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Research article

Limit-Cycle Behavior of the Exponentiated Exponential Autoregressive Model: Theory and Applications

Khitam M. Shakir* **and Hiba H. Abdullah**

Department of Mathematics, College of Education for Women, Tikrit University, Tikrit 3400, Iraq;
Khitam.Mahmood354@st.tu.edu.iq; hiba.h.a.83@tu.edu.iq

***Correspondence:** Khitam.Mahmood354@st.tu.edu.iq

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ABSTRACT

This study aims to design and evaluate a special nonlinear time series model, namely the exponentiated exponential autoregressive (EEAR) model, used to construct time series estimates that incorporate the cumulative distribution function of the raised exponential distribution. The underlying analytical constraints on the stability of these models and their limit cycles were obtained via local linearization techniques. The resulting models were developed by applying them to real data (monthly average of British pounds (in millions) spent by foreign visitors in the United Kingdom for the period 1-1-1988 to 31-12-2018) and indicated how efficiently they can capture the intricate dynamic character of data. The EEAR model was the best model according to information criteria, fit quality, and dynamically stable with respect to the different types of ordered models. The study represents a substantial improvement to the nonlinear time series model's theory, as well as new analytical resources to model complex economic and financial phenomena.

1. Introduction

Classical linear autoregressive models are not satisfactory to describe the complex dynamic patterns of real economic and financial time series when nonlinear feedback, asymmetry and oscillation is a factor. To overcome such limitations, several nonlinear autoregressive methods including threshold autoregressive (TAR), smooth transition autoregressive (STAR) and distribution-based autoregressive models have been advanced. Although these strategies can effectively capture the nonlinear dependence, the dynamical stability properties are usually numerically examined, and only minimal explicit analytical conditions for periodic behavior are available. Recent progress in nonlinear time series modeling promotes incorporating probabilistic transformations with dynamical system theory toward interpretable stability conditions. Inspired by this line, the current work advocates for the use of a nonlinear autoregressive model, based on the exponentiated exponential distribution, provided by the exponentiated exponential to facilitate a direct analysis of fixed-point stability and limit-cycle behavior through a unified tool.

Time series analysis begins with the identification phase, followed by the estimation phase, then the diagnostic checking phase, and finally, the forecasting phase [1]. The concept of the stability of dynamical (kinetic) systems, which consist of a system of differential equations, states that the system is stable when the solution path approaches a singular point or a closed trajectory, known as a limit cycle [2]. Numerous studies have focused on investigating and analyzing this type of stability, including but not limited to: ([3]-[11]).

Though the nonlinear autoregressive model literature is extensive, there are still no analytical results providing sufficient conditions for the existence and orbital stability of non-zero periodic solutions in autoregressive models driven by probabilistic transformations. Current work on cumulative distribution function-based models focuses primarily on numerical simulations rather than rigorous stability analysis. To address this, this article derives explicit analytical conditions for the existence of non-zero singularities and stable limit cycles in the EEAR(p) model, thereby linking probabilistic nonlinearities to long-term dynamic behavior.

Therefore, this study aims to derive a general mathematical formula for the conditions of terminal orbit stability in the EEAR(p) model and apply it to different examples to determine how the orbit shape changes according to the model parameters. Numerical testing will be used to confirm stability and plot the trajectories to track movement towards the orbit. The study will also test the model's ability to produce a stable terminal orbit when the order p is changed and identify which parameters create distinct periodic behavior.

Although there are already nonlinear autoregressive models such as STAR, TAR, Logistic AR, and Gompertz-AR models based almost entirely on smooth transition functions or deterministic nonlinear mappings, the proposed EEAR model provides a probabilistic nonlinearity constructed directly from a cumulative distribution function. This construction leads to creating a nonlinear feedback mechanism that admits non-zero singularities and permits the analytical derivation of explicit conditions for orbital stability and limit-cycle existence. Contrary to many distribution-based AR models that numerically check for stability, the current framework establishes sufficient analytical conditions linking the exponentiated exponential transformation to the long-term dynamic behavior of the system.

2. The Exponentiated Exponential distribution

Let x Represent the random variable, the CDF functions of Exponentiated Exponential with two parameters are [12]:

$$F(x,\beta,\delta) = (1 - exp(-\delta x))^{\beta},\ x \geq 0, \beta, \delta > 0. \tag{2.1}$$

And pdf function has form [12]:

$$f(x,\beta,\delta) = \beta\delta\, exp(-\delta x)\,(1 - exp(-\delta x))^{\beta-1}, \qquad x \geq 0, \beta, \delta > 0. \tag{2.2}$$

Using programming in MATLAB R2022a, the CDF and PDF functions are plotted for the Exponentiated Exponential distribution for different parameter values, as shown in Figures 1 and 2, respectively.

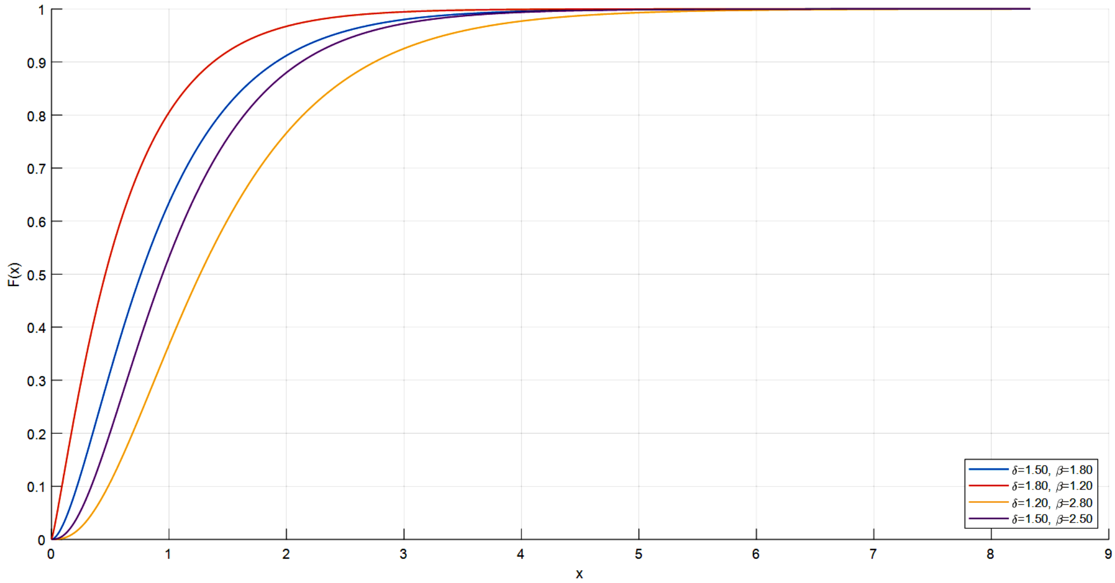

Figure 1. CDF function of Exponentiated Exponential distribution for different parameter values

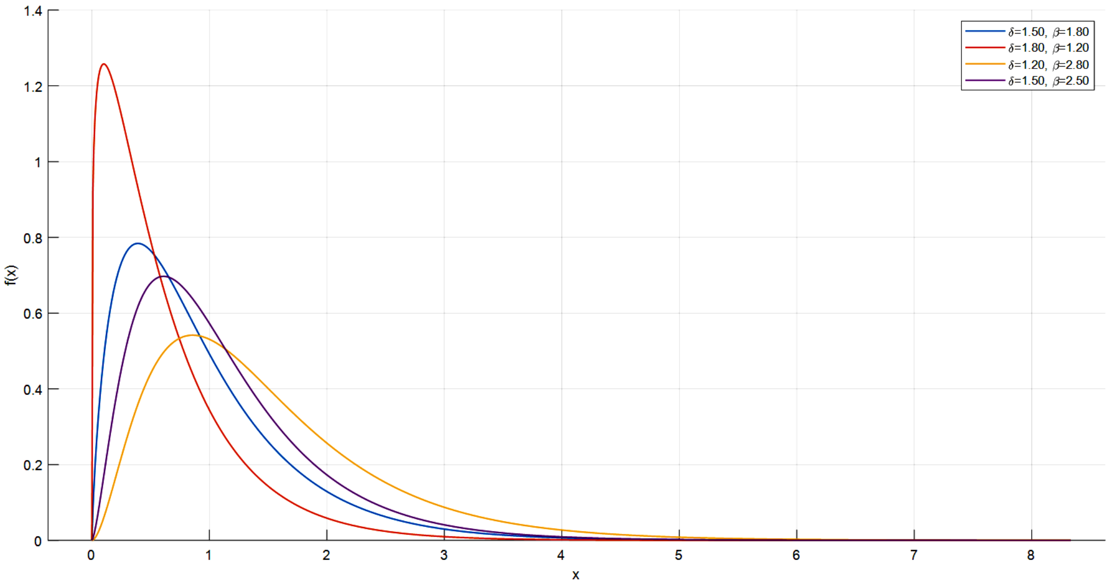

Figure 2. PDF function of Exponentiated Exponential distribution for different parameter values

