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Research article

A Novel Alpha Power Gumbel-X Family of Distributions with Exponential Baseline

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ABSTRACT

This study introduces the novel alpha power Gumbel-X (NAPG-X) family of distributions, developed through the T-X transformation with a logarithmic generalizer. The NAPG-Exponential (NAPGEX) distribution is studied as a sub-model, with key mathematical properties derived, including the probability density function, cumulative distribution function, moments, moment generating function, Rényi entropy, and order statistics. The hazard rate function exhibits versatile shapes including increasing-decreasing, reversed-J, and L-shaped, making it suitable for diverse reliability applications. Ten classical estimation methods are evaluated through extensive Monte Carlo simulations across varying sample sizes and three parameter combinations. Results demonstrate that maximum likelihood and Anderson-Darling consistently provide superior performance with minimal bias, mean relative errors and root mean square errors. Furthermore, the practical applicability of the NAPGEX distribution is validated using three real-life datasets. Comprehensive comparisons using various performance measures reveal that the NAPGEX significantly outperforms competing models. These findings establish the NAPG-X family as a flexible and powerful tool for modeling asymmetric, positively skewed lifetime datasets across several scientific disciplines

1. Introduction

Over the past several decades, classical distributions have been widely applied to data modeling in various fields, such as biomedical analysis, reliability engineering, economics, forecasting, astronomy, demography, and insurance. In proposing new statistical distributions, many general methods have since been introduced to create new families of distributions (Elbatal et al. [1]). One common approach is generalization, which introduces additional shape parameters from the generalized family, thereby increasing the distribution's flexibility. These new shape parameters adjust the tail weight of the resulting compound distribution, often adding skewness. Extending classical distributions is a longstanding practice in statistics, as crucial as other applied problems in the field. Such generalizations typically add location, scale, or shape parameters to the initial model, a focus area in statistics that has garnered considerable attention (Adubisi et al. [2]).

Numerous distributions have been developed over recent years. For example, Azzalini [3] introduced the skew-normal distribution by adding a parameter to increase the flexibility of the normal distribution. Mudholkar and Srivastava [4] applied a similar technique to the two-parameter Weibull distribution. Marshall and Olkin [5] developed a method to expand a family of distributions by adding parameters. Eugene et al. [6] presented the beta generalized distribution family with two additional shape parameters derived from the logit transformation of a beta random variable. Cordeiro and de Castro [7] explored a generalized family of distributions based on the Kumaraswamy distribution. Other generalizations, like the gamma and McDonald families, were presented by Zografos and Balakrishnan [8], Ristic and Balakrishnan [9], and Alexander et al. [10].

Furthermore, Alzaatreh et al. [11] introduced the T-X family, a novel method for creating continuous distribution families, while Mahdavi and Kundu [12] proposed another new method for deriving distributions. These approaches have led to the development of various family of distributions which include but not limited to a new class of continuous distributions called the generalized Burr X-G family by Aldahlan et al. [13], T-R {Y} power series family by Osatohanmwen et al. [14], Topp-Leone exponential-G family by Sanusi et al. [15], exponential T-X family by Ahmad et al. [16], Type II Quasi Lambert-G family by Hamedani et al. [17], Gull Alpha power family by Mutua et al. [18], generalized alpha exponent power family by Hussain et al. [19], Marshall–Olkin Weibull generated family by Klakattawi et al. [20], new Generalized Odd Fréchet-G family by Sadiq et al. [21], Ramos-Louzada Generator (RL-G) family by Okutu et al. [22], alpha power Marshall-Olkin-G family by Eghwerido et al. [23], New Type-1 Alpha power family by Tekle et al. [24], Alpha-power generalized odd generalized exponential-G family by Abdulkadir et al. [25], Sine pie-power odd-G family by Sapkota et al. [26], DUS Topp- Leone family by Ekemezie et al. [27], new flexible exponent power family by Shah et al. [28], Generalized Alpha-Beta-power family by Semary et al. [29] and new odd reparametrized exponential transformed-X family by Orji et al. [30].

Despite the availability of various distributions for diverse fields, there remains a need for more adaptable distributions suited to a broad range of situations. The novel alpha power Gumbel-X (NAPG-X) family of distributions proposed in this study represents advancement in this direction. This NAPG-X family shall extend classical distributions, by introducing additional parameters that allow greater control over the distribution's shape. To evaluate the inferential performance, ten classical estimation methods are investigated through extensive Monte Carlo simulations under multiple parameter settings and sample sizes.

This article is prearranged as follows: Section 2 introduces the novel alpha power Gumbel-X (NAPG-X) family of distributions. Section 3 focuses on the NAPG-Exponential (NAPGEX) distribution derived from NAPG-X family. Section 4 outlines the key statistical properties of the NAPGEX model. Section 5 describes the estimation methods considered, and Section 6 reports the results of the Monte Carlo simulation study used to assess the performance of the estimators. Section 7 demonstrates the practical usefulness of the NAPGEX distribution through applications to three real-life datasets. Section 8 concludes the article.

2. Genesis of the NAPG-X Family of Distributions

The new family of distributions in this study is developed from the extended Gumble distribution. The probability density function (pdf) of the novel alpha power Gumbel (NAPG) distribution is $r(t) = \varphi \log(\alpha) e^{-\varphi t} \alpha^{-e^{-\varphi t}}, \quad t \in \mathbb{R}, \alpha > 1, \varphi > 0$. The cumulative distribution function (cdf) by integrating $r(t)$ is given as $R(t) = \alpha^{-e^{-\varphi t}}$. Given that $T \sim NAPG(\alpha, \varphi)$ such that $T \in (a, b)$ for $-\infty \leq a < b < \infty$. Next, utilizing the procedure introduced by Alzaatreh et al. [11] with the log function $W[G(x;\xi)] = log\{-log[1 - G(x;\xi)]\}$ as the generalizer, the cdf of the new family of distributions is specified as

$$F(x;\xi) = \int_{a}^{\log\{-\log[1-G(x;\xi)]\}} r(t)dt = \alpha^{-\{-\log[1-G(x;\xi)]\}^{-\varphi}}, \tag{2.1}$$

