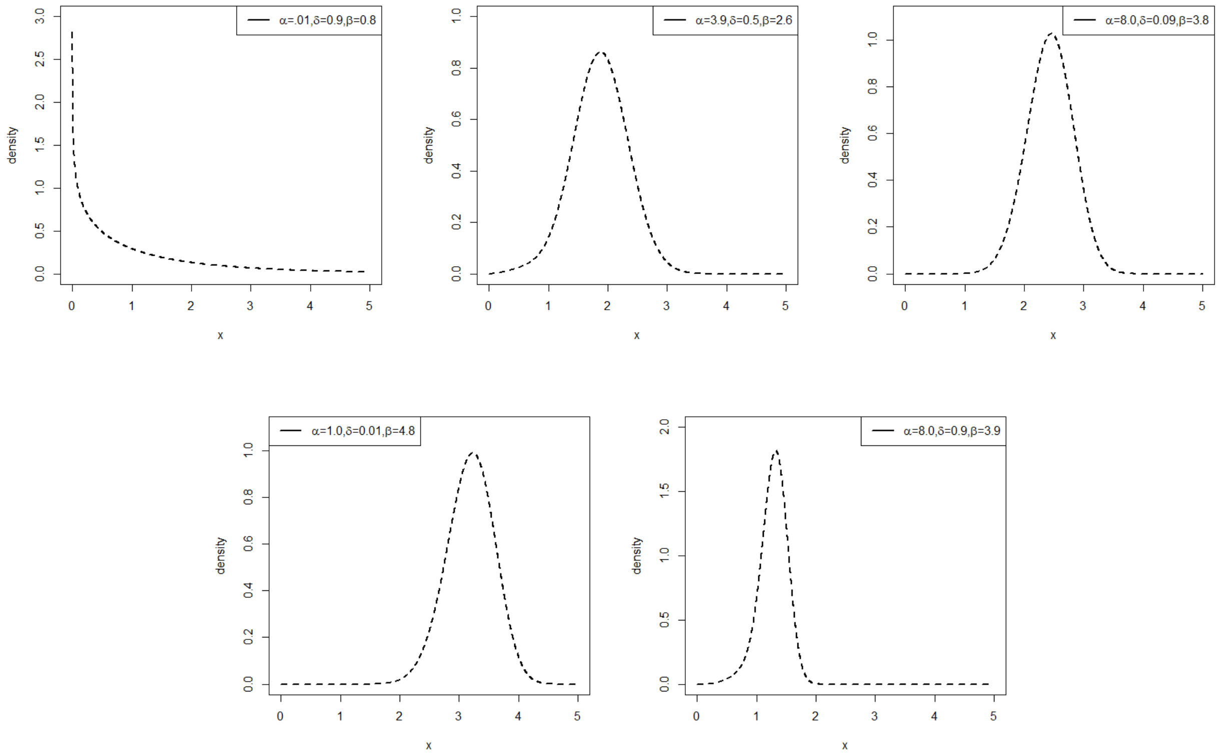

Figure 1. Density plots for PMD.

3. Statistical properties

In this section, we derive several significant statistical features of PMD. Quantile function, moments, and generating function, moments of residual life, R'enyi entropy, Lorenz, and Bonferroni are some characteristics. Curves and order statistics distribution.

3.1. *Quantile function of PMD*

It is important to point out that the quantile function has an important role in generating random data sets. The quantile function of the PMD is obtained by inverting Equation (2.3) as follows

$$F(x) = 1 - \frac{e^{-\delta x^{\beta}}\left(\delta^2\left(\frac{\alpha}{\delta^2} + \frac{1}{2}\alpha x^{2\beta} + \frac{\alpha x^{\beta}}{\delta} + 1\right)\right)}{\alpha + \delta^2} = u, \tag{3.1}$$

for $0 \leq u \leq 1$, that is, we solve the non-linear equation,

$$-\delta x^{\beta} + \ln\left(\left(\frac{\alpha}{\delta^2} + \frac{1}{2}\alpha x^{2\beta} + \frac{\alpha x^{\beta}}{\delta} + 1\right)\right) - \ln\left(\frac{(1-u)(\alpha+\delta^2)}{\delta^2}\right) = 0. \tag{3.2}$$

Thus, random numbers from the PMD can be obtained using Equation (3.2).

3.2. *Moments and related measures*

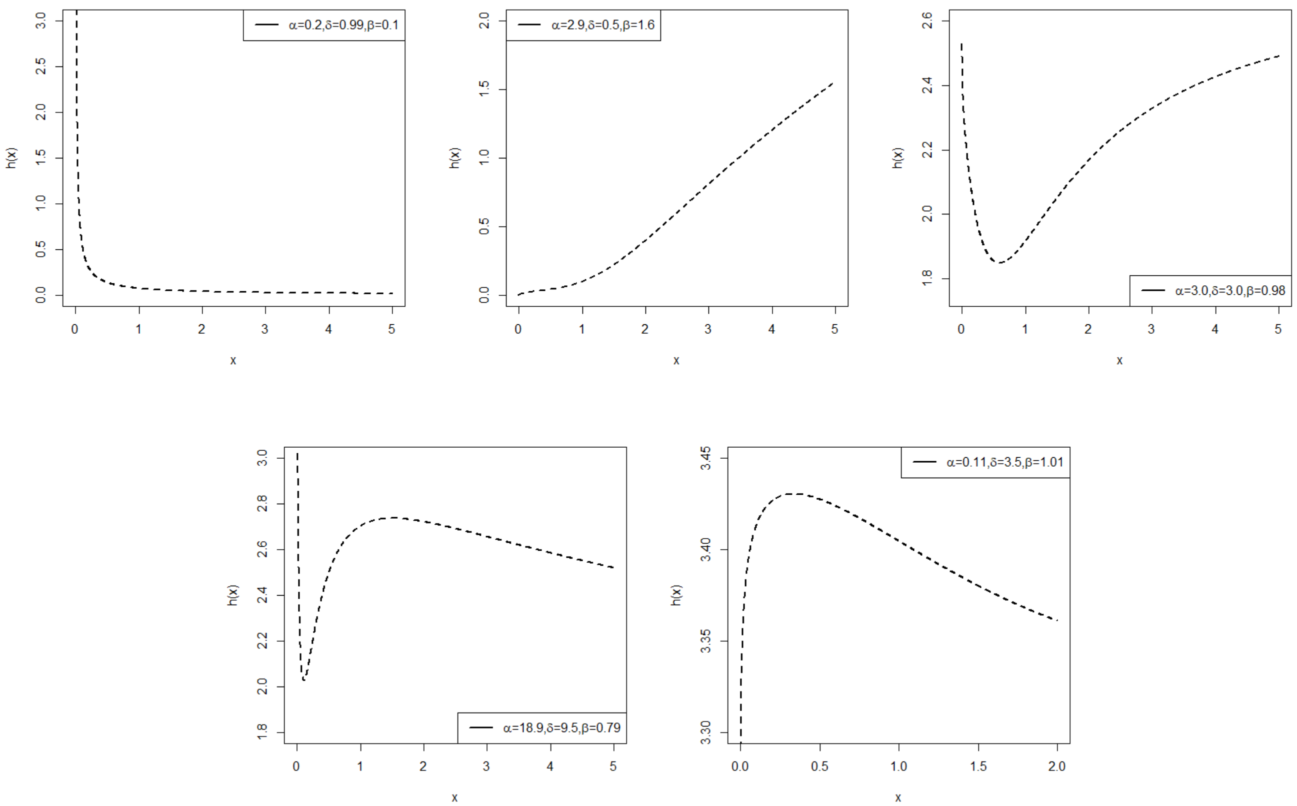

Figure 2. Hazard rate plots for the PMD.