The shape and scale parameters control the behavior of the distribution (Figures 1 and 2). As illustrated by the CDF curve, increasing the shape parameter results in a faster transition to one, representing a higher probability of large values, and as depicted by the PDF curve, density can be single-peaked or steeply sloping depending on the chosen parameter values. This behavioral variability is the basis for incorporating this distribution into the construction of a nonlinear time model, as it allows for considerable flexibility in representing data with jumps and asymmetry.

3. Local Linearization Technique

In recent decades, interest in nonlinear time-series models has grown because most real-world phenomena exhibit nonlinearity. This interest focuses on studying the nonlinear dynamics arising from the nonlinear relationships between time-sequential observations, with theoretical evidence explaining phenomena such as nonlinearity and jumping. Nonlinear dynamical systems are often represented by nonlinear differential equations and analyzed using approximation methods that transform them into linear systems, most notably local linear approximation, which relies on the physical structure of the system. The stability of these systems depends on the behavior of the solutions, whether they converge towards singular points or follow closed trajectories at periodic solutions. Nonlinear systems are characterized by fundamental properties including jumping, frequency-amplitude dependence, and the presence of limit cycles, as in the Vander Pol equation [1].

Local linear approximation is one of the most important techniques used to transform nonlinear systems into local linear systems in the vicinity of the singular point representing the system's constant solution. This technique relies on simplifying the dynamic behavior of the nonlinear system near its equilibrium points to analyze stability and understand the nature of the solutions. To achieve this, the nonlinear differential equation is transformed into a linear form through systematic steps that begin with reducing the equation's order, then representing it in state-space form, and finally identifying the constant point and the system's constant solution [2].

Definition (1) [14]:

The point $T_r \in R$ is called the limit point of the difference equation:

$$T_r = f(T_{r-1}) \ , \qquad T_0 \in R^K. \tag{3.1}$$

If it results from the repetition of the vector function f, where f^i denotes the i repetition of the function:

$$f^{(i)}(T_t) = \underbrace{f\left(f\left(\dots f(T_t)\right)\right)\dots)}_{i-times}.$$

Definition (2) [1]:

A point T^* is considered a periodic point with a period of $d \geq 1$ for Equation (3) if:

$$T^* \neq f^{(i)}(T^*) \ \ and \ T^* = f^{(d)}(T^*) \ \ \forall \ 1 \leq i < d.$$

where T^* is a fixed point for $f^{(d)}$.

The resulting set is called a period of order d. A point T_0 is called eventually periodic if it reaches a periodic point, and parallel periodic if its orbits converge towards the orbit of a periodic point according to the Euclidean criterion.

Definition (3) [14]:

A point ω is called singular point if it satisfies $\omega = f(\omega)$ and no other stable point exists in its vicinity, and the singular point for model is:

$$x(t) = f(x_{t-1}, x_{t-2}, \dots, x_{t-p}). \tag{3.2}$$

The point ω is stable if the paths closest to it converge when $t \to \infty$, and unstable if the convergence occurs when $t \to -\infty$.

Definition (4) [15]:

A limit cycle is defined as a closed, isolated path of a system that returns to its initial state after an integer q of steps. A limit cycle is stable if the paths closest to it converge when $t \to \infty$, and unstable if the convergence occurs when $t \to -\infty$. The smallest q that achieves this is called the cycle period, and a system is said to be orbitally stable if its limit cycle is stable.

The following steps illustrate the stages of analyzing a nonlinear time series using the dynamic method with local linearization approximation.

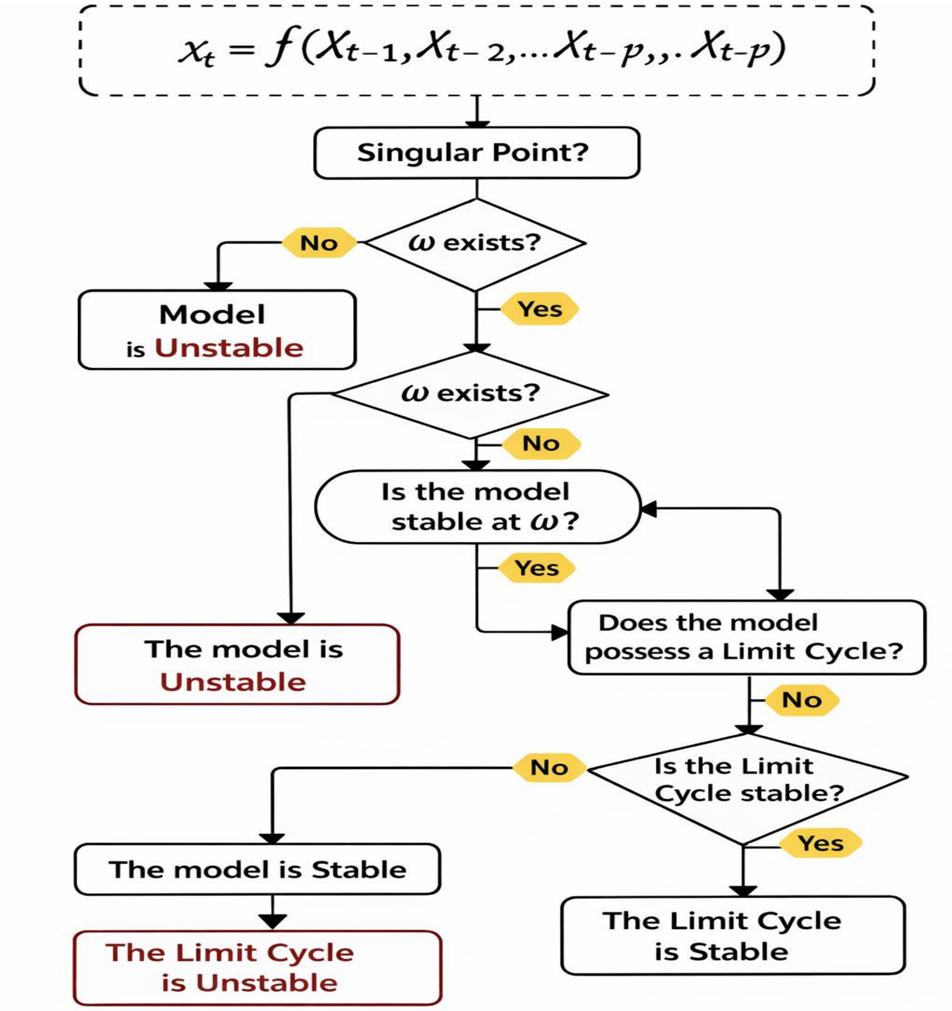

Figure 3. The stages of analyzing a nonlinear time series by local linearization approximation

Figure 3 shows the stages of nonlinear time series analysis, from dynamic formulation to state-space representation and stability analysis. The model is not considered a static statistical construct, and the trajectories and orbital behavior over time have been studied for such models. The integration of statistical modeling and dynamical systems theory is one of the main strengths of the study.