and the corresponding pdf by differentiating Equation (2.1) is specified as

$$f(x;\xi) = \frac{\varphi \log(\alpha) g(x;\xi) \alpha^{-\{-\log[1-G(x;\xi)]\}^{-\varphi}} \{-\log[1-G(x;\xi)]\}^{-(\varphi+1)}}{1-G(x;\xi)}, \tag{2.2}$$

where $G(x;\xi)$ and $g(x;\xi)$ are the cdf and pdf of any baseline distribution with vector parameter (ξ). The survival $s(x;\xi)$ and failure rate $h(x;\xi)$ functions are specified, respectively as

$$s(x;\xi) = 1 - \alpha^{-\{-\log[1-G(x;\xi)]\}^{-\varphi}} \tag{2.3}$$

and

$$h(x;\xi) = \frac{\varphi \log(\alpha) g(x;\xi) \alpha^{-\log[1-G(x;\xi)]}}{-\varphi} \{-\log[1-G(x;\xi)]\}^{-(\varphi+1)} 1 - G(x;\xi) 1 - \alpha^{-\log[1-G(x;\xi)]} - \varphi. \tag{2.4}$$

This family is to be known as the novel alpha power Gumbel-X (NAPG-X) family of distributions. It is important to state that

$r(t)$ is a valid pdf introduced by Hossam et al. [31] and $log\{-log[1-G(x;\xi)]\}$ satisfies the conditions outlined by Alzaatreh et al. [11]. Clearly, $\int_{-\infty}^{\infty}\varphi\log(\alpha)e^{-\varphi t}\alpha^{-e^{-\varphi t}}\,dt=1$ and

i. $W[G(x;\xi)]\in[a,b]$.
ii. $W[G(x;\xi)]$ is differentiable and monotonically non-decreasing.
iii. $W[G(x;\xi)]\to a$,as $x\to-\infty$ and $W[G(x;\xi)]\to b$ as $x\to+\infty$.

3. The NAPG-exponential (NAPGEX) Distribution

Here, the extension of the exponential (EX) distribution using the NAPG-X family is introduced. The cdf and pdf of the EX-distribution are respectively, specified as

$$G(x;\lambda)=1-e^{-\lambda x},x>0 \tag{3.1}$$

and

$$g(x;\lambda)=\lambda e^{-\lambda x}. \tag{3.2}$$

where $\lambda>0$ is the scale parameter. Inserting $G(x;\lambda)$ and $g(x;\lambda)$ into Equations (2.1) and (2.2), the cdf and pdf of the NAPGEX distribution are derived. Let X denotes a non-negative continuous random variable such that $X\sim NAPGEX\,(\xi)$, where $\xi=(\varphi,\alpha,\lambda)$. The simplified forms of the cdf and pdf are expressed as

$$F(x;\xi)=\alpha^{-(\lambda x)^{-\varphi}}\quad,x>0 \tag{3.3}$$

and

$$f(x;\xi)=\varphi\log(\alpha)\lambda\alpha^{-(\lambda x)^{-\varphi}}(\lambda x)^{-(\varphi+1)}. \tag{3.4}$$

respectively. where $\varphi>0,\alpha>1$ are the shape parameters and $\lambda>0$ is the scale parameter. The validity of the pdf is derived as follows:

$$\int_0^{\infty}\varphi\log(\alpha)\lambda\alpha^{-(\lambda x)^{-\varphi}}(\lambda x)^{-(\varphi+1)}dx,$$

Let $z=\lambda x\implies dz=\lambda dx$, then
$dx=\frac{1}{\lambda}dz$.
Hence,

$$\varphi\log(\alpha)\int_0^{\infty}\alpha^{-(z)^{-\varphi}}(z)^{-(\varphi+1)}dz,$$

Let $u=(z)^{-\varphi}\implies du=-\varphi z^{-(\varphi+1)}dz$, then $z^{-(\varphi+1)}dz=-\frac{1}{\varphi}du$.
Given the limits: $z\to0,u\to\infty$ and $z\to\infty,u\to0$, we have

$$\varphi\log(\alpha)\int_{\infty}^{0}\alpha^{-u}\left(-\frac{1}{\varphi}\right)du,$$

Simplifying the preceding function, we have

$$\log(\alpha)\int_0^{\infty}e^{-u\log(\alpha)}du=\frac{\log(\alpha)}{\log(\alpha)}=1.$$

The pdf of the NAPGEX is valid. Likewise, the cdf of the NAPGEX distribution satisfies the boundary conditions of a distribution as follows: As $x\to0$, $\lim_{x\to0}F(x;\xi)=0$, and $x\to\infty$, $\lim_{x\to\infty}F(x;\xi)=1$ given that $\alpha>1$.

Figure 1 presents the density function plots of the NAPGEX distribution which depicts unimodal and positive-skewed shapes while Figure 2 depict the plots of the failure rate function to be increasing-decreasing, reversed-J shaped and L-shaped, respectively.

4. Properties of the NAPGEX Distribution

Here, the statistical properties of the NAPGEX distribution are presented. These include the reliability analysis, quantile function, moments, moment generating function, Rényi entropy and order statistics.