Moments are essential statistical metrics that characterize the form and distribution of a probability function. The first moment denotes the central trend, while the second moment, variance, measures dispersion. Higher-order moments, including skewness and kurtosis, elucidate asymmetry and tail behavior. These moments are essential in statistical modeling, aiding in the characterization of data distributions, evaluation of model fit, and facilitation of decision-making across many applicable domains. The r^{th} moment of the PMD is defined as follows

$$\begin{aligned}\mu_r = E(X^r) &= \int_0^\infty x^r f(x)dx = \frac{\beta\delta^3}{2(\alpha+\delta^2)}\left[\alpha\int_0^\infty x^{r+3\beta-1}e^{-\delta x^\beta}dx + 2\int_0^\infty x^{r+\beta-1}e^{-\delta x^\beta}dx\right] \\ &= \frac{\delta^{-\frac{r}{\beta}}\left(\alpha\Gamma\left(\frac{r}{\beta}+3\right)+2\delta^2\Gamma\left(\frac{r+\beta}{\beta}\right)\right)}{2(\alpha+\delta^2)}.\end{aligned} \tag{3.3}$$

Consequently, from (3.3) and by setting $r = 1$ and 2, we can be provide the 1^{st} (μ_1) which define the Mean for the PMD.

Furthermore, the variance (Var) and coefficient of variation (CV) of the PMD can be expressed as

$$\text{Var} = \mu_2 - \mu_1^2,$$

and

$$\text{CV} = \frac{\sqrt{\mu_2 - \mu_1^2}}{\mu_1}.$$

The skewness (γ_3) and kurtosis (γ_4) measures of the PMD are obtained to be

$$\gamma_3 = \frac{\mu_3 - 3\mu_2 + 2\mu_1^3}{(\mu_2 - \mu_1^2)^{3/2}},$$

and

$$\gamma_4 = \frac{\mu_4 - 4\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1^4}{(\mu_2 - \mu_1^2)^2}.$$

Table 1. Various statistical measures for the PMD at $\beta = 1.5$.

	α	Mean	Var	CV	γ_3	γ_4
δ=1.25	0.25	0.9095	0.4124	0.7061	1.1122	1.3363
	0.5	1.0088	0.489	0.6932	0.982	0.837
	0.75	1.0866	0.5352	0.6733	0.8673	0.5105
δ=1.5	0.25	0.7734	0.2977	0.7054	1.1428	1.5003
	0.5	0.8423	0.3505	0.7028	1.0625	1.1231
	0.75	0.8997	0.3872	0.6916	0.9715	0.8035
δ=1.75	0.25	0.6793	0.2279	0.7029	1.1519	1.5759
	0.5	0.7286	0.2645	0.706	1.1098	1.3255
	0.75	0.7714	0.2924	0.701	1.0435	1.0504
δ=2	0.25	0.6098	0.1822	0.7	1.1531	1.6018
	0.5	0.6462	0.2081	0.7059	1.1355	1.4572
	0.75	0.6787	0.229	0.705	1.0909	1.2405
δ=2.25	0.25	0.5562	0.1505	0.6974	1.1559	1.6116
	0.5	0.5837	0.1691	0.7046	1.1478	1.5366
	0.75	0.6089	0.1849	0.7062	1.1206	1.3782

Tables (1) and (2) represent different measures of PMD by employing numerous choices of α, δ and β. These results show that as α tends to increase and for fixed values of δ and β, the Mean and Var of PMD are increasing, while γ_3 and γ_4 are decreasing. Further, for fixed values of α and β, both values of γ_3 and γ_4 are increasing, and the value of Mean and Var decrease as δ increases. Consequently, the PMD is a flexible distribution for explaining different data sets.

The r^{th} incomplete moments of the PMD are defined as follows

$$\varphi_r(t) = \int_0^t x^r f(x)dx = \frac{\beta\delta^3}{2(\alpha+\delta^2)}\left[\alpha\int_0^t x^{r+3\beta-1}e^{-\delta x^\beta}dx + 2\int_0^t x^{r+\beta-1}e^{-\delta x^\beta}dx\right]$$

Table 2. Various statistical measures for the PMD at $\beta = 2.5$.

	α	Mean	Var	CV	γ_3	γ_4
δ=1.25	0.25	0.8875	0.1555	0.4443	0.4093	-0.1668
	0.5	0.9451	0.1737	0.4409	0.3281	-0.3481
	0.75	0.9903	0.1832	0.4322	0.242	-0.4576
δ=1.5	0.25	0.8056	0.1273	0.4429	0.4223	-0.1086
	0.5	0.8476	0.1417	0.4441	0.3812	-0.2443
	0.75	0.8825	0.151	0.4403	0.3207	-0.3599
δ=1.75	0.25	0.7456	0.1081	0.4409	0.4227	-0.0826
	0.5	0.7769	0.1192	0.4444	0.4081	-0.1707
	0.75	0.8041	0.1272	0.4436	0.3695	-0.2707
δ=2	0.25	0.6992	0.0943	0.4392	0.4182	-0.0739
	0.5	0.7231	0.1028	0.4435	0.4199	-0.1237
	0.75	0.7445	0.1095	0.4445	0.398	-0.2012
δ=2.25	0.25	0.6619	0.0839	0.4376	0.4122	-0.0739
	0.5	0.6806	0.0906	0.4422	0.4064	-0.0961
	0.75	0.6977	0.0960	0.4441	0.4036	-0.1514

$$= \frac{\delta^{-\frac{r}{\beta}}\left(\alpha\delta^{\frac{r}{\beta}+3}\left(-t^{3\beta+r}\right)E_{-\frac{r}{\beta}-2}\left(t^{\beta}\delta\right) - 2\delta^{\frac{r}{\beta}+3}t^{\beta+r}E_{-\frac{r}{\beta}}\left(t^{\beta}\delta\right) + \alpha\Gamma\left(\frac{r}{\beta}+3\right) + 2\delta^{2}\Gamma\left(\frac{r+\beta}{\beta}\right)\right)}{2\left(\alpha+\delta^{2}\right)}, \tag{3.4}$$

where $E_n(z) = \int_1^{\infty} \frac{e^{-zt}}{t^n} dt$.
One of the most important uses of the incomplete moments is to determine two critical inequality curves called the Bonferroni and Lorenz curves. They are determined, respectively, by the PMD as follows

$$B(p) = \frac{1}{p\mu}\int_0^{x_q} xf(x)dx = \frac{1}{p\mu}\varphi_1(x_p), \; F(x_p) = p,$$
$$L(p) = \frac{1}{\mu}\int_0^{x_p} xf(x)dx = \frac{1}{\mu}\varphi_1(x_p).$$

3.3. Moment generating function

By applying the series expansion

$$e^{tx} = \sum_{i=0}^{\infty} \frac{t^i x^i}{i!},$$

and following the same steps leading to equation (3.3), we obtain MGF as follows

$$M(t) = E(e^{tx}) = \sum_{i=0}^{\infty} \frac{t^i}{i!} E(X^i) = \sum_{i=0}^{\infty} \frac{t^i}{i!} \frac{\delta^{-\frac{i}{\beta}} \left(\alpha\Gamma\left(\frac{i}{\beta}+3\right) + 2\delta^2\Gamma\left(\frac{i+\beta}{\beta}\right)\right)}{2\left(\alpha+\delta^2\right)}. \tag{3.5}$$

3.4. Moment of residual and reversed residual life

The s^{th} moment of the residual life, say $\omega_s(t)$ of X follows PMD is expressed as

$$\omega_s(t) = E\left[(X-t)^s \mid X > t\right] = \frac{1}{1-F(t)} \int_t^{\infty} (x-t)^s f(x)dx = \frac{1}{1-F(t)} \sum_{r=0}^{s} \binom{s}{r} (-t)^{s-r} \int_t^{\infty} x^r f(x)dx$$

$$= \frac{1}{1-F(t)} \sum_{r=0}^{s} \binom{s}{r} (-t)^{s-r} \frac{t^r \left(\delta t^{\beta}\right)^{-\frac{r}{\beta}} \left(\alpha\Gamma\left(\frac{r}{\beta}+3, t^{\beta}\delta\right) + 2\delta^2\Gamma\left(\frac{r+\beta}{\beta}, t^{\beta}\delta\right)\right)}{2\left(\alpha+\delta^2\right)},$$

where $\Gamma(a,z) = \int_z^{\infty} t^{a-1}e^{-t}dt$.