4. The Autoregressive Exponentiated Exponential Model and its stability

The use of all symbols throughout this paper is consistent as follows: The process $\{X_t\}$ denotes the observed time series; p represents the autoregressive order; and δ and β denote the shape and scale parameters of the exponentiated exponential distribution, respectively. The function $G(\cdot)$ denotes the corresponding cumulative distribution function. All vectors and matrices introduced in the state-space representation are defined explicitly at their first occurrence to avoid ambiguity.

$$X_t = \sum_{i=1}^{p} \left[\ell_i + \mathcal{E}_i (1-\exp(-\delta x_{t-1}))^{\beta}\right] X_{t-i} + \varepsilon_t \quad , \varepsilon_t \sim iid\ N(0, \sigma_\varepsilon^2). \tag{4.1}$$

Here $\{\ell_i\}$ and $\{\mathcal{E}_i\}$ are the autoregressive model parameters, for $i = 1,2,\dots,p$

$F(x, \beta, \delta)$ is the exponentiated exponential distribution function. Form (4.1) describes a collection of beneficial dynamic properties:

1. $\lim_{\alpha \to +\infty} (1-\exp(-\delta x_{t-1}))^{\beta} = 0.$
2. $\lim_{X_{t-1} \to 0} (1-\exp(-\delta x_{t-1}))^{\beta} = 0.$
3. $\lim_{\gamma \to +\infty} (1-\exp(-\delta x_{t-1}))^{\beta} = 1.$
4. $\lim_{X_{t-1} \to +\infty} (1-\exp(-\delta x_{t-1}))^{\beta} = 1.$

The $EEAR(p)$ model can be expressed in state-space as follows:

$$X_t = \begin{bmatrix} \ell_1 & \ell_2 & \cdots & \ell_{p-1} & \ell_p \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} X_{t-1} + \frac{\begin{bmatrix} \mathcal{E}_1 & \mathcal{E}_2 & \cdots & \mathcal{E}_{p-1} & \mathcal{E}_p \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}{(1-\exp(-\delta x))^{-\beta}} X_{t-1} + \epsilon_t, \tag{4.2}$$

where the coefficient matrix and vectors are defined accordingly, representing a Markov chain $\{X_t\}$ with a parameter matrix of dimension $(p \times p)$ and vectors ϵ_t and X_t and X_{t-1} defined as:

$$X_t = \begin{bmatrix} x_t \\ x_{t-1} \\ \vdots \\ x_{t-p+1} \end{bmatrix}, \quad X_{t-1} = \begin{bmatrix} x_{t-1} \\ x_{t-2} \\ \vdots \\ x_{t-p} \end{bmatrix}, \quad \epsilon_t = \begin{bmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where ϵ_t represents the white noise innovation vector.

The model can be written in the compact form:

$$X_t = F(X_{t-1}) = T(X_{t-1}) + L(X_{t-1}, \epsilon_t),$$

where $\mathrm{T}(\mathrm{X}_{t-1})$ denotes the linear component of the model, while $L(X_{t-1}, \epsilon_t)$ represents the nonlinear component, defined respectively as:

$$T(X_{t-1}) = \begin{bmatrix} \ell_1 & \ell_2 & \cdots & \ell_{p-1} & \ell_p \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} X_{t-1} = K X_{t-1},$$

$$L(X_{t-1}, \epsilon_t) = \frac{\begin{bmatrix} \mathcal{E}_1 & \mathcal{E}_2 & \cdots & \mathcal{E}_{p-1} & \mathcal{E}_p \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}{(1-\exp(-\delta X_{t-1}))^{-\beta}} X_{t-i} + \epsilon_t.$$

The linear component $T(X_{t-1})$ satisfies the geometric ergodicity conditions for Markov chains. In other words, the eigenvalues of the matrix in $T(X_{t-1})$ can be found as follows:

$$|K - \lambda I| = 0$$

This can be verified by solving the characteristic equation:

$$\lambda^p - \sum_{i=1}^{p} h_j \lambda^{p-j} = 0. \tag{4.3}$$

The linear component is stable if all roots of Equation (4.3) lie inside the unit circle, meaning:

$$|\lambda_i| < 1, i = 1,2, \dots, p.$$

For the nonlinear component $L(X_{t-1}, \epsilon_t)$ can be possess geometric ergodicity properties.

To study the specific conditions for the stability of the $EEAR(p)$ model to identify the non-zero isolated fixed point of the model, set $\omega = f(\omega)$, where $x_t = x_{t-1} = x_{t-2} \dots x_{t-p} = \omega$, to get a final form:

$$\omega = \frac{-1}{\beta} ln\left(1 - \sqrt[\delta]{k}\right), \quad k = \frac{1 - \sum_{i=1}^{p} \ell_i}{\sum_{i=1}^{p} \mathcal{E}_i}. \tag{4.4}$$

If and only if $\ln\left(1 - \sqrt[\delta]{k}\right) < 0$, the non-zero singularity ω exists and is real (by definition, it belongs to the real numbers and exists in the sense that the value is smaller than infinity).

The non-zero singularity ω of the $EEAR(P)$ model, if it exists, is stable if and only if all roots of the Equation (4.3). It is located within the unit circle, where

$$h_1 = 1 - \beta\, \mathcal{E}_1 \left(1 - \sqrt[\delta]{\frac{1-\ell_1}{\mathcal{E}_1}}\right) \ln\left(1 - \sqrt[\delta]{\frac{1-\ell_1}{\mathcal{E}_1}}\right), \tag{4.5}$$

$$h_j = \ell_j + k\mathcal{E}_j \qquad j = 2,3, \dots p. \tag{4.6}$$

Unlike classical autoregressive models, the stability conditions derived here explicitly depend on the exponentiated exponential transformation. The nonlinear feedback introduced by the probabilistic mapping alters the Jacobian structure, leading to stability conditions that cannot be reduced to purely linear eigenvalue constraints.

$$(|h_i| < 1\,, i = 1,2, \dots, p), \tag{4.7}$$

where h_i is the root of the characteristic equation.

Stability conditions limit cycle of $EEAR(P)$ model.

Fixed-point stability must be distinguished from the stability of periodic orbits. Fixed-point stability concerns the convergence of trajectories to a single equilibrium, whereas limit-cycle stability refers to the convergence of trajectories toward a closed invariant orbit.

At this stage, the stability conditions of the limit cycle (if any) of the $EEAR(P)$ model will be studied by finding the stability conditions of the model of first order, i.e., the $EEAR(1)$. The following theorem explains these conditions in terms of the model parameters.

Theorem (1): Assume that the $EEAR(1)$ model admits a non-zero periodic orbit and that the nonlinear mapping is continuously differentiable in a neighborhood of this orbit. Then, the limit cycle is orbitally stable if and only if the modulus of the eigenvalue of the associated Jacobian matrix is strictly less than one.

$$\left| \prod_{i=1}^{q} \left[\ell_1 + \mathcal{E}_1 - \delta \mathcal{E}_1 \sqrt[\delta]{\frac{1-\ell_1}{\mathcal{E}_1}} \right]^i \right| < 1. \tag{4.8}$$

Proof:

Let's assume that the end cycle of the model has a cycle $q > 1$, q which is in the form:

$$x_t, x_{t+1}, x_{t+2}, \dots, x_{t+q} = x_t.$$

It represents a closed and isolated path. By taking an open neighborhood with a sufficiently small radius for each endpoint of the cycle, where a slight disturbance occurs at all these points, added to their magnitude by a small change in the position of each point at the beginning of the cycle, using a computational substitution equation by replacing both (x_t) with $(x_t + \omega_t)$ and (x_{t-1}) with $(x_{t-1} + \omega_{t-1})$, where ω_t is so small that $\omega_t^2 \to 0$ for all $n \geq 2$, then substituting the model in Equation (4.3) by first-order substitution and canceling the white disturbance effect, ε_t yields:

$$x_t + \omega_t = [\ell_1 + \mathcal{E}_1(1 - \delta \exp(-\delta x) + \gamma\delta\beta\omega_{t-1} \exp(-\delta x))](x_{t-1} + \omega_{t-1})$$

$$x_t + \omega_t = [\ell_1 + \mathcal{E}_1(1 - \delta \exp(-\delta x) + \gamma\delta\beta\omega_{t-1} \exp(-\delta x))]x_{t-1}$$

$$+[\ell_1 + \mathcal{E}_1(1 - \delta \exp(-\delta x) + \gamma\delta\beta\omega_{t-1} \exp(-\delta x))]\omega_{t-1}.$$