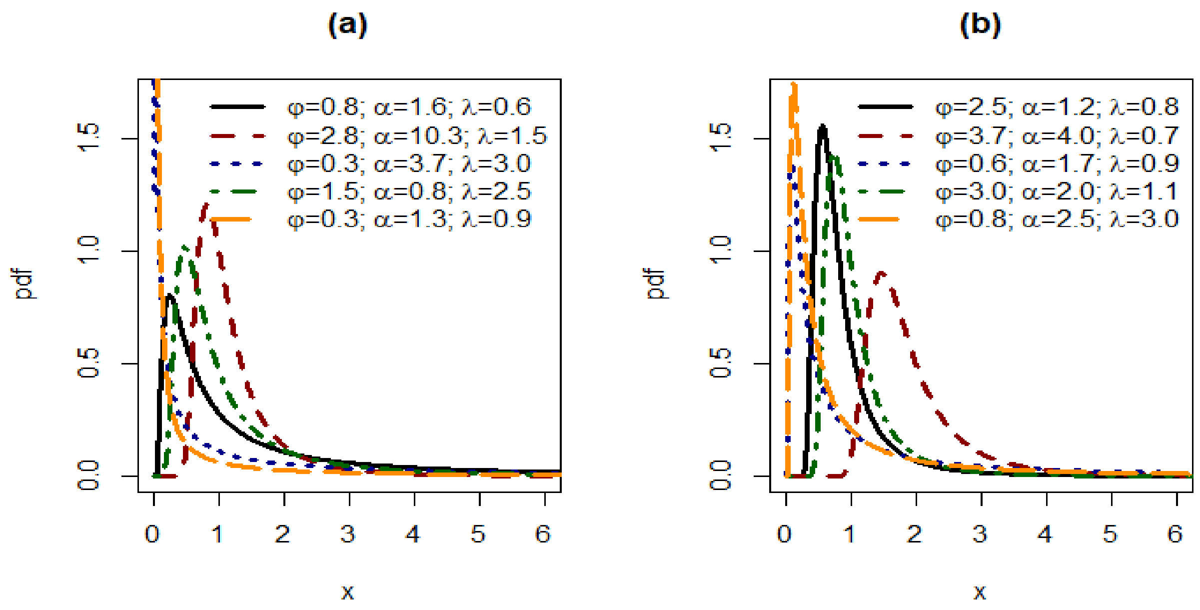

Figure 1. The density function (pdf) plots of the NAPGEX distribution.

4.1. Reliability Analysis

The survival $s(x;\xi)$ and failure rate $h(x;\xi)$ functions of the NAPGEX distribution are expressed as

$$S(x;\xi) = 1 - \alpha^{-(\lambda x)^{-\varphi}} \tag{4.1}$$

and

$$h(x;\xi) = \frac{\varphi\log(\alpha)\lambda\alpha^{-(\lambda x)^{-\varphi}}(\lambda x)^{-(\varphi+1)}}{1-\alpha^{-(\lambda x)^{-\varphi}}}. \tag{4.2}$$

Likewise, the cumulative failure rate $H(x;\xi)$, reversed failure rate $R(x;\xi)$ and odds rate $O(x;\xi)$ are respectively expressed as

$$H(x;\xi) = -\log\left[1-\alpha^{-(\lambda x)^{-\varphi}}\right], \tag{4.3}$$

$$R(x;\xi) = \frac{\varphi\log(\alpha)\lambda\alpha^{-(\lambda x)^{-\varphi}}(\lambda x)^{-(\varphi+1)}}{\alpha^{-(\lambda x)^{-\varphi}}}, \tag{4.4}$$

$$O(x;\xi) = \frac{\alpha^{-(\lambda x)^{-\varphi}}}{1-\alpha^{-(\lambda x)^{-\varphi}}}. \tag{4.5}$$

4.2. Quantile Function

The quantile function $Q(u)$ also called the inverse of the cumulative density function (cdf) is another way a probability distribution can be well defined in a simple form.

$$G(u) = F^{-1}(x), \tag{4.6}$$

where u~uniform (0, 1). The quantile function (qf) of the NAPGEX distribution based on Equation (4.6) is specified as follows

$$Q(u;\xi) = \frac{1}{\lambda}\left[\left(\frac{-\log u}{\log\alpha}\right)^{-\frac{1}{\varphi}}\right], \quad 0 < u < 1. \tag{4.7}$$

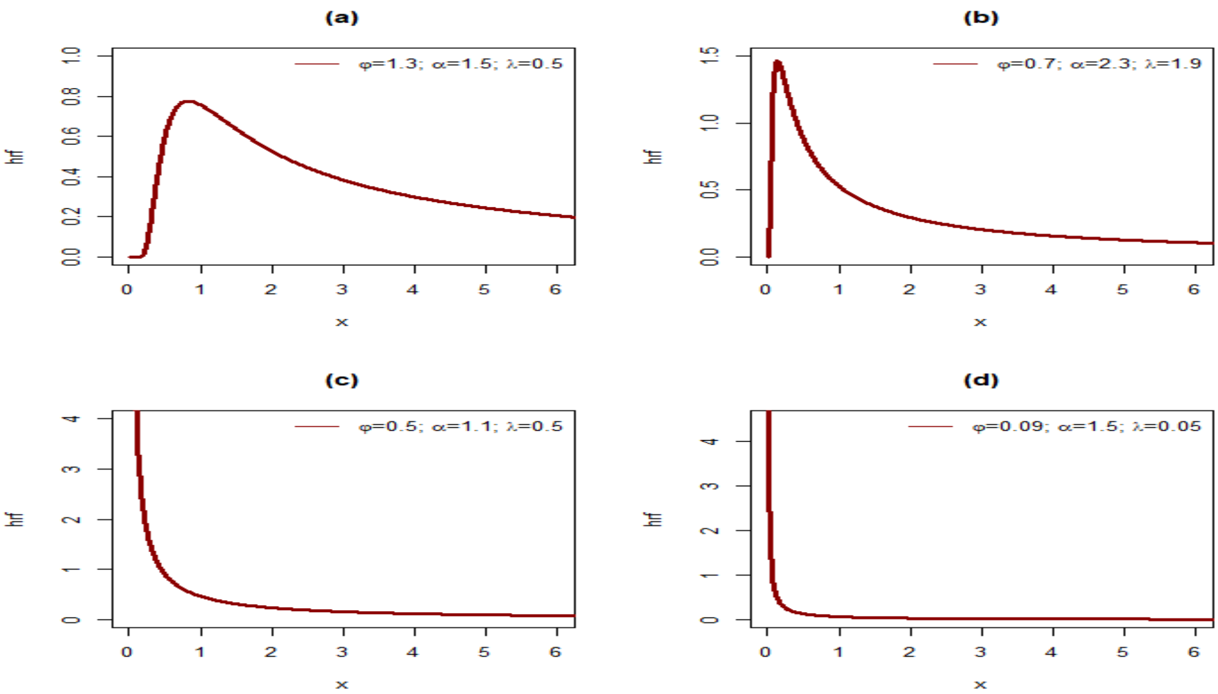

Figure 2. The failure rate function (frf) plots of the NAPGEX distribution.