As a result, the s^{th} moment of the reversed residual life ($\vartheta_s(t)$) of X is

$$\vartheta_s(t) = E\left[(t-X)^s \mid X \le t\right] = \frac{1}{F(t)} \int_0^{t} (t-x)^s f(x)dx = \frac{1}{F(t)} \sum_{r=0}^{s} \binom{s}{r} (t)^{s-r}(-1)^r \int_0^{t} x^r f(x)dx$$

$$= \frac{1}{F(t)} \sum_{r=0}^{s} \binom{s}{r} (t)^{s-r}(-1)^r \varphi_r(t). \tag{3.6}$$

3.5. Rényi entropy

The corresponding Rényi entropy of X can be defined as

$$I_R(w) = \frac{1}{1-w} \log\left(\int_0^{\infty} [f(x;\alpha,\beta,\delta)]^w dx\right), w \neq 1, w > 0. \tag{3.7}$$

Then, for PMD, we have

$$\int_0^{\infty} [f(x;\alpha,\beta,\delta)]^w = \frac{\beta^w\delta^{3w}\alpha^w}{2^w\left(\alpha+\delta^2\right)^w} \int_0^{\infty} x^{w(\beta-1)+2\beta w} \left(1+\frac{2}{\alpha}x^{-2\beta}\right)^w e^{-w\delta x^{\beta}} dx$$

$$= \frac{\beta^w\delta^{3w}\alpha^w}{2^w\left(\alpha+\delta^2\right)^w} \sum_{i=0}^{\infty} (-1)^i \binom{w+i-1}{i} \left(\frac{2}{\alpha}\right)^i \int_0^{\infty} x^{w(\beta-1)+2\beta w-2i\beta} e^{-w\delta x^{\beta}} dx$$

$$= \sum_{i=0}^{\infty} (-1)^i \binom{w+i-1}{i} \frac{\beta^{w-1}\alpha^{w-i}}{2^{w-i}\left(\alpha+\delta^2\right)^w} \delta^{\frac{2\beta i+w-1}{\beta}} w^{\frac{2\beta i-3\beta w+w-1}{\beta}} \Gamma\left(\frac{3\beta w-w-2i\beta+1}{\beta}\right).$$

Thus, the Rényi entropy of the PMD distribution is expressed as

$$I_R(w) = \frac{1}{1-w} \log\left[\sum_{i=0}^{\infty} (-1)^i \binom{w+i-1}{i} \frac{\beta^{w-1}\alpha^{w-i}}{2^{w-i}\left(\alpha+\delta^2\right)^w} \delta^{\frac{2\beta i+w-1}{\beta}} w^{\frac{2\beta i-3\beta w+w-1}{\beta}} \Gamma\left(\frac{3\beta w-w-2i\beta+1}{\beta}\right)\right].$$

3.6. Order atatistics

Order statistics are essential in statistical analysis as they provide insights into the distribution of ranked data points. The values acquired from a sample ordered in ascending or descending order are shown, with the minimum and maximum serving as the first and last order statistics, respectively. These statistics are crucial for reliability analysis, risk assessment, and extreme value theory since they facilitate the modeling of minimum or maximum observations within a dataset. Their applications include domains like as engineering, finance, and actuarial science, where comprehension of extremes and percentiles is crucial for decision-making and risk management.

The PDF of the k^{th} order statistic from the PMD distribution is given by

$$
\begin{aligned}
f_{k:n}(x) &= \frac{n!f(x)}{(k-1)!(n-k)!}[F(x)]^{k-1}[1-F(x)]^{n-k} \\
&= \frac{n!f(x)}{(k-1)!(n-k)!}\sum_{p=0}^{n-k}(-1)^p\binom{n-k}{p}[F(x)]^{p+k-1} \\
&= \frac{n!}{(k-1)!(n-k)!}\sum_{p=0}^{n-k}(-1)^p\binom{n-k}{p}\left[1-\frac{e^{-\delta x^\beta}\left(\delta^2\left(\frac{\alpha}{\delta^2}+\frac{1}{2}\alpha x^{2\beta}+\frac{\alpha x^\beta}{\delta}+1\right)\right)}{\alpha+\delta^2}\right]^{p+k-1} \\
&\times \frac{\beta\delta^3 x^{\beta-1}\left(\alpha x^{2\beta}+2\right)e^{-\delta x^\beta}}{2\left(\alpha+\delta^2\right)} \\
&= \frac{n!}{(k-1)!(n-k)!}\sum_{p=0}^{n-k}\sum_{l=0}^{\infty}(-1)^{p+l}\binom{n-k}{p}\binom{p+k-1}{l} \\
&\times \frac{\beta\delta^{3+2l} x^{\beta-1}\left(\alpha x^{2\beta}+2\right)e^{-(l+1)\delta x^\beta}}{2\left(\alpha+\delta^2\right)^{l+1}}\left(\frac{\alpha}{\delta^2}+\frac{1}{2}\alpha x^{2\beta}+\frac{\alpha x^\beta}{\delta}+1\right)^l .
\end{aligned}
$$

The CDF of the k^{th} order statistic from the PMD distribution is determined as follows

$$
\begin{aligned}
F_{k:n}(x) &= \sum_{r=k}^{n}\tbinom{n}{r}(F(x))^r(1-F(x))^{n-r} \\
&= \frac{2^{k-n}\Gamma(n+1)\left(\frac{e^{\delta\left(-x^\beta\right)}\left(-2\delta^2-\alpha\left(\delta x^\beta\left(\delta x^\beta+2\right)+2\right)\right)}{2\left(\alpha+\delta^2\right)}+1\right)^k\left(\frac{e^{\delta\left(-x^\beta\right)}\left(2\delta^2+\alpha\left(\delta x^\beta\left(\delta x^\beta+2\right)+2\right)\right)}{\alpha+\delta^2}\right)^{n-k} H}{\Gamma(-k+n+1)},
\end{aligned}
$$

where $H = \tilde{F}_1\left(1, k-n; k+1; 1-\frac{2e^{x^\beta\delta}\left(\delta^2+\alpha\right)}{2\delta^2+\alpha\left(\delta\left(\delta x^\beta+2\right)x^\beta+2\right)}\right)$ is regularized hypergeometric function.

4. Estimation methods of PMD

Seven methods for calculating the PMD's parameters α, δ, and β are covered in this section. Determining the distribution parameters using conventional estimating techniques has been made available to various writers. Likely, ([7],[30], [8], [12], [33],[6], [18], [5], [20]).

4.1. Maximum likelihood estimation

Let $\{x_1, \ldots, x_n\}$ be a random sample of size n from PMD(α, δ, β). Then, the corresponding log-likelihood function can be given as

$$l(\alpha,\delta,\beta) = n\ln\beta + 3n\ln\delta + (\beta-1)\sum_{i=1}^{n}\ln(x_i) - \delta\sum_{i=1}^{n}x_i^{\beta} + \sum_{i=1}^{n}\ln(\alpha x_i^{2\beta}+2) - 2n\ln(\alpha+\delta^2). \tag{4.1}$$

Let $\hat{\alpha}_{MLE}$, $\hat{\delta}_{MLE}$, and $\hat{\beta}_{MLE}$ denote the maximum likelihood estimators of α, δ, and β, respectively. They are derived by solving the three non-linear equations accordingly.

$$\frac{\partial l(\alpha,\delta,\beta)}{\partial\alpha} = \sum_{i=1}^{n}\frac{x_i^{2\beta}}{\alpha x_i^{2\beta}+2} - \frac{2n}{\alpha+\delta^2} = 0,$$

$$\frac{\partial l(\alpha,\delta,\beta)}{\partial\delta} = \frac{3n}{\delta} - \sum_{i=1}^{n}x_i^{\beta} - \frac{4n\delta}{\alpha+\delta^2} = 0,$$

and

$$\frac{\partial l(\alpha,\delta,\beta)}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\ln(x_i) - \delta\sum_{i=1}^{n}x_i^{\beta}ln(x_i) + 2\alpha\sum_{i=1}^{n}\frac{x_i^{2\beta}\ln x_i}{\alpha x_i^{2\beta}+2} = 0.$$

4.2. Least square and weighted least square estimators

Let $x_1, ..., x_n$ be a random sample of size n from PMD and $x_{(1)} < ... < x_{(n)}$ represent the order statistics of the random sample from the PMD. The least-square estimator(LSE) (see [36]) of α, δ and β (say, $\hat{\alpha}_{LSE}$, $\hat{\delta}_{LSE}$ and $\hat{\beta}_{LSE}$) can be resulted by minimizing

$$\sum_{i=1}^{n}\left[F(x_{(i)}|\alpha,\delta,\beta) - \frac{i}{n+1}\right]^2,$$

where $F(x|\alpha,\delta,\beta)$ is (2.3). Now, another technique of estimation, say the weighted least square estimators (WLSEs) of α, δ, and β, note $\hat{\alpha}_{WLSE}$, $\hat{\delta}_{WLSE}$ and $\hat{\beta}_{WLSE}$ is expressed by minimize the equation

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{(i)}|\alpha,\delta,\beta) - \frac{i}{n+1}\right]^2.$$

4.3. Maximum product of spacings

Let

$$D_i(\alpha,\delta,\beta) = F(x_{(i)}|\alpha,\delta,\beta) - F(x_{(i-1)}|\lambda,\theta);\quad i = 1, ..., n+1,$$

with

$$F(x_{(0)}|\alpha,\delta,\beta) = 0, \quad \text{and} \quad F(x_{(n+1)}|\alpha,\delta,\beta) = 1.$$

It is clear that $\sum_{i=1}^{n+1} D_i(\alpha,\delta,\beta) = 1$.