Since

$$x_t = [\ell_1 + \mathcal{E}_1(1 - \delta \exp(-\delta x) + \gamma\delta\beta\omega_{t-1} \exp(-\delta x))]x_{t-1}.$$

Then we get:

$$\omega_t = [\ell_1 + \mathcal{E}_1(1 - \delta \exp(-\delta x) + \gamma\delta\beta\omega_{t-1} \exp(-\delta x))]\omega_{t-1},$$
$$\omega_t = \ell_1\omega_{t-1} + \mathcal{E}_1\delta \exp(-\delta x)\,\omega_{t-1} + \gamma\delta\beta\omega_{t-1}^2 \exp(-\delta x).$$

From assumption $\omega_t^2 \to 0$ for all $n \geq 2$, led to $\gamma\delta\beta\omega_{t-1}^2 \exp(-\delta x) = 0$, implies to:

$$\omega_t = \ell_1\omega_{t-1} + \mathcal{E}_1\delta \exp(-\delta x)\,\omega_{t-1},$$
$$\omega_t = [\ell_1 + \mathcal{E}_1\delta \exp(-\delta x)]\omega_{t-1}.$$

Since $\exp(-\delta x) = \sqrt[\delta]{k}$, implies to:

$$\omega_t = \left[\ell_1 + \mathcal{E}_1\delta\sqrt[\delta]{k}\right]\omega_{t-1},$$

we replace each (t) with $(t + 1)$

$$\omega_{t+1} = \left[\ell_1 + \mathcal{E}_1\delta\sqrt[\delta]{k}\right]\omega_t$$
$$\omega_{t+1} = \left[\ell_1 + \mathcal{E}_1\delta\sqrt[\delta]{k}\right]\omega_t$$
$$\omega_{t+1} = \left[\ell_1 + \mathcal{E}_1\delta\sqrt[\delta]{k}\right]^2\omega_{t-1}$$
$$\vdots$$
$$\omega_{t+q} = \left[\ell_1 + \mathcal{E}_1\delta\sqrt[\delta]{k}\right]^q\omega_t$$
$$\frac{\omega_{t+q}}{\omega_t} = \left[\ell_1 + \mathcal{E}_1\delta\sqrt[\delta]{k}\right]^q.$$

According to the stability requirements of the limit cycle, if:

$$\left|\frac{\omega_{t+q}}{\omega_t}\right| < 1.$$

Hence

$$\left|\left[\ell_1 + \mathcal{E}_1\delta\sqrt[\delta]{k}\right]^q\right| < 1$$
$$\left|\prod_{i=1}^{q+1}\left[\ell_1 + \mathcal{E}_1\delta\sqrt[\delta]{k}\right]^i\right| < 1$$
$$\left|\prod_{i=1}^{q} T(x_{t+q-i})\right| < 1$$
$$\left|\prod_{i=1}^{q+1}\left[\ell_1 + \mathcal{E}_1\delta\sqrt[\delta]{\frac{1-\ell_1}{\mathcal{E}_1}}\right]^i\right| < 1.$$

This issue represents a special case of the $EEAR(1)$ model, i.e., when the model is of first order. The next step is to provide a generalization of this issue to the $EEAR(P)$ model when $p > 1$. In this case, the model is written as a state space as follows:

$$X_t = AX_{t-1} + \epsilon_t,$$

where

$$X_t = \begin{bmatrix} x_t \\ x_{t-1} \\ \vdots \\ x_{t-p+1} \end{bmatrix}, X_{t-1} = \begin{bmatrix} x_{t-1} \\ x_{t-2} \\ \vdots \\ x_{t-p+1} \end{bmatrix}, \epsilon_t = \begin{bmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

These are vectors defined in space $\mathbb{R}^p$, and A represents a matrix of magnitude $p \times p$ defined as follows:

$$A = \begin{bmatrix} S_1 & \cdots & S_{p-1} & S_p \\ 1 & \ddots & 0 & 0 \\ 0 & & 0 & 0 \\ \ldots & & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix},$$

where

$S_1 = \ell_1 + \mathcal{E}_1(1 - \exp(-\delta X_{t-1}))^\beta,$

$S_{p-1} = \ell_{p-1} + \mathcal{E}_{p-1}(1 - \exp(-\delta X_{t-1}))^{\beta}$,
$, S_p = \ell_p + \mathcal{E}_p(1 - \exp(-\delta X_{t-1}))^{\beta}$.

Theorem (2): The terminus of the $EEAR(p)$ model is orbitally stable if all eigenvalues of matrix A are less than one, i.e., if:

$$A = A_q . A_{q-1} \dots A_1 = \prod_{i=1}^{q} A_i,$$

where

$$A = \begin{bmatrix} a_{1,1}^{(i)} & \cdots & a_{1,p-1}^{(i)} & a_{1,p}^{(i)} \\ & & 0 & 0 \\ 1 & \ddots & 0 & 0 \\ 0 & & \vdots & \vdots \\ \dots & & & \\ 0 & \cdots & 1 & 0 \end{bmatrix}, i = 1,2, \dots, p.$$

Also

$$a_{1,1}^{(i)} = \ell_1 + \mathcal{E}_1(1 - \exp(-\delta x_{t-1}))^{\beta} + \sum_{i=1}^{p} \delta\beta \exp(-\delta x_{t-1}) (1 - \exp(-\delta x_{t-1}))^{\beta-1},$$

$$a_{1,l}^{(i)} = \ell_l + \mathcal{E}_l(1 - \exp(-\delta x_{t-1}))^{\beta}, \quad l = 2,3, \dots, p.$$

Proof:

Let the model $EEAR(P)$ be expressed as a state space as follows:

$$\begin{bmatrix} x_t \\ x_{t-1} \\ \vdots \\ x_{t-p+1} \end{bmatrix} = \begin{bmatrix} n_{1,1} & \cdots & n_{1,p-1} & n_{1,p} \\ & & 0 & 0 \\ 1 & \ddots & 0 & 0 \\ 0 & & \vdots & \vdots \\ \dots & & & \\ 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ x_{t-2} \\ \vdots \\ x_{t-p} \end{bmatrix}.$$

where $n_{1,i} = \ell_i + \mathcal{E}_i(1 - \exp(-\delta x_{t-1}))^{\beta}, \ i = 1,2,3, \dots, p.$

Let's assume the limit cycle of the model has a cycle $q > 1$, where q is in the form:

$$x_t, x_{t+1}, x_{t+2}, \dots, x_{t+q} = x_t,$$

It represents a closed and isolated path. By taking an open neighborhood with a sufficiently small radius for each endpoint of the cycle, where a slight disturbance occurs at all these points, added to their magnitude by a small change in the position of each point at the beginning of the cycle, using a computational substitution equation by replacing both (x_t) with $(x_t + \omega_t)$ and (x_{t-i}) with $(x_{t-1} + \omega_{t-1})$, where ω_t is so small that $\omega_t^2 \to 0$ for all $n \geq 2$, then substituting the model in Equation (4.3) by first-order substitution and canceling the white disturbance effect, ε_t yields:

$$x_t + \omega_t = \begin{bmatrix} n_{1,1} & \cdots & n_{1,p-1} & n_{1,p} \\ & & 0 & 0 \\ 1 & \ddots & 0 & 0 \\ 0 & & \vdots & \vdots \\ \dots & & & \\ 0 & \cdots & 1 & 0 \end{bmatrix} (x_{t-1} + \omega_{t-1}).$$