4.3. Ordinary moment

The r^{th} moment of the random variable X about the origin that assumes the NAPGEX distribution is expressed as:

$$E(X^r) = \int_0^{+\infty} x^r f(x)dx = \varphi\log(\alpha)\lambda^{-\varphi}\int_0^{\infty} x^r \alpha^{-(\lambda x)^{-\varphi}} x^{-(\varphi+1)}dx. \qquad (4.8)$$

Let $z = (\lambda x)^{-\varphi}$,

$dz = -\varphi\lambda^{-\varphi}x^{-(\varphi+1)}dx \Rightarrow x^{-(\varphi+1)}dx = -\frac{1}{\varphi}\lambda^{\varphi}dz$ and $x^r = \lambda^{-r}z^{-r/\varphi}$.

Given the limits: $x \to 0 \Rightarrow z \to \infty$ and $x \to \infty \Rightarrow z \to 0$. Equation (4.8) can be expressed as

$$E(X^r) = \varphi\log(\alpha)\lambda^{-\varphi}\int_{\infty}^{0} \lambda^{-r}z^{-r/\varphi}\alpha^{-z} - \frac{1}{\varphi}\lambda^{\varphi}dz. \qquad (4.9)$$

Simplifying the preceding equation, we have

$$E(X^r) = \log(\alpha)\lambda^{-r}\int_0^{\infty} z^{-r/\varphi}\alpha^{-z}dz. \qquad (4.10)$$

Let $\alpha^{-z} = e^{-z\log(\alpha)}$ then, Equation (4.10) can be expressed as

$$E(X^r) = \log(\alpha)\lambda^{-r}\int_0^{\infty} z^{-r/\varphi}e^{-z\log(\alpha)}dz. \qquad (4.11)$$

The function in Equation (4.11) can be rewritten as

$$E(X^r) = \log(\alpha)\lambda^{-r}\int_0^{\infty} z^{\left(1-\frac{r}{\varphi}\right)-1}e^{-z\log(\alpha)}dz. \qquad (4.12)$$

Utilizing the gamma integral: $\frac{\Gamma(p)}{q^p} = \int_0^{\infty} z^{p-1}e^{-qz}dz, p > 0, q > 0$. Given that $p = 1 - \frac{r}{\varphi}$ and $q = \log\alpha$, Equation (4.12) is expressed as

$$E(X^r) = \log(\alpha)\lambda^{-r}\frac{\Gamma\left(1-\frac{r}{\varphi}\right)}{(\log\alpha)^{1-\frac{r}{\varphi}}}. \qquad (4.13)$$

Therefore, the r^{th} moment of the NAPGEX distribution is specified as

$$E(X^r) = \lambda^{-r}(\log\alpha)^{\frac{r}{\varphi}}\Gamma\left(1-\frac{r}{\varphi}\right), \quad r < \varphi. \qquad (4.14)$$

The NAPGEX distribution has finite moments only up to order $r < \varphi$. The expressions of the first four moments of the NAPGEX distribution by inserting r = 1,2,3,4 into Equation (4.14), are specified as

$$\acute{\mu}_1 = \lambda^{-1}(\log\alpha)^{\frac{1}{\varphi}}\Gamma\left(1-\frac{1}{\varphi}\right),$$

$$\acute{\mu}_2 = \lambda^{-2}(\log\alpha)^{\frac{2}{\varphi}}\Gamma\left(1-\frac{2}{\varphi}\right),$$

$$\acute{\mu}_3 = \lambda^{-3}(\log\alpha)^{\frac{3}{\varphi}}\Gamma\left(1-\frac{3}{\varphi}\right),$$

$$\acute{\mu}_4 = \lambda^{-4}(\log\alpha)^{\frac{4}{\varphi}}\Gamma\left(1-\frac{4}{\varphi}\right).$$

The summary statistics obtain from the NAPGEX distribution is presented in Table 1. The first four moments, coefficient of variation (CV), standard deviation (SD) and deviation index (DI) are obtained using selected values of the NAPGEX distribution parameters. This descriptive table compares four NAPGEX parameter groupings (Gp): $A = (\alpha = 2.5, \varphi = 2.7, \lambda = 3.0)$, $B = (\alpha = 1.9, \varphi = 2.5, \lambda = 2.5)$, $C = (\alpha = 2.5, \varphi = 3.0, \lambda = 2.5)$ and $D = (\alpha = 1.9, \varphi = 2.7, \lambda = 2.5)$, all four have modest means around (0.16-0.22) but large relative dispersion.

Table 1. Summary statistics of the NAPGEX distribution.

	A	B	C	D
$\acute{\mu}_1$	0.1639	0.2056	0.2171	0.2078
$\acute{\mu}_2$	0.0708	0.0993	0.0704	0.0989
$\acute{\mu}_3$	0.0378	0.0629	0.0704	0.0603
$\acute{\mu}_4$	0.0379	0.0557	0.0574	0.0501
CV	1.2771	1.1619	1.1723	1.2546
SD	0.2094	0.2389	0.2546	1.1723
DI	0.2675	0.2776	0.2984	0.2984

The CV is greater than one in every case, meaning the SD exceeds the mean. Gp C and D show the highest DI, indicating greater sensitivity to upper-tail values, while Gp A, shows the smallest mean and lowest DI. This shows that the NAPGEX is flexible enough to represent different centers and tail sensitivities especially positive and asymmetric data.

4.4. Moment generating function

The moment generating function (MGF) of a random variable X, say $M_x(t)$ is expressed as

$$M_X(t) = \int_0^{+\infty} e^{tx} f(x)dx. \tag{4.15}$$

Using the power-series expansion on Equation (4.15) leads to

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{+\infty} x^r f(x)dx = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r, \tag{4.16}$$

where $\mu_r = E(X^r)$.

Inserting Equation (4.14) into Equation (4.16), the MGF of the NAPGEX distribution is specified as

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\lambda^{-r}(\log\alpha)^{\frac{r}{\varphi}}\Gamma\left(1-\frac{r}{\varphi}\right), \tag{4.17}$$

The MGF does not exist in close form and diverges for t > 0.