The maximum product spacing (MPS) estimators of α, δ, and β ($\hat{\alpha}_{MPS}$, $\hat{\delta}_{MPS}$ and $\hat{\beta}_{MPS}$), can be obtained by maximizing

$$J(\alpha,\delta,\beta) = \left[\prod_{i=1}^{n+1} D_i(\alpha,\delta,\beta)\right]^{1/(n+1)}. \tag{4.2}$$

4.4. Cramer-von Mises minimum distance estimators

The Cramer-von Mises-type minimum distance estimators (CMEs) $\hat{\alpha}_{CME}$, $\hat{\delta}_{CME}$ and $\hat{\beta}_{CME}$ are obtained by minimizing

$$V(\alpha,\delta,\beta) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{(i)}|\alpha,\delta,\beta) - \frac{2i-1}{2n}\right]^2. \tag{4.3}$$

4.5. Anderson-Darling and right-tail Anderson-Darling

Anderson-Darling estimators (ADEs) $\hat{\alpha}_{ADE}$, $\hat{\delta}_{ADE}$ and $\hat{\beta}_{ADE}$ of α, δ and β are calculated by minimizing

$$D(\alpha,\delta,\beta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\ln F(x_{(i)}|\alpha,\delta,\beta) + \ln S(x_{(n+1)}|\alpha,\delta,\beta)\right\}.$$

In the same sense, the Right-tail Anderson-Darling estimators (RADEs), $\hat{\alpha}_{RADE}$, $\hat{\delta}_{RADE}$ and $\hat{\beta}_{RADE}$ are determined by minimizing

$$R(\alpha,\delta,\beta) = \frac{n}{2} - 2\sum_{i=1}^{n}\ln F(x_{(i)}|\alpha,\delta,\beta) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\ln S(x_{(n+1)}|\alpha,\delta,\beta).$$

5. Numerical simulation

Here, we provide some results from a Monte Carlo (MC) simulation study to show how the proposed estimation procedures in Section 4 work. Under a given parameter set and over $N = 1000$ times, we obtain a random observation taken from the PMD with size n by applying the solution of the non-linear equation (3.2). After that, the average biases (ABs), the associated mean squared errors (MSEs), and the mean relative errors (MREs) are computed. The results are reported in Tables (3)-(5) represent the result. Table (3)-(5) shows that the ABs, MSEs, and MREs decrease as n increases based on all estimation methods, which ensures that the proposed estimators are consistent and asymptotically unbiased. In addition, the MPSEs are a better method of estimating the PMD by taking the MSE as an optimality criterion. Also, it is shown that the MSEs rise for all estimate techniques as α, δ, and β rise.

6. Risk measures

Risk exposure is described through probability distributions. Actuaries and risk managers often use such important risk indicators to assess the degree to which their organizations are vulnerable to certain risks that result from changes in underlying factors like stock prices, interest rates, or exchange rates. In the literature, numerous risk measures and their properties have been provided. For example on may refer to [9], [23], [19], [39] and [28], and references therein. This section covers value at risk, tail values at risk, and tail variance premium for the PMD.

6.1. Value at risk

The value at risk (R_1) is calculated for the PMD by finding the inverse of Equation (2.4) concerning x as follows

$$e^{-\delta x^{\beta}}\left(\frac{\alpha}{\delta^2}+\frac{1}{2}\alpha x^{2\beta}+\frac{\alpha x^{\beta}}{\delta}+1\right)=\frac{(1-p)(\alpha+\delta^2)}{\delta^2}.$$

6.2. Tail values at risk

Another significant risk indicator is the tail value at risk (R_2). It calculates the anticipated loss given an occurrence outside a certain probability threshold. It is determined for the PMD by using the following relation

$$R_2=\frac{1}{(1-p)}\int_{R_1}^{\infty}x\,f(x)\,dx=\frac{\delta^{-1/\beta}\left(\alpha\Gamma\left(3+\frac{1}{\beta},\delta R_1^{\beta}\right)+2\delta^2\Gamma\left(1+\frac{1}{\beta},\delta R_1^{\beta}\right)\right)}{2\left(\alpha+\delta^2\right)(1-p)},$$

where $\Gamma(a,z)=\int_z^{\infty}t^{a-1}e^{-t}dt$.

6.3. Tail variance premium

Another noteworthy statistic that plays a crucial role in the area of insurance sciences is the tail variance premium (R_3), and the following relation determines it

$$R_3=R_2+pR_4,$$

where

$$R_4=\frac{1}{(1-p)}\int_{R_1}^{\infty}x^2 f(x)dx-(R_2)^2.$$

Then, it is determined for the PMD as follows

$$\begin{aligned}R_3=&\frac{\delta^{-2/\beta}}{4(p-1)^2\left(\alpha+\delta^2\right)^2}\left[-p\left(\alpha\Gamma\left(3+\frac{1}{\beta},\delta R_1^{\beta}\right)+2\delta^2\Gamma\left(1+\frac{1}{\beta},\delta R_1^{\beta}\right)\right)^2\right.\\&-2(p-1)p\left(\alpha+\delta^2\right)\left(\alpha\Gamma\left(3+\frac{2}{\beta},\delta R_1^{\beta}\right)+2\delta^2\Gamma\left(\frac{\beta+2}{\beta},\delta R_1^{\beta}\right)\right)\\&\left.-2(p-1)\left(\alpha+\delta^2\right)\delta^{1/\beta}\left(\alpha\Gamma\left(3+\frac{1}{\beta},\delta R_1^{\beta}\right)+2\delta^2\Gamma\left(1+\frac{1}{\beta},\delta R_1^{\beta}\right)\right)\right].\end{aligned}$$

6.4. Numerical computations for risk measures

This part presents the numerical results for R_1, R_2, and R_3 of the PMD and TMD across different parametric values. The parameters were computed using the machine learning methodology. The three risk measures were calculated based on the results of 1000 repetitions.

Tables 6 and 7 for the two comparable models include the numerical findings of these measures. We presented the findings in Figures 3 and 4 for visual comparisons. These tables and figures indicate that the PMD is denser than the MD, making it very appropriate for accommodating heavy-tailed data sets.

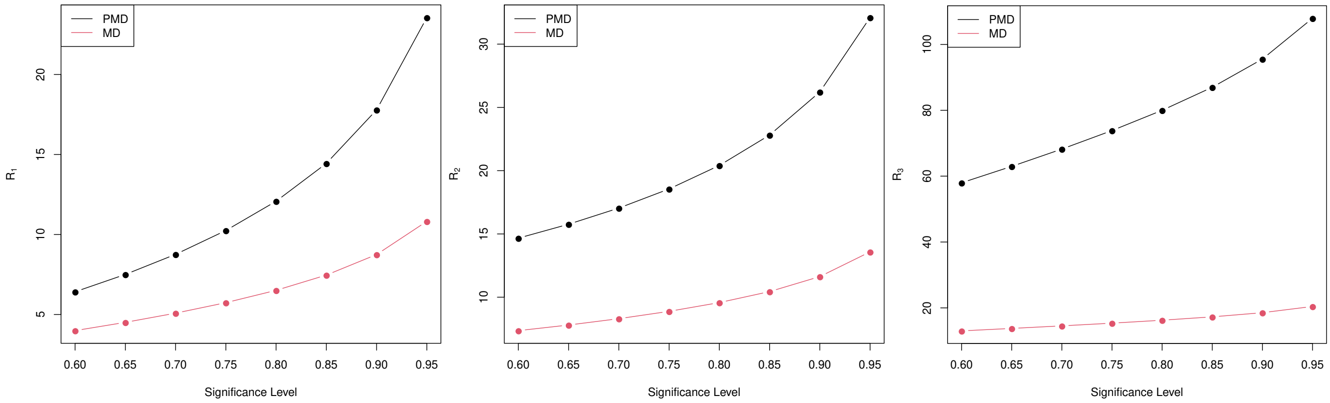

Figure 3. Graphical plots of computational value of R_1, R_2 and R_3 in Table 6.