Since x_t, ω_t, x_{t-1} and ω_{t-1} are all vectors defined in the R^p space, and

$$n_{1,1} = \ell_1 + \mathcal{E}_1(1 - \exp(-\delta x_{t-1}))^{\beta} + \sum_{i=1}^{p} \delta\beta \exp(-\delta x_{t-1}) (1 - \exp(-\delta x_{t-1}))^{\beta-1},$$

$$n_{1,l} = \ell_l + \mathcal{E}_l(1 - \exp(-\delta x_{t-1}))^{\beta}, \qquad l = 2,3, \dots, p.$$

By performing binomial approximations and some algebraic operations, we obtain:

$$\omega_t = A_0 \omega_{t-1}, \qquad A_0 = \begin{bmatrix} a_{1,1}^{(0)} & \cdots & a_{1,p-1}^{(0)} & a_{1,p}^{(0)} \\ & & 0 & 0 \\ 1 & \ddots & 0 & 0 \\ 0 & & \vdots & \vdots \\ \dots & & & \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Also

$$a_{1,1}^{(0)} = \ell_1 + \mathcal{E}_1(1-\exp(-\delta x_{t-1}))^{\beta} + \sum_{i=1}^{p} \delta\beta \exp(-\delta x_{t-1})\,(1-\exp(-\delta x_{t-1}))^{\beta-1},$$

$$a_{1,l}^{(0)} = \ell_l + \mathcal{E}_l(1-\exp(-\delta x_{t-1}))^{\beta}, \quad l = 2,3,\dots,p.$$

By repetition, we obtain:

$$\omega_{t+1} = A_1\omega_t, \qquad A_1 = \begin{bmatrix} a_{1,1}^{(1)} & \cdots & a_{1,p-1}^{(1)} & a_{1,p}^{(1)} \\ 1 & & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ \cdots & & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Also

$$a_{1,1}^{(1)} = \ell_1 + \mathcal{E}_1(1-\exp(-\delta x_{t}))^{\beta} + \sum_{i=1}^{p} \delta\beta \exp(-\delta x_{t})\,(1-\exp(-\delta x_{t}))^{\beta-1},$$

$$a_{1,l}^{(1)} = \ell_l + \mathcal{E}_l(1-\exp(-\delta x_{t}))^{\beta}, \quad l = 2,3,\dots,p.$$

By repeating for q of times, we get:

$$\omega_{t+q} = A_q\omega_{t+q-1} = A_qA_{q-1}\omega_{t+q-2} = \cdots = A_qA_{q-1}\dots A_1\omega_t,$$

$$\omega_{t+q} = \prod_{i=1}^{q} A_i\omega_t, \qquad A_q = \begin{bmatrix} a_{1,1}^{(q)} & \cdots & a_{1,p-1}^{(q)} & a_{1,p}^{(q)} \\ 1 & & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ \cdots & & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}. \qquad (4.9)$$

Also

$$a_{1,1}^{(q)} = \ell_1 + \mathcal{E}_1\big(1-\exp\big(-\delta x_{t+q-1}\big)\big)^{\beta} + \sum_{i=1}^{p} \delta\beta \exp\big(-\delta x_{t+q-1}\big)\,\big(1-\exp\big(-\delta x_{t+q-1}\big)\big)^{\beta-1},$$

$$a_{1,l}^{(q)} = \ell_l + \mathcal{E}_l\big(1-\exp\big(-\delta x_{t+q-1}\big)\big)^{\beta}, \quad l = 2,3,\dots,p.$$

5. Real Data Application

The data used in the applied aspect of the study consisted of the monthly average of British pounds (in millions) spent by foreign visitors in the United Kingdom period from January 1, 1988, to December 31, 2018 [14].

The experimental data is a real economic/financial price series, involving consecutive observations of monthly prices over a relatively long period. This series is characterized by fluctuations in the levels of its quantities — fast-reaching peaks and troughs, as well as periods of relative stability this leads to the notion that periodic shocks are a regular occurrence. These responses reflect non-linear dynamics; variance across time is not consistent and can be found across economic or financial series, in particular commodities or assets. Moreover, the lack of a consistent linear trend and the continuous variation in the time amplitude of fluctuations prove that the traditional linear AR model is insufficient to adequately capture the fluctuations, and therefore, we choose the EEAR model with the non-linear exponential transformation.

Statistically, the data are hardly stationary around a fixed mean, and their distribution tends to skewness and asymmetry, with the possibility of outliers during certain periods. This features data for studying linear components in conjunction with nonlinear feedback as in the EEAR(p) model.

First, A time series is the initial step in analyzing data. We import the information into MATLAB R2022a and plot our plotting on this data. Figure 4 presents the time curve of average monthly spending (millions of pounds sterling) of international visitors in the UK.

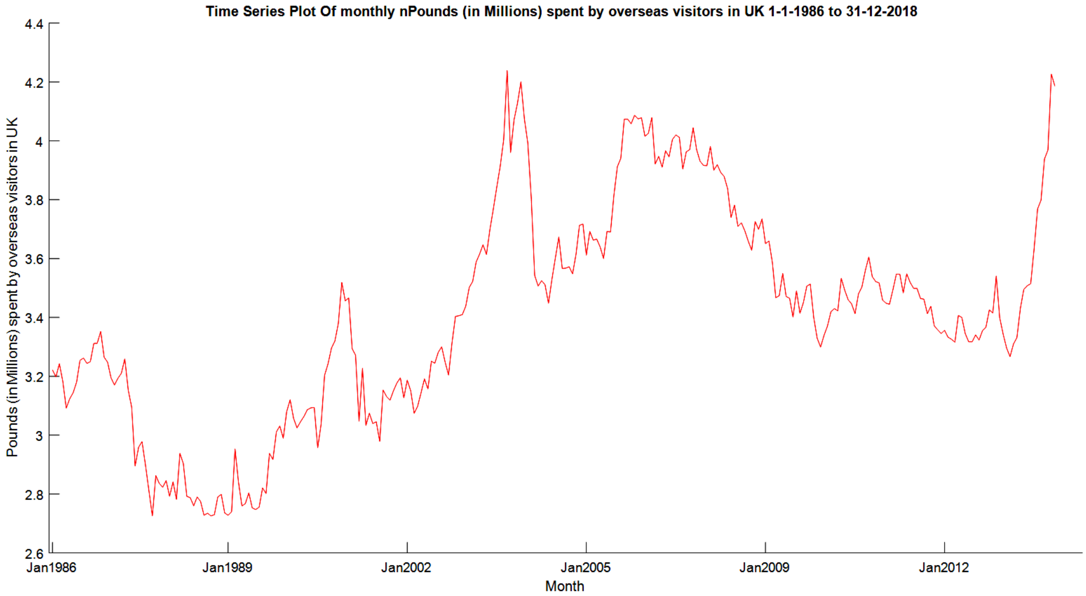

Figure 4. monthly average of British pounds (in millions) spent by foreign visitors in the United Kingdom

Figure 4 shows the residuals derived from fitting the EEAR model to the data. This image demonstrates that the residuals vary around zero, suggesting it succeeded in obtaining most of the regular structure of the original series. The residuals are not perfectly stable over time, with higher variance of the residuals evident. Although this shows that the model predicts the dynamics of the series, it does allow weak time dependencies or possibly random effects of a heterogeneous nature. The data in the experiment indicates the average over time, supporting the model's validity from an academic perspective, but also the data has an important characteristic: irregular noise which cannot be completely filtered out.

Secondly, one uses the time series that illustrates above to turn it into the "residuals series." The last one mathematically defines this by the following equation:

$$r_t = \log\left(\frac{x_t}{x_{t-1}}\right)$$

where r_t is the residuals series, and where x corresponds to $t-1$ and t observations respectively.

The MATLAB instruction for changing this series is: $r_t = \text{price2ret}(x_t)$. Figure 5 shows the residuals series of this transformation.