4.5 Rényi entropy

The Rényi entropy (Rényi, 1961) is a measure which defines the amount of uncertainty in a system. It is expressed as

$$I_\delta(X) = \frac{1}{1-\delta}\log\int_0^{+\infty} f(x)^\delta dx, \delta \geq 0 \text{ and } \delta \neq 1, \tag{4.18}$$

Hence, utilizing the density function (pdf) in Equation (3.4), we have

$$\int_0^\infty f(x)^\delta dx = [\varphi\log(\alpha)\lambda]^\delta \int_0^\infty \alpha^{-\delta(\lambda x)^{-\varphi}} (\lambda x)^{-\delta(\varphi+1)} dx, \tag{4.19}$$

Let $z = (\lambda x)^{-\varphi}$,

$$dz = -\varphi\lambda^{-\varphi}x^{-(\varphi+1)}dx \Rightarrow dx = -\frac{1}{\varphi}\lambda^{\varphi}x^{(\varphi+1)}dz.$$

Substituting the above expressions into Equation (4.19) and simplifying, the integral reduces to

$$\int_0^\infty f(x)^\delta dx = \frac{[\varphi\log(\alpha)]^\delta}{\varphi}\lambda^{\delta-1}\int_0^\infty z^{\frac{\delta(\varphi+1)-1}{\varphi}-1} e^{-\delta z\log(\alpha)}dz, \tag{4.20}$$

Utilizing the gamma integral: $\frac{\Gamma(p)}{q^p} = \int_0^\infty z^{p-1} e^{-qz}dz, p > 0, q > 0$. Given that $p = \frac{\delta(\varphi+1)-1}{\varphi}$ and $q = \delta\log\alpha$, Equation (4.20) is expressed as

$$\int_0^\infty f(x)^\delta dx = \frac{[\varphi\log(\alpha)]^\delta}{\varphi}\lambda^{\delta-1}\frac{\Gamma\left(\frac{\delta(\varphi+1)-1}{\varphi}\right)}{(\delta\log\alpha)^{\frac{\delta(\varphi+1)-1}{\varphi}}}. \tag{4.21}$$

Therefore, the Rényi entropy for the NAPGEX distribution is specified as

$$I_\delta(X) = \frac{1}{1-\delta}\log\left\{\frac{(\varphi\lambda)^{\delta-1}[\log(\alpha)]^{\frac{\delta-1}{\varphi}}}{(\delta)^{\frac{\delta(\varphi+1)-1}{\varphi}}}\Gamma\left(\frac{\delta(\varphi+1)-1}{\varphi}\right)\right\}. \tag{4.22}$$

The Rényi entropy for the NAPGEX distribution depends logarithmically on λ and α, polynomial on φ and it exists for $\delta > \frac{1}{\varphi+1}$.

4.6 Order statistics

Let $x_1, x_2, \dots, x_n$ denote a random sample from any continuous distribution and $x_{1:n}, x_{2:n}, \dots, x_{n:n}$ are order statistics (O.S) obtained from the sample. The p^{th} O.S, say $f_{p:n}(x)$ is expressed as:

$$f_p(x) = \frac{1}{B(p,n-p+1)}[F(x)]^{p-1}[1-F(x)]^{n-p}f(x), \tag{4.23}$$

where $B(.,.)$ denote the beta function. Since $0 \leq F(x) \leq 1$ for $x > 0$. Inserting Equations (3.3) and (3.4) into Equation (4.23), the O.S for the NAPGEX distribution is expressed as

$$f_p(x) = \frac{1}{B(p,n-p+1)}\left[\alpha^{-(\lambda x)^{-\varphi}}\right]^{p-1}\left[1-\alpha^{-(\lambda x)^{-\varphi}}\right]^{n-p} \varphi\log(\alpha)\lambda\alpha^{-(\lambda x)^{-\varphi}}(\lambda x)^{-(\varphi+1)}. \tag{4.24}$$

Simplifying Equation (4.24) leads to

$$f_p(x) = \frac{1}{B(p,n-p+1)}\varphi\lambda\log(\alpha)[\alpha^{-(\lambda x)^{-\varphi}}]^p(\lambda x)^{-(\varphi+1)}\left[1-\alpha^{-(\lambda x)^{-\varphi}}\right]^{n-p}. \tag{4.25}$$

Hence, the O.S of the NAPGEX distribution is specified as

$$f_p(x) = \frac{1}{(\lambda x)^{(\varphi+1)}B(p,n-p+1)}\varphi\lambda\log(\alpha)[\alpha^{-(\lambda x)^{-\varphi}}]^p\left[1-\alpha^{-(\lambda x)^{-\varphi}}\right]^{n-p}. \tag{4.26}$$

More so, the minimum and maximum O.S can be obtained by inserting $p = 1$ and $p = n$ in Equation (4.26). Thus, expressed as

$$f_1(x) = \frac{1}{(\lambda x)^{(\varphi+1)}B(1,n)}\varphi\lambda\log(\alpha)\alpha^{-(\lambda x)^{-\varphi}}\left[1-\alpha^{-(\lambda x)^{-\varphi}}\right]^{n-1}, \tag{4.27}$$

$$f_n(x) = \frac{1}{(\lambda x)^{(\varphi+1)} B(n,1)} \varphi\lambda\log(\alpha)[\alpha^{-(\lambda x)^{-\varphi}}]^n. \quad (4.28)$$

5. Estimation Techniques

This section presents various classical estimation methods for the NAPGEX parameters (Pa.): maximum likelihood estimation (MLE), maximum product of spacing estimation (MPSE), least squares estimation (LSE), weighted LSE (WLSE), Cramer-Von Mises estimation (CVME), Anderson Darling estimation (ADE), right-tailed ADE (RTADE), percentile estimation (PE), Kolmogorov estimation (KE) and method of moments (MOM) estimation.

5.1. The MLE

The estimates of the NAPGEX parameters are found using the MLE via maximization of the log-likelihood function. Consider a random sample $x_1, x_2, \ldots, x_n$ of size (n) observed from the NAPGEX model with unknown parameter vector $\zeta = (\varphi, \alpha.\lambda)^T$. The likelihood function $L(\xi)$ is given as

$$L(\zeta) = \prod_{i=1}^{n} f(x_i;\zeta) = \prod_{i=1}^{n} \left[\varphi\log(\alpha)\lambda\alpha^{-(\lambda x)^{-\varphi}}(\lambda x)^{-(\varphi+1)}\right]. \quad (5.1)$$

The log-likelihood function, say $\ell(\zeta)$ is given as

$$\ell(\zeta) = n\log\varphi + n\log\lambda + n\log(\log\alpha) - \sum_{i=1}^{n}(\lambda x_i)^{-\varphi}\log(\alpha) - (\varphi+1)\sum_{i=1}^{n}\log(\lambda x_i).$$