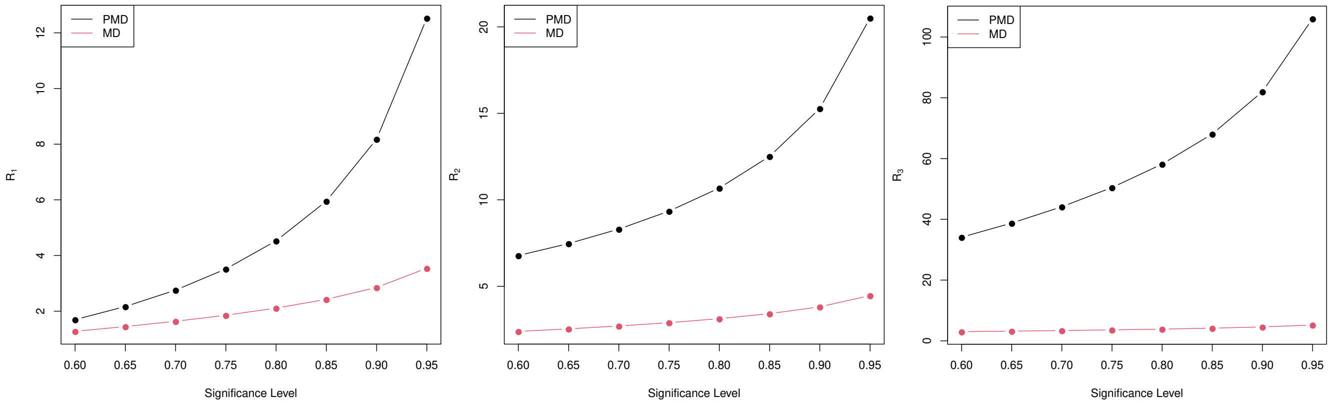

Figure 4. Graphical plots of computational value of R_1, R_2 and R_3 in Table 7.

7. Real data analysis

In this part of the work, we considered the third-party motor insurance collected in 1977 in Sweden. The data represents the total amount paid by the insurance company. It is available in [10]. Also, the dataset is studied by [29] and [32]. All the data points are divided by 10000 for computational purposes. The values of the data set are.

Table (9) provides some statistics of the motor insurance data. The PMD is compared with the various models such as TPM, univariate Poisson gamma (UP-GA), which is introduced by Abdelghani et al. [1], Normal, exponential geometric (EG), which Adamidis proposes [2], Gompertz, exponential (EXP) and Power Lindley (PL) which is proposed by [21] distributions. The PDFs of the competing models are given as follows

1. UP-GA:
$$f(x) = \frac{\theta \lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{(e^{\theta}-1)\Gamma(\alpha)} e^{\theta H_{\alpha,\lambda}(x)}; \quad x > 0, \quad \alpha, \lambda, \theta > 0.$$
where $H_{\alpha,\lambda}(x)$ is the CDF of gamma distribution.
2. Normal:
$$f(x) = \frac{1}{\lambda\sqrt{2\pi}} e^{-\dfrac{(x-\alpha)^2}{2\lambda^2}}, \quad x \in \mathbb{R}, \ \alpha \in \mathbb{R}, \lambda > 0.$$

3. EG:

$$f(x) = \frac{p\lambda e^{-\lambda x}}{(p + (1-p)e^{-\lambda x})^2}; \quad x > 0, \quad \lambda > 0, \quad 0 < p < 1.$$

4. Gompertz:

$$f(x) = \alpha\beta e^{-\beta(e^{\alpha x}-1)+\alpha x}, \quad x > 0, \ \alpha, \beta > 0.$$

5. EXP:

$$f(x) = \lambda e^{-\lambda x}; \quad x > 0, \quad \lambda > 0.$$

6. PL:

$$f(x) = \frac{\alpha\beta^2}{\beta+1}(1 + x^{\alpha})\, x^{\alpha-1}\, e^{-\beta x^{\alpha}}; \quad x > 0, \quad \alpha, \beta > 0.$$

Table (10) reports the results of the MLEs of unknown parameters with the log-likelihood (ll) value. To select the best model for modeling the insurance loss data set, the results of the Akaike Information Criterion (A_1), Akaike Information Criterion correction (A_2, Hannan Quinn Information Criterion (A_3), Bayesian Information Criterion (A_4), and Kolmogorov-Smirnov (K-S) statistics with associated p-values are reported in Table (11). Table (11) shows that the PMD model is the best and most suitable candidate for modeling the data set. The estimated PDF, CDF, survival function (SF), the scaled total time on a test (TTT), the probability-probability (PP), and box plots are sketched in Figures (6), (5), and (7), respectively.

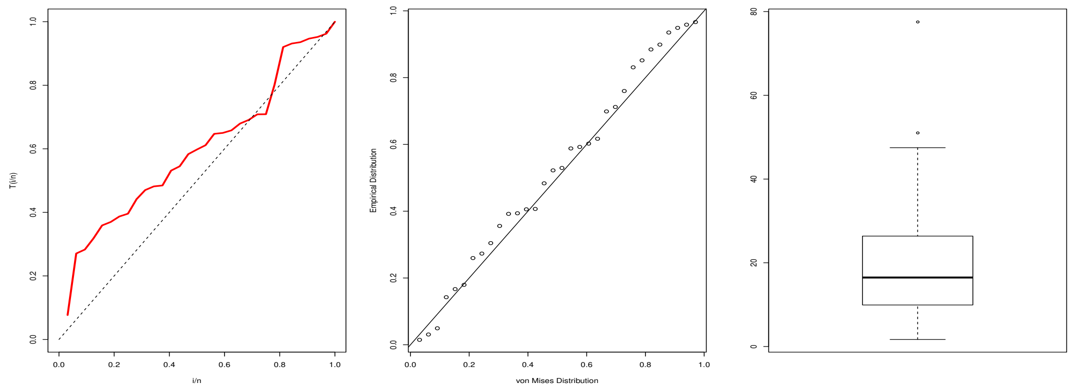

Figure 5. TTT, PP and box curves using motor insurance data set.

The results for estimates of the unknown parameters for the PMD distribution using various proposed estimation procedures are provided in Table (12).

8. Conclusion

This paper presents a unique probability distribution derived from the power transformation approach, offering a more flexible and resilient framework for modeling empirical data. We methodically generated and examined several statistical and mathematical aspects, such as moments, quantile functions, and order statistics, to demonstrate the theoretical underpinnings of the suggested model. Additionally, we examined several parameter estimation methods, evaluating their precision and efficacy using extensive simulation analyses. The findings indicate that the suggested estimating techniques provide dependable and consistent estimates, illustrating the practical feasibility of the model. To enhance its application, we calculated