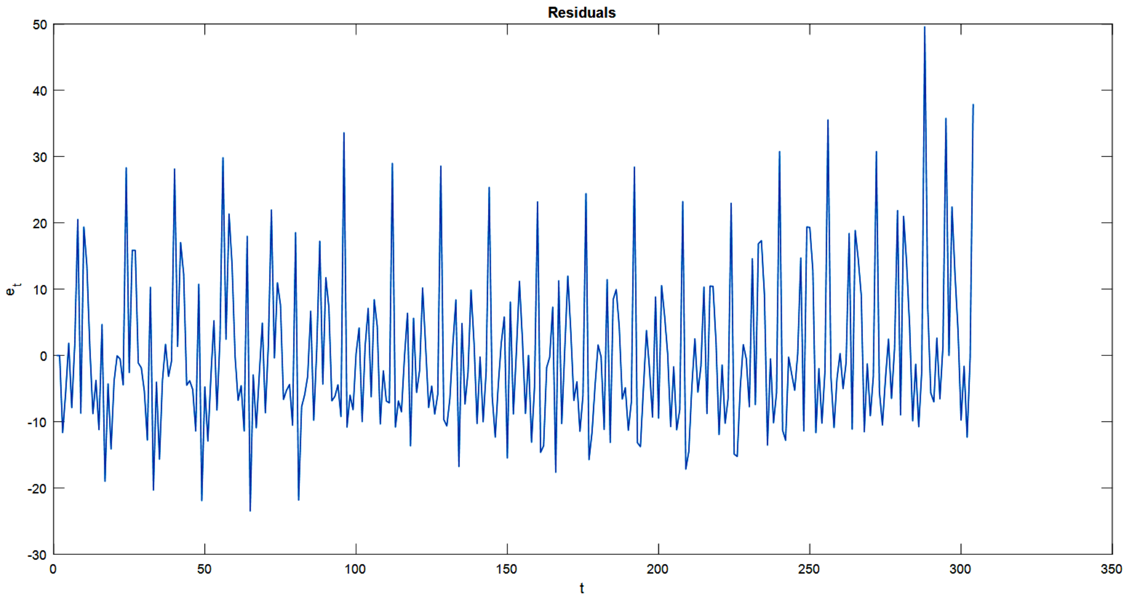

Figure 5. Plot the Residuals series.

Figure 5, which displays the autocorrelation function (ACF) of the model residuals, reveals notable peaks at certain time shifts, where some values exceed the statistical confidence threshold. This indicates that the residuals are not pure white noise but retain some temporal dependence. The emergence of this pattern confirms that the original series contains complex nonlinear components and that the EEAR model, despite its high power, does not impose complete stativity on the residuals. Academically, this behavior is interpreted as evidence of residual internal dynamics, which is expected in highly complex nonlinear models. [**15-16**], respectively, for the residual sequences with 40 lags, as shown in Figure 6.

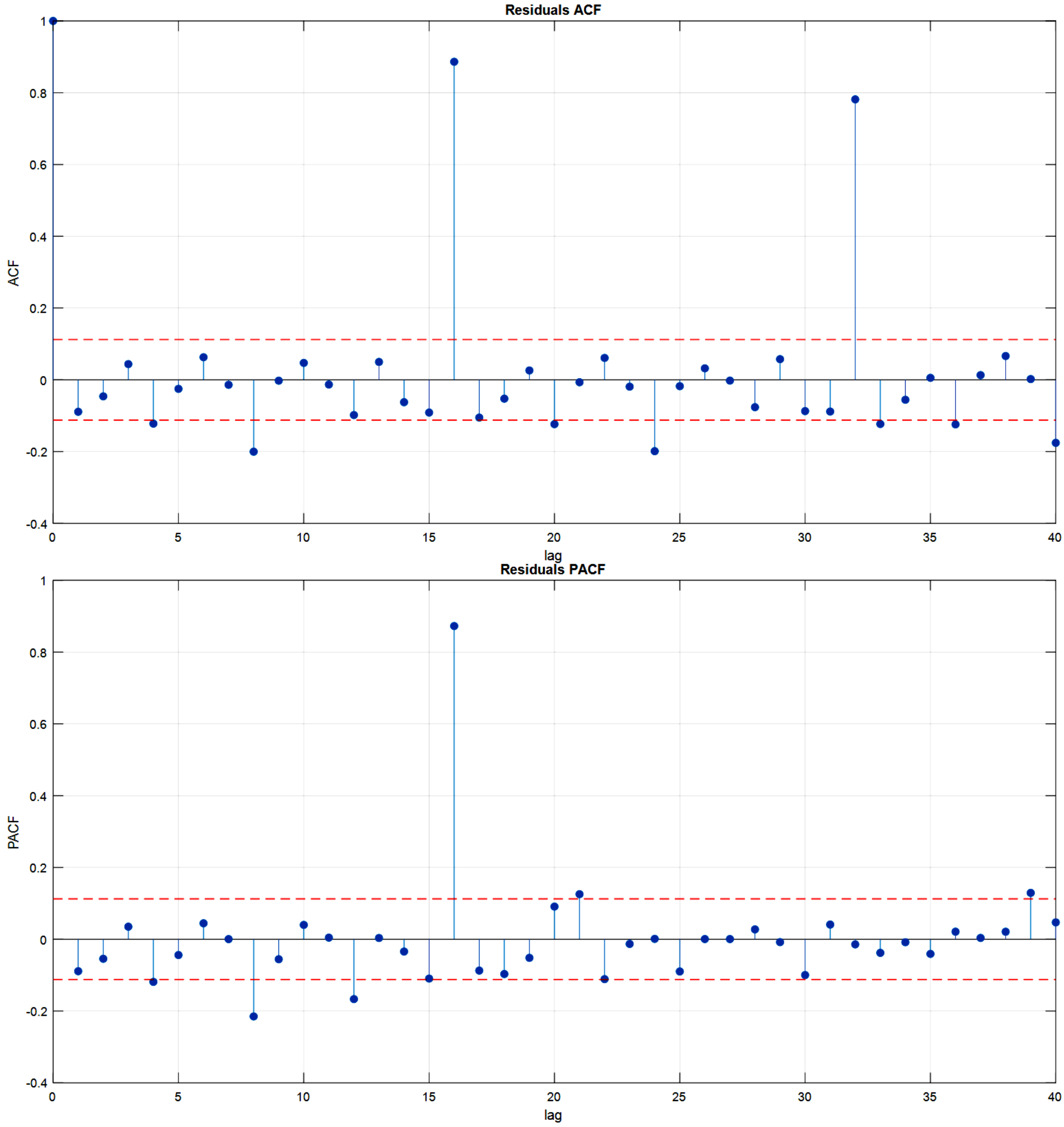

Figure 6. plot ACF, PACF for the Residuals sequences with 40 lags

Figure 6 shows the partial autocorrelation function (PACF) for the residuals, which a gradual decrease is also apparent with increasing time shift, although some values remain significant at the first order. This means that the direct influence of some past values persists despite model fitting but weakens rapidly over time. In theoretical terms, this figure shows that the model accounts for most of the direct dependence, while secondary effects of a nonlinear or stochastic nature remain. As shown in the source [16], this behavior is acceptable for nonlinear models where achieving a completely white residual is not required, as is the case in strict linear models, although there are several time differences outside the confidence interval.

Thirdly, Estimating the model parameters for ten orders. In this section, the error variance and information criteria for each order are calculated [17-18]. Also, the h values are determined, and the stability condition is verified according to Equations (4.5) and (4.6) by applying Equation (4.7). Details of the results are presented in Tables 1 (parameters), 2 (information criteria), and 3 (model selection and stability).