The associated score function $U(\zeta) = \left(\frac{\partial\ell(\zeta)}{\partial\varphi}, \frac{\partial\ell(\zeta)}{\partial\alpha}, \frac{\partial\ell(\zeta)}{\partial\lambda}\right)^T$ components are

$$U_\varphi(\zeta) = \frac{n}{\varphi} + \log\alpha\sum_{i=1}^{n}(\lambda x_i)^{-\varphi}\log(\lambda x_i) - \sum_{i=1}^{n}\log(\lambda x_i) = 0,$$

$$U_\alpha(\zeta) = \frac{n}{\alpha\log\alpha} - \frac{1}{\alpha}\sum_{i=1}^{n}(\lambda x_i)^{-\varphi} = 0,$$

and

$$U_\lambda(\zeta) = \frac{n}{\lambda} + \frac{\varphi\log\alpha}{\lambda}\sum_{i=1}^{n}(\lambda x_i)^{-\varphi} - \frac{n(\varphi+1)}{\lambda} = 0.$$

The parameter estimates MLE_φ, MLE_α and MLE_λ can be obtained by solving the components of the score vector simultaneously.

5.2. The MPSE

The MPSE estimates of the NAPGEX parameters are found by maximizing the log-geometric mean of the product spacing specified as

$$\Phi(\varphi,\alpha.\lambda) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i(\varphi,\alpha.\lambda), \quad (5.2)$$

where $D_i = F(x_{i:n}|\varphi,\alpha.\lambda) - F(x_{i-1:n}|\varphi,\alpha.\lambda)$, with $F(x_{0:n}|\varphi,\alpha.\lambda) = 0$, $F(x_{n+1:n}|\varphi,\alpha.\lambda) = 1$ and $\sum_{i=1}^{n+1} D_i(\varphi,\alpha.\lambda) = 1$.

5.3. The ADE and RTADE

The ADE estimates of the NAPGEX parameters are found by minimizing the Anderson-Darling statistic specified as

$$\mathrm{AD}(\varphi,\alpha.\lambda) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[\log F(x_{i:n}|\varphi,\alpha.\lambda) + \log S(x_{n+1-i:n}|\varphi,\alpha.\lambda)]. \quad (5.3)$$

Likewise, the RTADE estimates are found by minimizing the right-tailed ADE statistic specified as

$$RT\mathrm{AD}(\varphi,\alpha.\lambda) = \frac{2}{n} - 2\sum_{i=1}^{n}F(x_{i:n}|\varphi,\alpha.\lambda) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S(x_{n+1-i:n}|\varphi,\alpha.\lambda). \quad (5.4)$$

where $F(x_{i:n}|\varphi,\alpha.\lambda)$ and $S(x_{n+1-i:n}|\varphi,\alpha.\lambda) = \left(1 - F(x_{n+1-i:n}|\varphi,\alpha.\lambda)\right)$ are the cdf and survival function of the NAPGEX model, and $x_{i:n}$ denotes the order statistic.

5.4. The CVME

The CVME estimates of the NAPGEX parameters are found by minimizing the CVM statistic specified as

$$\mathrm{Cv}(\varphi,\alpha.\lambda) = \frac{1}{12n} + \sum_{i=1}^{n}[F(x_{i:n}|\varphi,\alpha.\lambda) - \delta_i]^2, \quad (5.5)$$

where $\delta_i = \frac{2(i-1)+1}{2n}$.

5.5. The LSE and WLSE

The LSE estimates of the NAPGEX parameters are found by minimizing the least squares statistic specified as

$$\mathrm{LS}(\varphi, \alpha.\lambda) = \sum_{i=1}^{n}[F(x_{i:n}|\varphi, \alpha.\lambda) - \mathrm{K}_i]^2, \tag{5.6}$$

where $\mathrm{K}_i = \frac{i}{n+1}$.

Likewise, the WLSE estimates are found by minimizing the weighted least squares statistic specified as

$$\mathrm{LS}(\varphi, \alpha.\lambda) = \sum_{i=1}^{n}\Upsilon(\mathrm{i},\mathrm{n})[F(x_{i:n}|\varphi, \alpha.\lambda) - \mathrm{K}_i]^2, \tag{5.7}$$

where $\Upsilon(\mathrm{I},\mathrm{n}) = \frac{(n+1)^2(n+2)}{i(n-i+1)}$.

5.6. The PE

The PE estimates of the NAPGEX parameters are found by minimizing the percentile statistic specified as

$$\mathrm{P}(\varphi, \alpha.\lambda) = \sum_{i=1}^{n}[x_i - Q(u_i|\varphi, \alpha.\lambda)]^2 \tag{5.8}$$

where $\mathrm{u}_i = \frac{i}{n+1}$ and $Q(u_i|\varphi, \alpha.\lambda)$ are the unbiased estimator of $F(x_i|\varphi, \alpha.\lambda)$ and quantile function of the NAPGEX model, respectively.

5.7. The KOLE

The KOLE estimates of the NAPGEX parameters are found by minimizing the Kolmogorov statistic specified as

$$\mathrm{KO}(\varphi, \alpha.\lambda) = \max_{1\le I\le N}\left[\frac{I}{N} - F(x_{i:n}|\varphi, \alpha.\lambda), F(x_{i:n}|\varphi, \alpha.\lambda) - \mathrm{K}_i\right]. \tag{5.9}$$

where $\mathrm{K}_i = \frac{i-1}{n}$.

5.7. The MOM

The MOM estimates of the NAPGEX parameters are found by equating the first two sample moments with their theoretical moments. The MOM functions are specified as

$$\frac{1}{n}\sum_{i=1}^{n}x_i - \acute{\mu}_1 = 0 \text{ and } \frac{1}{n}\sum_{i=1}^{n}x_i^2 - \acute{\mu}_2 = 0, \tag{5.10}$$

where $\acute{\mu}_1 = \lambda^{-1}(\log\alpha)^{\frac{1}{\varphi}}\Gamma\left(1-\frac{1}{\varphi}\right)$ and $\acute{\mu}_2 = \lambda^{-2}(\log\alpha)^{\frac{2}{\varphi}}\Gamma\left(1-\frac{2}{\varphi}\right)$ denote the first and second theoretical moments.