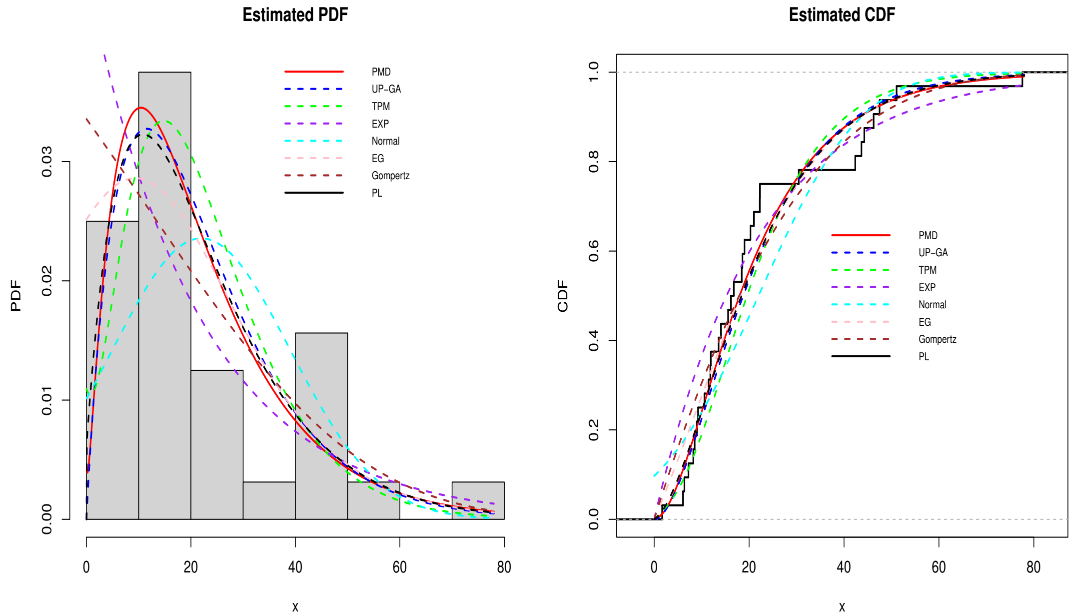

Figure 6. Estimated curves of PDF and CDF for proposed fitting models using motor insurance data set.

essential risk metrics, underscoring our distribution's significance in actuarial and financial domains. Furthermore, we performed an empirical study using a genuine insurance loss dataset, assessing our model's goodness of fit compared to competing distributions. The results indicate that the proposed model excels in capturing data features such as skewness and heavy tails, highlighting its use in risk assessment, reliability analysis, and statistical modeling. Our suggested distribution is a substantial advancement in probability and statistical modeling. Future studies may investigate its applicability across several fields, such as banking, engineering, and environmental studies, while expanding its theoretical qualities to multivariate contexts and Bayesian inference frameworks.

Authors' Contributions

All authors have worked equally to write and review the manuscript.

Data Availability Statement

The data that supports the findings of this study are available within the article.

Conflicts of Interest

The authors declare no conflict of interest.

Table 3. ABs, MSEs and MREs of PMD at (α, δ, β)=(0.75, 1.25, 1.5).

n	Method	$\hat{\alpha}$			$\hat{\delta}$			$\hat{\beta}$		
		AB	MSE	MRE	AB	MSE	MRE	AB	MSE	MRE
30	MLE	0.9857	1.1164	1.3143	0.0992	0.1934	0.0793	0.0718	0.1359	0.0478
	LSE	0.6305	1.3312	0.8406	0.1384	0.2402	0.1107	0.0717	0.1755	0.0478
	WLSE	0.4891	1.0722	0.6522	0.0710	0.1591	0.0568	0.0186	0.1319	0.0124
	CME	1.0389	1.6722	1.3852	0.1472	0.2496	0.1177	0.0250	0.1821	0.0166
	MPS	0.5262	1.0662	0.7024	0.1558	0.1321	0.1246	0.1207	0.1042	0.0805
	ADE	0.6405	1.1210	0.8540	0.1287	0.1710	0.1029	0.0201	0.1438	0.0134
	RADE	0.8705	1.1925	1.1607	0.0774	0.1738	0.0619	0.0304	0.1537	0.0202
50	MLE	0.4406	0.4341	0.5875	0.2708	0.0733	0.2166	0.1739	0.0602	0.1159
	LSE	0.2335	0.6235	0.3114	0.0373	0.0758	0.0299	0.0697	0.0870	0.0465
	WLSE	0.1323	0.4277	0.1764	0.0183	0.0645	0.0147	0.0323	0.0772	0.0215
	CME	0.3911	1.2170	0.5214	0.0916	0.1206	0.0733	0.0173	0.0913	0.0115
	MPS	0.2862	0.3186	0.3816	0.0860	0.0625	0.0688	0.0901	0.0597	0.0601
	ADE	0.2588	0.4425	0.3450	0.0603	0.0697	0.0482	0.0104	0.0796	0.0069
	RADE	0.4562	0.5744	0.6083	0.0705	0.0873	0.0564	0.0085	0.0866	0.0056
100	MLE	0.3711	0.0602	0.4948	0.0786	0.0316	0.0629	0.0587	0.0359	0.0391
	LSE	0.0478	0.1426	0.0637	0.0014	0.0363	0.0011	0.0155	0.0520	0.0103
	WLSE	0.0415	0.0526	0.0553	0.0073	0.0259	0.0058	0.0017	0.0352	0.0011
	CME	0.0959	0.1433	0.1279	0.0108	0.0392	0.0086	0.0211	0.0697	0.0141
	MPS	0.0980	0.0447	0.1307	0.0551	0.0203	0.0441	0.0584	0.0257	0.0389
	ADE	0.0491	0.0573	0.0655	0.0252	0.0265	0.0201	0.0034	0.0368	0.0022
	RADE	0.0835	0.1412	0.1114	0.0124	0.0361	0.0099	0.0102	0.0345	0.0068
300	MLE	0.0329	0.0102	0.0438	0.0188	0.0059	0.0150	0.0441	0.0121	0.0294
	LSE	0.0445	0.0423	0.0593	0.0105	0.0096	0.0084	0.0056	0.0146	0.0037
	WLSE	0.0149	0.0087	0.0199	0.0125	0.0056	0.0100	0.0102	0.0079	0.0068
	CME	0.0416	0.0475	0.0555	0.0126	0.0119	0.0101	0.0033	0.0149	0.0022
	MPS	0.0385	0.0075	0.0513	0.0271	0.0052	0.0217	0.0319	0.0067	0.0213
	ADE	0.0276	0.0108	0.0368	0.0022	0.0074	0.0018	0.0086	0.0083	0.0057
	RADE	0.0370	0.0444	0.0493	0.0006	0.0108	0.0005	0.0157	0.0147	0.0105
500	MLE	0.0263	0.0039	0.0350	0.0089	0.0040	0.0071	0.0127	0.0057	0.0085
	LSE	0.0177	0.0198	0.0236	0.0023	0.0067	0.0019	0.0003	0.0104	0.0002
	WLSE	0.0107	0.0033	0.0142	0.0034	0.0033	0.0027	0.0019	0.0050	0.0012
	CME	0.0172	0.0287	0.0229	0.0017	0.0074	0.0014	0.0021	0.0123	0.0014
	MPS	0.0204	0.0023	0.0273	0.0200	0.0030	0.0160	0.0197	0.0046	0.0131
	ADE	0.0057	0.0045	0.0076	0.0019	0.0038	0.0015	0.0029	0.0059	0.0019
	RADE	0.0184	0.0231	0.0246	0.0029	0.0071	0.0023	0.0103	0.0112	0.0068

Table 4. ABs, MSEs and MREs of PMD at (α, δ, β)=(1, 1.5, 1.75).