Table 1. Estimated parameters, variance, and k- value at each order

EEAR(p)	$\sum_{i=1}^{p} a_i$	$\sum_{i=1}^{p} b_i$	β	δ	σ^2	k
EEAR(1)	0.9157	0.9157	−0.4386	−4.0926	244.74	−12.969
EEAR(2)	1.2327	−2.1028	0.0490	−16.6603	143.96	0.1107
EEAR(3)	1.0049	−0.0538	−0.5231	14.1684	214.97	0.0911
EEAR(4)	0.9188	0.0330	−0.3659	−2.8203	215.64	2.458
EEAR(5)	0.9481	0.0158	−0.3300	0.1565	205.53	3.284
EEAR(6)	0.9679	0.0054	0.1941	−0.0276	197.27	5.944

Table 2. Evaluate information criteria at each order

EEAR(p)	L	AIC	BIC
EEAR(1)	−1263.20	2536.4	2555
EEAR(2)	−1178.90	2371.8	2397.8
EEAR(3)	−1235.35	2488.7	2522.1
EEAR(4)	−1231.70	2485.4	2526.2
EEAR(5)	−1220.45	2466.9	2515
EEAR(6)	−1210.25	2450.5	2505.9

Table 3. Evaluate the h_i- value at each order

EEAR(p)	h_1	h_2	h_3	h_4	h_5	h_6	Stability or not
EEAR(1)	1.0032−0.0149i	---	---	---	---	---	No
EEAR(2)	1.0106−0.0831i	−0.2034	---	---	---	---	No
EEAR(3)	1.0049−0.0617i	0.2420	0.1441	---	---	---	No
EEAR(4)	0.9824+0.2158i	0.2182	0.1766	−0.0042	---	---	No
EEAR(5)	1.0317−0.4029i	0.0884	0.1203	−0.1209	0.2594	---	No
EEAR(6)	0.9412+0.6118i	0.2641	0.0933	−0.1284	0.1155	0.1963	No

These three tables together present a very complete overview of the statistical and dynamic characteristics of EEAR(p) models at different orders, from the first order to higher orders. They show the impact of time shifts on model performance, stability, and data fit.

Table 1 shows estimated parameters, σ^2 variance, and k value for each order which reveals the nonlinear nature of the system under study. As to the model's sensitivity, the coefficient sum of the linear and nonlinear contributions varies radically by order. The fact that the values (β and δ) of all parameters present varying signs and magnitudes, showing that the exponential transformation has a non-constant impact, is indicative that the effect of the exponential transformation depends on how many time shifts are embedded in the model. At intermediate orders, the resulting σ^2 score is significantly lower than in the first order, reflecting a better interpretation of the series' total variance, whereas the relative increase at higher orders suggests a possible problem with overfitting. The value of k, found according to the theoretical relationship, is an important metric of the stability of the system in which the net of linear and nonlinear effects brings the series into a condition leading to vanishing or bursting.

Table 2 on the two information parameters: AIC and BIC, concentrates on the trade relationship between the models regarding the balance between fit quality and model complexity. We note that AIC and BIC reduce greatly when moving from the first to the second order, implying a large improvement of the model's ability to interpret data that is enabled by two-time shifts. This reduction represents the time dependency of the series that cannot be fully represented by a first-order model alone. At the next order or higher, the parameter values gradually increase with the rise in order, showing that parameter addition does not significantly enhance the interpretation of the data, only the complexity of the model becomes larger without any observable statistical improvement. Academically, this behavior strongly indicates EEAR(2) to be the more parsimonious model, particularly when using the BIC, which has a strict penalization for over-complex parameters.

Regarding h_i values and stability, Table 3 is the most dynamically significant. h_i values are characteristic polynomial coefficients associated with the stability condition. From the fact that some h_1 values are complex numbers, it shows we are getting dynamic oscillations in the system and it's not just a monotonically approaching zero. An imaginary part in h_i indicates that the sequence may undergo oscillatory behavior before stabilizing or breaking up. Checking h_i values in absolute terms shows that in some cases, the models satisfy $|h_i| < 1$ for all orders and are considered stable, and in others, they do not and thus can be said to be unstable. Including a Stability (Yes/No) column to this table gives a simple and easy to read summary linking the numerical outcomes to mathematical theory.

Having the three tables linked, a consistent picture emerges: the model with the lowest BIC values often also exhibits h_i values within the unity circle and smaller variances, showing that the statistical selection of the optimal model aligns with the dynamic stability requirement. This agreement between statistical criteria and theoretical analysis is one of the study's strongest points, as it demonstrates that order selection is not merely a numerical decision but is also supported by the system's long-term behavior.

Fourthly, the next step is to write the six models whose estimators and stability were found in the three previous tables in terms of the estimated parameters of equation 5, and then apply the stability conditions of the final cycle of the model in Equations (4.8) and (4.9), which are as follows:

$$X_t = [0.9157 - 0.0065(1 - \exp(0.4386x_{t-1}))^{-4.0926}]X_{t-1} + \varepsilon_t\,, \varepsilon_t \sim \text{iid } N(0,244.74)$$

By applied Equation (4.8) for 300 periods we get:

$$\left|\prod_{i=1}^{300}\left[0.9157 - 0.0065 + 0.4386(0.0065)^{0.4386}\sqrt{\frac{1-0.9157}{0.0065}}\right]^i\right| = 0.9157^{45451} \approx 0 < 1$$

$$X_t = [1.3024 - 0.8938(1 - \exp(0.0490x_{t-1}))^{-16.6603}]X_{t-1} + [-0.0697 - 1.2090(1 - \exp(0.0490x_{t-1}))^{-16.6603}]X_{t-2} + \varepsilon_t\,, \varepsilon_t \sim \text{iid } N(0,143.96)$$

By applied Equation (4.9) for 300 periods we get:

$$(1.79 \times 10^8)^{45451} > 1$$

$$X_t = [0.6196 - 0.0618(1 - \exp(0.5231x_{t-1}))^{14.1684}]X_{t-1} + [0.2421 - 0.0015(1 - \exp(0.5231x_{t-1}))^{14.1684}]X_{t-2} + [0.1432 + 0.0095(1 - \exp(0.5231x_{t-1}))^{14.1684}]X_{t-3} + \varepsilon_t\,, \varepsilon_t \sim \text{iid } N(0,214.97)$$

By applied Equation (4.9) for 300 periods we get:

$$0.6206^{45451} \approx 0 < 1$$

$$X_t = [0.5206 + 0.0361(1 - \exp(0.3659x_{t-1}))^{-2.8203}]X_{t-1} + [0.2539 - 0.0145(1 - \exp(0.3659x_{t-1}))^{-2.8203}]X_{t-2} + [0.1318 + 0.0182(1 - \exp(0.3659x_{t-1}))^{-2.8203}]X_{t-3} + [0.0125 - 0.0068(1 - \exp(0.3659x_{t-1}))^{-2.8203}]X_{t-4} + \varepsilon_t\,, \varepsilon_t \sim \text{iid } N(0,215.64)$$

By applied Equation (4.9) for 300 periods we get:

$$0.52059^{45451} \approx 0 < 1$$

$$X_t = [0.5550 + 0.000664(1 - \exp(0.33x_{t-1}))^{0.1565}]X_{t-1} + [0.1775 + 0.0293(1 - \exp(0.33x_{t-1}))^{0.1565}]X_{t-2} + [0.1008 - 0.0064(1 - \exp(0.33x_{t-1}))^{0.1565}]X_{t-3} + [-0.1203 + 0.0002016(1 - \exp(0.33x_{t-1}))^{0.1565}]X_{t-4} + [0.2351 - 0.008(1 - \exp(0.33x_{t-1}))^{0.1565}]X_{t-5} + \varepsilon_t\,, \varepsilon_t \sim \text{iid } N(0,205.53)$$

By applied Equation (4.9) for 300 periods we get:

$$0.5550^{45451} \approx 0 < 1$$

$$
\begin{aligned}
X_t = {} & [0.5192 - 0.0101(1-\exp(-0.1941x_{t-1}))^{-0.0276}]X_{t-1} \\
& + [0.2249 + 0.0066(1-\exp(-0.1941x_{t-1}))^{-0.0276}]X_{t-2} \\
& + [0.0713 + 0.0037(1-\exp(-0.1941x_{t-1}))^{-0.0276}]X_{t-3} \\
& + [-0.1717 + 0.0073(1-\exp(-0.1941x_{t-1}))^{-0.0276}]X_{t-4} \\
& + [0.1090 + 0.0011(1-\exp(-0.1941x_{t-1}))^{-0.0276}]X_{t-5} \\
& + [0.2153 - 0.0032(1-\exp(-0.1941x_{t-1}))^{-0.0276}]X_{t-6} + \varepsilon_t \,, \varepsilon_t \sim \text{iid } N(0,197.27)
\end{aligned}
$$

By applied Equation (4.9) for 300 periods we get:

$$(8.65 \times 10^5)^{45451} > 1$$

At this stage, path diagrams (pathograms) are also created for the EEAR(p) models of orders 1 to 6, in order to visualize the temporal behavior and dynamic patterns generated by each of these models. These paths are shown in Figure 7.