More so, iterative method via *Optim function* (R-program software) is used in estimating the parameters of the NAPGEX distribution in all the estimation methods.

6. Simulations

In this section, simulated datasets are used to evaluate the efficiency and accuracy of several estimators for the parameters of the NAPGEX distribution. The primary objective to determine the most reliable and robust estimator through extensive simulation study conducted over different sample sizes (n). This approach allows for a practical assessment of estimator performance under conditions similar to real-world applications. The simulation procedure is outlined as follows

i. Random samples are generated using the quantile function given in Equation (4.7):

$$Q(u;\xi) = \frac{1}{\lambda}\left[\left(\frac{-\log u}{\log\alpha}\right)^{-\frac{1}{\varphi}}\right],$$

where $\xi = (\varphi, \alpha.\lambda)$ represents the vector of model parameters.

ii. Three initial parameter combination (COM) are considered: $COM1$ ($\varphi = 0.5, \alpha = 1.3, \lambda = 0.9$), $COM2$ ($\varphi = 1.0, \alpha = 2.0, \lambda = 1.5$), and $COM3$ ($\varphi = 1.7, \alpha = 1.5, \lambda = 0.5$). The three parameter combinations were chosen to represent heavy, moderate, and light tailed scenarios, thereby assessing the robustness of the proposed estimators across different distributional shapes and scales.

iii. For each parameter combination, random sample sizes $n = 25, 75, 150$ and 250 are generated. For each sample size, the simulation was replicated 5,000 times to ensure stable estimation of bias, root mean square errors (RMSEs) and mean relative errors (MREs).

The performance of each estimator is assessed using the bias, RMSEs and MREs. For any estimator $\hat{\xi} = \hat{\varphi}, \hat{\alpha}. \hat{\lambda}$, the following performance metrics are computed as:

$$Bias = \frac{1}{5000}\sum_{i=1}^{5000}(\hat{\xi}_i - \xi),\ RMSE = \sqrt{\frac{1}{5000}\sum_{i=1}^{5000}(\hat{\xi}_i - \xi)^2},\ \text{and}\ MRE = \frac{1}{5000}\sum_{i=1}^{5000}|\hat{\xi}_i - \xi|/\xi.$$

These metrics enable a comprehensive comparison of estimator performance across different sample sizes and parameter settings.

Tables 2, 3 and 4 present the results of an extensive simulation study conducted under three distinct parameter combinations to evaluate the performance of several classical estimators for the NAPGEX parameters. The estimators were assessed using Bias, Mean Relative Error (MRE), and Root Mean Square Error (RMSE). Across all parameter combinations and sample sizes, the MLE, ADE, RTADE, CVME, LSE, and WLSE estimators consistently delivered the most robust and reliable performance.

Table 2. The simulation results with COM 1.

Method	**Pa.**	$\boldsymbol{n = 25}$			$\boldsymbol{n = 75}$			$\boldsymbol{n = 150}$			$\boldsymbol{n = 250}$		
		Bias	MRE	RMSE	Bias	MRE	RMSE	Bias	MRE	RMSE	Bias	MRE	RMSE
MLE	φ	0.030	0.060	0.095	0.010	0.020	0.049	0.005	0.010	0.033	0.003	0.006	0.025
	α	0.007	0.005	0.109	0.011	0.008	0.064	0.011	0.009	0.046	0.012	0.009	0.035
	λ	0.056	0.062	0.125	0.062	0.069	0.089	0.062	0.069	0.080	0.063	0.070	0.077
ADE	φ	0.010	0.019	0.091	0.004	0.008	0.051	0.002	0.004	0.036	0.001	0.002	0.027
	α	0.023	0.018	0.115	0.016	0.013	0.069	0.013	0.010	0.049	0.012	0.009	0.037
	λ	0.056	0.062	0.109	0.062	0.069	0.089	0.058	0.065	0.077	0.055	0.061	0.069
RTADE	φ	0.021	0.042	0.114	0.007	0.014	0.060	0.004	0.008	0.042	0.002	0.004	0.032
	α	0.019	0.015	0.116	0.015	0.011	0.068	0.013	0.010	0.049	0.012	0.009	0.038
	λ	0.048	0.053	0.104	0.056	0.062	0.077	0.059	0.065	0.074	0.057	0.063	0.070
MPSE	φ	-0.029	0.085	0.058	-0.016	0.048	0.031	-0.010	0.034	0.020	-0.007	0.026	0.015
	α	0.052	0.123	0.040	0.031	0.071	0.024	0.023	0.050	0.018	0.020	0.039	0.015
	λ	0.053	0.116	0.059	0.058	0.085	0.064	0.060	0.079	0.067	0.060	0.073	0.066
CVME	φ	0.033	0.065	0.117	0.011	0.022	0.061	0.006	0.011	0.041	0.003	0.006	0.031
	α	0.010	0.008	0.120	0.012	0.010	0.071	0.012	0.009	0.051	0.011	0.009	0.039
	λ	0.052	0.057	0.110	0.062	0.069	0.087	0.060	0.067	0.077	0.057	0.063	0.072
LSE	φ	0.000	0.001	0.104	0.001	0.001	0.058	0.001	0.001	0.040	0.000	0.000	0.031
	α	0.031	0.024	0.124	0.019	0.015	0.073	0.015	0.012	0.052	0.013	0.010	0.040
	λ	0.052	0.058	0.107	0.062	0.068	0.085	0.061	0.067	0.077	0.057	0.064	0.073
WLSE	φ	0.005	0.011	0.098	0.004	0.008	0.053	0.003	0.005	0.036	0.001	0.003	0.027
	α	0.027	0.020	0.119	0.016	0.012	0.069	0.012	0.010	0.049	0.011	0.008	0.036
	λ	0.054	0.060	0.106	0.060	0.067	0.087	0.056	0.063	0.079	0.051	0.057	0.075
PE	φ	0.050	0.296	0.100	0.039	0.255	0.078	0.037	0.239	0.075	0.035	0.228	0.071
	α	0.151	0.371	0.116	0.185	0.423	0.142	0.222	0.493	0.171	0.256	0.557	0.197
	λ	0.005	0.387	0.005	-0.003	0.546	0.003	-0.022	0.568	0.025	-0.029	0.598	0.032
KOLE	φ	0.041	0.113	0.082	0.015	0.062	0.030	0.008	0.042	0.015	0.004	0.032	0.009
	α	-0.015	0.111	0.012	0.003	0.069	0.002	0.007	0.050	0.005	0.008	0.038	0.006
	λ	0.072	0.153	0.080	0.068	0.104	0.075	0.065	0.092	0.072	0.060	0.080	0.067
MOM	φ	0.207	0.414	0.248	0.185	0.369	0.221	0.165	0.330	0.199	0.144	0.287	0.176
	α	-0.081	0.063	0.157	-0.068	0.053	0.147	-0.054	0.041	0.136	-0.041	0.032	0.131
	λ	0.011	0.012	0.128	0.015	0.017	0.124	0.014	0.016	0.119	0.019	0.021	0.112

Table 3. The simulation results with COM 2.