n	Method	$\hat{\alpha}$			$\hat{\delta}$			$\hat{\beta}$		
		AB	MSE	MRE	AB	MSE	MRE	AB	MSE	MRE
30	MLE	1.0126	1.9753	1.0126	0.0990	0.3235	0.0660	0.0820	0.1459	0.0468
	LSE	0.8314	2.9802	0.8314	0.1666	0.4988	0.1110	0.0614	0.3301	0.0350
	WLSE	0.5498	1.8719	0.5498	0.1617	0.3207	0.1078	0.0650	0.2582	0.0371
	CME	1.1209	5.3741	1.1209	0.1554	0.5118	0.1036	0.0263	0.3772	0.0150
	MPS	0.6751	1.7101	0.6751	0.2261	0.2747	0.1507	0.1788	0.1694	0.1021
	ADE	0.6382	2.7230	0.6382	0.0972	0.3314	0.0648	0.0265	0.2622	0.0151
	RADE	0.8953	5.0925	0.8953	0.1588	0.4408	0.1058	0.0152	0.2636	0.0087
50	MLE	0.6502	1.9341	0.6502	0.0543	0.1668	0.0362	0.0345	0.1473	0.0197
	LSE	0.2580	2.8194	0.2580	0.0403	0.1788	0.0269	0.0993	0.1684	0.0567
	WLSE	0.4131	1.7525	0.4131	0.0653	0.1321	0.0435	0.0050	0.1370	0.0028
	CME	0.9403	5.0313	0.9403	0.0576	0.1896	0.0384	0.0327	0.1897	0.0186
	MPS	0.5643	1.4844	0.5643	0.1475	0.1299	0.0983	0.1262	0.1128	0.0721
	ADE	0.4998	1.9652	0.4998	0.1029	0.1443	0.0686	0.0400	0.1374	0.0229
	RADE	0.7473	4.7662	0.7473	0.0960	0.1613	0.0640	0.0316	0.1514	0.0180
100	MLE	0.3711	0.0602	0.4948	0.0621	0.0764	0.0414	0.0608	0.0779	0.0347
	LSE	0.1236	0.7174	0.1236	0.0157	0.0772	0.0104	0.0471	0.0931	0.0269
	WLSE	0.1203	0.5626	0.1203	0.0174	0.0607	0.0116	0.0343	0.0690	0.0196
	CME	0.3738	1.2439	0.3738	0.0653	0.1089	0.0435	0.0004	0.1227	0.0002
	MPS	0.2920	0.3388	0.2920	0.0907	0.0504	0.0605	0.1034	0.0525	0.0590
	ADE	0.1071	0.6124	0.1071	0.0129	0.0616	0.0086	0.0809	0.0868	0.0462
	RADE	0.2447	0.8525	0.2447	0.0186	0.0946	0.0124	0.0077	0.0920	0.0044
300	MLE	0.0547	0.0411	0.0547	0.0267	0.0125	0.0178	0.0158	0.0156	0.0090
	LSE	0.0221	0.1098	0.0221	0.0053	0.0229	0.0035	0.0255	0.0307	0.0145
	WLSE	0.0439	0.0384	0.0439	0.0161	0.0117	0.0107	0.0138	0.0137	0.0079
	CME	0.0206	0.1305	0.0206	0.0103	0.0259	0.0068	0.0296	0.0351	0.0169
	MPS	0.1252	0.0280	0.1252	0.0394	0.0104	0.0262	0.0596	0.0120	0.0340
	ADE	0.0065	0.0487	0.0065	0.0072	0.0150	0.0048	0.0296	0.0246	0.0169
	RADE	0.1453	0.1153	0.1453	0.0421	0.0325	0.0280	0.0206	0.0349	0.0117
500	MLE	0.0113	0.0315	0.0113	0.0123	0.0107	0.0082	0.0269	0.0117	0.0154
	LSE	0.0152	0.0601	0.0152	0.0040	0.0134	0.0027	0.0118	0.0168	0.0067
	WLSE	0.0088	0.0232	0.0088	0.0021	0.0075	0.0014	0.0104	0.0115	0.0059
	CME	0.0090	0.0663	0.0090	0.0092	0.0139	0.0061	0.0224	0.0206	0.0128
	MPS	0.0615	0.0210	0.0615	0.0240	0.0069	0.0160	0.0255	0.0111	0.0146
	ADE	0.0192	0.0241	0.0192	0.0078	0.0089	0.0052	0.0217	0.0124	0.0124
	RADE	0.0615	0.0652	0.0615	0.0137	0.0137	0.0091	0.0057	0.0172	0.0032

Table 5. ABs, MSEs and MREs of PMD at (α,δ,β)=(1.8, 2.5, 2.75).

n	Method	$\hat{\alpha}$			$\hat{\delta}$			$\hat{\beta}$		
		AB	MSE	MRE	AB	MSE	MRE	AB	MSE	MRE
30	MLE	0.0260	0.0013	0.0144	0.1151	0.2986	0.0460	0.4978	0.4423	0.1810
	LSE	0.0159	0.0035	0.0088	0.2689	0.7407	0.1075	0.1986	0.7833	0.0722
	WLSE	0.0273	0.0050	0.0151	0.4215	0.6421	0.1686	0.2370	1.0165	0.0862
	CME	0.0315	0.0056	0.0175	0.5011	0.4726	0.2004	0.4499	0.6732	0.1636
	MPS	0.0168	0.0008	0.0093	0.1265	0.2829	0.0506	0.1851	0.4270	0.0673
	ADE	0.8705	1.3562	0.4836	1.0175	1.1861	0.4070	0.6024	1.9488	0.2190
	RADE	1.0956	5.2943	1.0923	0.4009	0.6911	0.3571	0.2652	0.2636	0.0087
50	MLE	0.0145	0.0004	0.0080	0.0898	0.2364	0.0359	0.6810	0.4293	0.2476
	LSE	0.0203	0.0025	0.0113	0.1201	0.3858	0.0480	0.6180	0.6031	0.2247
	WLSE	0.03103	0.0023	0.0172	0.2513	0.4227	0.1005	0.3728	0.4752	0.1355
	CME	0.02090	0.0042	0.0116	0.0350	0.3723	0.0140	0.0544	0.4711	0.0197
	MPS	0.0245	0.0006	0.0136	0.37768	0.2125	0.1510	0.3097	0.2141	0.1126
	ADE	1.3860	1.0551	0.7700	0.7495	0.8075	0.2998	0.4478	1.9374	0.1628
	RADE	1.3660	1.3051	1.0228	0.9995	1.0575	0.5498	0.9678	2.1874	0.4128
100	MLE	0.0078	0.0003	0.0043	0.1504	0.1730	0.0601	0.2922	0.3335	0.1062
	LSE	0.02019	0.0020	0.0112	0.1767	0.2621	0.0707	0.8521	0.5167	0.3098
	WLSE	0.0168	0.0021	0.0093	0.1047	0.3412	0.04190	0.2569	0.3283	0.0934
	CME	0.0450	0.0037	0.0250	0.1324	0.3387	0.0529	0.0458	0.3973	0.0166
	MPS	0.0255	0.0002	0.0142	0.0028	0.1479	0.0011	0.1997	0.2051	0.0726
	ADE	1.6119	1.0188	0.8955	0.9167	0.7861	0.3667	1.2723	0.9152	0.4626
	RADE	1.8619	1.2688	1.3455	1.3667	1.0361	0.6167	1.5223	1.1652	0.7126
300	MLE	0.0138	0.0003	0.0076	0.1907	0.1345	0.0763	0.3381	0.2560	0.1229
	LSE	0.0026	0.0015	0.0014	0.06601	0.2029	0.0264	0.0466	0.4551	0.0169
	WLSE	0.0118	0.0014	0.0065	0.0183	0.2523	0.0073	0.2238	0.2785	0.0814
	CME	0.0035	0.0031	0.0019	0.0717	0.2474	0.0286	0.1209	0.3600	0.0439
	MPS	0.01202	0.0002	0.0066	0.0849	0.1255	0.0339	0.2636	0.1527	0.0958
	ADE	1.3526	0.9154	0.7514	0.9206	0.7474	0.3682	0.5686	0.7939	0.2067
	RADE	1.6026	1.1615	0.7514	1.0706	0.9974	0.6182	0.8186	1.0439	0.4567
500	MLE	0.0072	0.0002	0.0040	0.0342	0.0870	0.0136	0.1530	0.1203	0.0556
	LSE	0.0020	0.0015	0.0011	0.1965	0.1318	0.0786	0.07387	0.3873	0.0268
	WLSE	0.01050	0.0010	0.0058	0.0082	0.0925	0.0032	0.2799	0.2284	0.1017
	CME	0.0131	0.0022	0.0073	0.1099	0.1769	0.0439	0.3233	0.2743	0.1175
	MPS	0.0138	0.0001	0.0076	0.2010	0.0790	0.0804	0.2735	0.1074	0.0994
	ADE	1.5673	0.4748	0.8707	0.9104	0.7169	0.3641	0.3749	0.4499	0.1363
	RADE	1.8173	0.7248	1.1207	1.1604	0.9669	0.6141	0.6249	0.6999	0.3863

Table 6. Numerical computations of R_1, R_2 and R_3 for the PMD and MD.

Distribution	Parameters	Significance Level	R_1	R_2	R_3
PMD	$\alpha = 0.25,\ ,\delta = 0.5,\ \beta = 0.75$	0.60	6.42203	14.66596	57.98484
		0.65	7.50493	15.76808	62.97213
		0.70	8.76048	17.04336	68.22528
		0.75	10.25036	18.55553	73.84361
		0.80	12.07897	20.41185	79.99503
		0.85	14.44345	22.81451	86.99481
		0.90	17.78991	26.22053	95.55478
		0.95	23.55705	32.10349	107.94750
MD	$\alpha = 0.25,\ \delta = 0.5$	0.60	3.99507	7.35744	13.00944
		0.65	4.51089	7.80154	13.75866
		0.70	5.08561	8.30304	14.53503
		0.75	5.74066	8.88260	15.35582
		0.80	6.51178	9.57464	16.24935
		0.85	7.46538	10.44305	17.26975
		0.90	8.74856	11.63023	18.53438
		0.95	10.82286	13.58311	20.43073

Table 7. Numerical computations of R_1, R_2 and R_3 for the PMD and MD.