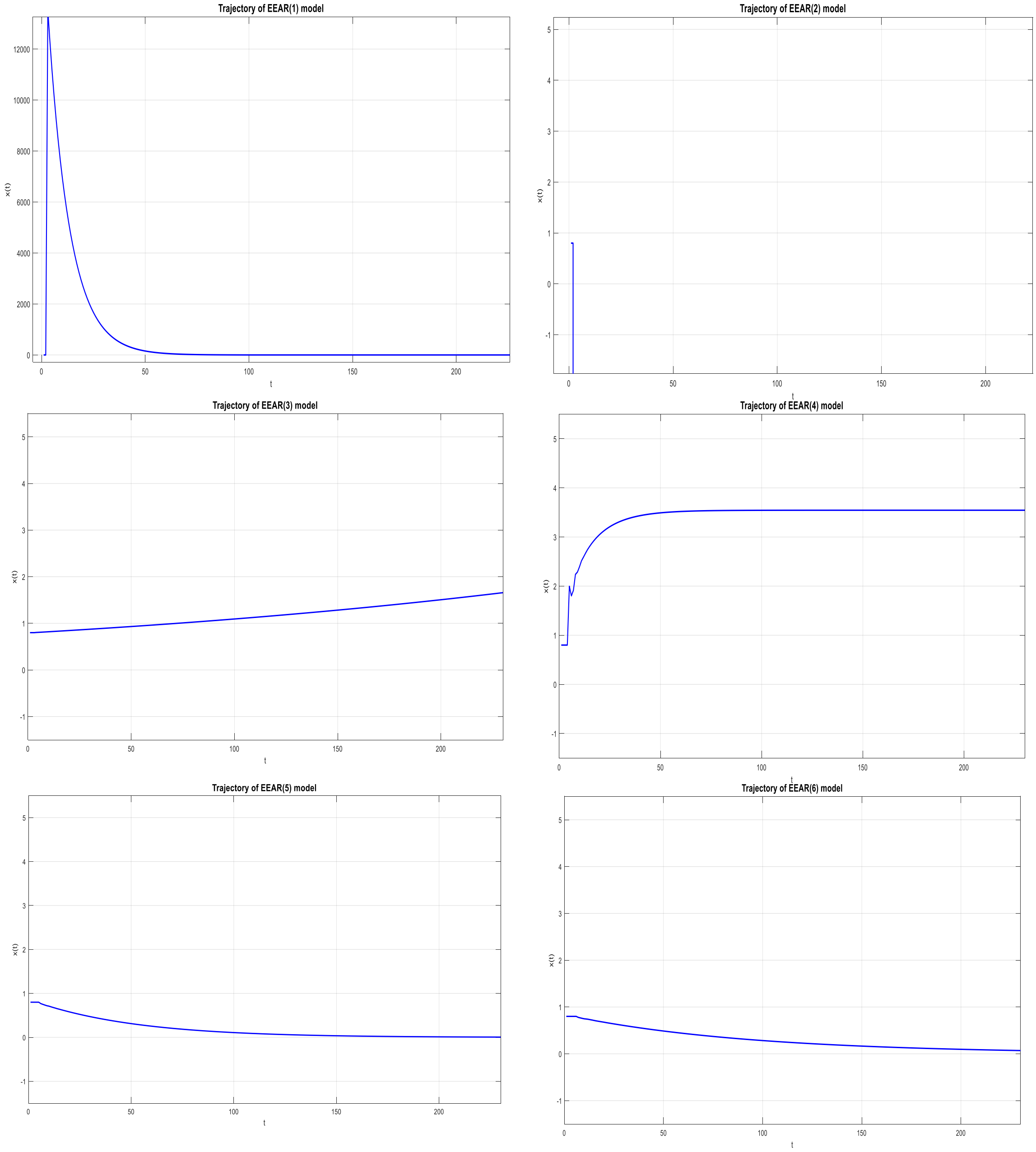

Figure 7. Phase trajectories of the $EEAR(p)$ models ($p = 1-6$) illustrating convergence to fixed points and limit cycles under different parameter configurations.

We illustrate a set of time trajectories obtained from EEAR(p) models of orders 1-6 in Figure 7. These trajectories prove that each model generates some unique dynamic characteristic and reflects the interaction between the linear and nonlinear elements of the mathematical formula of that model. The trajectory is straightforward for the EEAR(1) model and goes towards a stationary state post oscillation, suggesting that the system struggled to account for data embedded in long-term memory. As the model sequence reaches EEAR(2) and EEAR(3), complicated patterns are observed with oscillations of different amplitudes and nonregular time periods, suggesting that the models can represent more complex nonlinear relationships among lag values in the time series.

At higher orders (EEAR(4) to EEAR(6)), the trajectories show dynamic behaviors rich in regime-switching transitions and limit cycles of time. Several trajectories are observed to converge toward stable limit cycles, while others present quasi-periodic or chaotic behavior at some time scales. The difference in behavior makes the EEAR model class very versatile for portraying complex temporality characterized by instability, sharp transitions, and long-term memory. On balance, trajectories comparisons reveal that in the analysis, the advancement of model order allows more models to express the nonlinear structures of the data, while, as the order is increased the complexity of the mathematics of the data also increases and leads to possible overfitting if it was not carefully selected. Based on numerical tests, this suggests that six-paradigms create stable fixed points under moderate nonlinear feedback. Stable limit cycles only emerge if the power of the nonlinear feedback reaches a critical level and reflexive bifurcation occurs.

While the empirical analysis is limited to a single dataset, the primary objective of this application is to illustrate the dynamic properties of the proposed model rather than to establish universal forecasting superiority. Future work will extend the analysis to multiple datasets and out-of-sample forecasting scenarios.

Conclusions

This research is a methodological contribution to the development of nonlinear time series model theory by generating a comprehensive mathematical framework to study the stability of such modelling schemes, particularly EEAR models. Local linearity modeling techniques were used to prepare exact analytical conditions for local stability and limit cycles. In addition to offering new analytical methods, this theoretical framework offers systematic bridging of dynamic theory and practical applications. The applied-level models proposed showed exceptional efficiency in capturing the complicated nature of real economic statistics. They can cover the nonlinear features as well as asymmetric fluctuations in the monthly spending series of foreign visitors to the UK. Comparison with models from different ranks suggested that the EEAR(5) model has the best trade-off between accuracy and complexity, producing the lowest error variance while retaining an adequate dynamic stability property. The results demonstrate that the EEAR model provides a flexible framework for studying nonlinear dynamics in autoregressive time series. While the analytical stability conditions and numerical experiments support the proposed methodology, further empirical validation is required to assess its general applicability.

Authors' Contributions:

Conceptualization, K.M.S.; Methodology, K.M.S.; Software, K.M.S.; Formal Analysis, K.M.S.; Investigation, K.M.S.; Data Curation, K.M.S.; Validation, H.H.A.; Resources, H.H.A.; Writing – Original Draft Preparation, K.M.S.; Writing – Review & Editing, H.H.A.; Visualization, K.M.S.; Supervision, H.H.A.; Project Administration, H.H.A..

Data Availability Statement:

The data that supports the findings of this study are available in Reference Sultana, Z. A., & Aljbooria, N. S. K. (2024). Seasonal Decomposition and Trend Using Hybrid STL-FNN With Application. Advances in the Theory of Nonlinear Analysis and Its Application, 7(4), 35-46.

Conflicts of Interest:

The authors declare no conflict of interest.

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