Distribution	Parameters	Significance Level	R_1	R_2	R_3
PMD	$\alpha = 2,\ ,\delta = 1.5,\ \beta = 0.5$	0.60	1.69791	6.77850	34.11604
		0.60	2.16748	7.47203	38.77419
		0.65	2.75806	8.30876	44.14104
		0.70	3.51707	9.34645	50.46666
		0.75	4.52832	10.68383	58.17927
		0.80	5.95469	12.51303	68.08405
		0.85	8.18180	15.28311	82.02290
		0.90	12.53011	20.51401	106.04209
MD	$\alpha = 2,\ ,\delta = 1.5$	0.60	1.27819	2.39278	3.01885
		0.60	1.44820	2.54008	3.20065
		0.65	1.63803	2.70661	3.39830
		0.70	1.85488	2.89906	3.61870
		0.75	2.11046	3.12915	3.87229
		0.80	2.42690	3.41813	4.17950
		0.85	2.85328	3.81350	4.58555
		0.90	3.54361	4.46482	5.23332

Table 8. The values for the Swedish motor insurance data set.

77.585	11.839	14.084	9.006	13.581	51.080	8.255	22.292	20.295	8.538
19.038	10.597	22.261	46.244	18.603	47.495	9.253	44.278	18.455	11.936
6.378	43.676	30.465	16.781	11.469	21.026	15.583	6.085	16.182	42.369
1.688	7.229								

Table 9. Basic statistics of Swedish motor insurance data set

Mean	**Median**	**Std.Dev**	Q_1	Q_3	γ_3	γ_4
21.989	16.482	17.202	10.261	24.335	1.365	1.357

Table 10. The estimation parameters with log-likelihood (ll) values using numerous proposed models

Model	**Parameters**			**ll**
PMD	$\hat{\alpha}$=6.2772	$\hat{\delta}$=0.2689	$\hat{\beta}$=0.7918	-127.303
TPM	$\hat{\alpha}$=0.1286	$\hat{\delta}$=0.1781		-128.850
UP-GA	$\hat{\alpha}$ = 1.9610	$\hat{\lambda}$=0.0938	$\hat{\theta}$=0.3015	-127.670
Normal	$\hat{\alpha}$=21.988	$\hat{\lambda}$=16.931		-135.939
EG	$\hat{\lambda}$=0.0771	$\hat{p}$=0.3260		-129.100
Gompertz	$\hat{\alpha}$=0.0154	$\hat{\lambda}$=0.0335		-129.732
EXP	$\hat{\lambda}$=0.0454			-130.897
PL	$\hat{\alpha}$=1.0149	$\hat{\beta}$=0.0831		-128.710

Table 11. The goodness of fit tests for Swedish motor insurance data set.

Distribution	A_1	A_2	A_3	A_4	**K-S**	p**-value**
PMD	260.607	261.464	262.065	264.004	0.1379	0.531
TPM	261.700	262.114	262.672	264.632	0.1695	0.2834
UP-GA	261.341	262.198	262.799	265.738	0.1560	0.3775
Normal	275.879	276.292	276.850	278.810	0.2428	0.0380
EG	262.201	262.615	263.173	265.132	0.1509	0.4183
Gompertz	263.464	263.878	264.436	266.396	0.1615	0.3371
EXP	263.794	263.927	264.280	265.260	0.2104	0.1010
PL	261.420	261.833	262.391	264.351	0.1504	0.4220

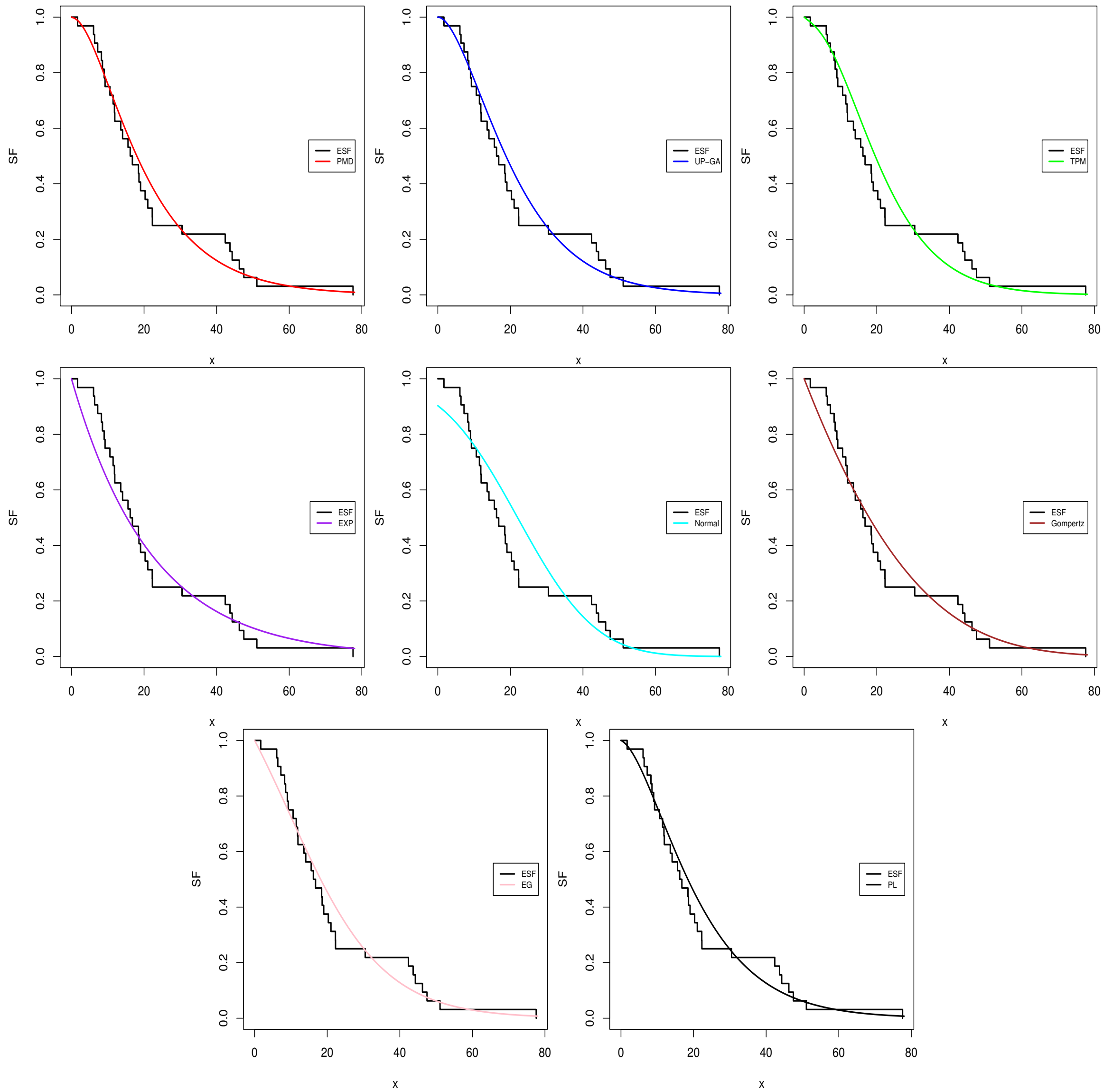

Figure 7. Plots of the ESF and fitted survival functions for proposed fitting models using motor insurance data set.

Table 12. The estimates of α, δ and β under various estimation procedures.

Par	MLE	LSE	WLSE	CME	MPS	ADE	RTADE
$\hat{\alpha}$	6.2772	1.5616	1.6435	1.5456	1.7709	1.5103	2.6942
$\hat{\delta}$	0.2689	0.3919	0.4939	0.3856	0.5119	0.3670	0.3928
$\hat{\beta}$	0.7918	0.6356	0.5593	0.6465	0.6140	0.6653	0.6576

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