Method	**Pa.**	$n = 25$			$n = 75$			$n = 150$			$n = 250$		
		Bias	MRE	RMSE	Bias	MRE	RMSE	Bias	MRE	RMSE	Bias	MRE	RMSE
MLE	φ	0.060	0.060	0.189	0.020	0.020	0.097	0.010	0.010	0.067	0.006	0.006	0.051
	α	0.038	0.019	0.213	0.055	0.027	0.139	0.054	0.027	0.106	0.054	0.027	0.088
	λ	0.042	0.028	0.166	0.054	0.036	0.108	0.055	0.037	0.087	0.055	0.036	0.077
ADE	φ	0.019	0.019	0.182	0.008	0.008	0.102	0.004	0.004	0.071	0.002	0.002	0.054
	α	0.050	0.025	0.215	0.059	0.030	0.143	0.057	0.028	0.110	0.054	0.027	0.090
	λ	0.046	0.031	0.165	0.056	0.037	0.108	0.056	0.038	0.088	0.053	0.036	0.076
RTADE	φ	0.042	0.042	0.228	0.014	0.014	0.120	0.008	0.008	0.083	0.004	0.004	0.063
	α	0.046	0.023	0.207	0.057	0.029	0.135	0.056	0.028	0.106	0.054	0.027	0.088
	λ	0.038	0.025	0.185	0.053	0.035	0.119	0.054	0.036	0.095	0.053	0.035	0.080
MPSE	φ	-0.058	0.170	0.058	-0.031	0.096	0.031	-0.020	0.067	0.020	-0.015	0.052	0.015
	α	0.083	0.213	0.042	0.076	0.144	0.038	0.067	0.112	0.034	0.063	0.093	0.031
	λ	0.064	0.175	0.042	0.062	0.114	0.042	0.060	0.092	0.040	0.058	0.080	0.038
CVME	φ	0.065	0.065	0.234	0.022	0.022	0.121	0.011	0.011	0.082	0.006	0.006	0.062
	α	0.027	0.014	0.223	0.051	0.026	0.142	0.052	0.026	0.108	0.053	0.026	0.089
	λ	0.028	0.018	0.168	0.050	0.033	0.110	0.053	0.035	0.090	0.053	0.035	0.078
LSE	φ	0.001	0.001	0.208	0.001	0.001	0.116	0.001	0.001	0.081	0.000	0.000	0.061
	α	0.048	0.024	0.218	0.057	0.029	0.143	0.056	0.028	0.109	0.055	0.027	0.090
	λ	0.047	0.032	0.172	0.056	0.037	0.113	0.056	0.037	0.092	0.055	0.036	0.079
WLSE	φ	0.011	0.011	0.196	0.008	0.008	0.106	0.005	0.005	0.073	0.003	0.003	0.055
	α	0.052	0.026	0.215	0.058	0.029	0.141	0.054	0.027	0.108	0.051	0.025	0.088
	λ	0.050	0.034	0.168	0.056	0.037	0.110	0.054	0.036	0.088	0.051	0.034	0.075
PE	φ	-0.057	0.417	0.057	-0.081	0.360	0.081	-0.088	0.337	0.088	-0.092	0.322	0.092
	α	0.034	0.537	0.017	-0.016	0.495	0.008	-0.053	0.511	0.027	-0.060	0.548	0.030
	λ	0.278	0.908	0.185	0.376	1.089	0.250	0.379	1.104	0.253	0.425	1.138	0.283
KOLE	φ	0.089	0.237	0.089	0.031	0.125	0.031	0.016	0.084	0.016	0.009	0.063	0.009
	α	-0.025	0.225	0.013	0.030	0.137	0.015	0.041	0.106	0.021	0.045	0.088	0.023
	λ	0.070	0.185	0.047	0.062	0.121	0.042	0.059	0.098	0.040	0.056	0.084	0.037
MOM	φ	0.367	0.367	0.470	0.274	0.274	0.346	0.221	0.221	0.284	0.186	0.186	0.243
	α	-0.113	0.057	0.274	-0.064	0.032	0.209	-0.040	0.020	0.181	-0.018	0.009	0.162
	λ	0.009	0.006	0.227	0.015	0.010	0.189	0.026	0.017	0.173	0.029	0.019	0.157

Under COM 1, these estimators exhibit remarkably low biases, minimal MREs, and small RMSEs across all sample sizes, demonstrating numerical stability and strong accuracy even for moderate sample sizes. In contrast, the MPSE shows only moderate efficiency with moderately inflated relative errors, while the PE and MOM estimators perform very poorly, producing large biases, MREs, and RMSEs that render them unsuitable for dependable inference. For COM 2, which involves moderately larger parameter values and a slightly more complex data-generating mechanism, the strong-performing estimators (MLE, ADE, RTADE, CVME, LSE, and WLSE) continue to show high accuracy, though with marginally increased variability. Meanwhile, the performance of the PE and MOM estimators deteriorates further, yielding unstable estimates with unacceptably high MREs and RMSEs, reinforcing their inadequacy for NAPGEX parameter estimation.

Under COM 3, representing the most challenging setting due to heavier tails and sharper curvature in the underlying distribution, all estimators experience increased estimation difficulty. Nevertheless, the MLE, ADE, RTADE, CVME, LSE, and WLSE remain distinctly superior, maintaining substantially lower biases, MREs, and RMSEs, with accuracy improving steadily as sample size increases. The MPSE becomes increasingly sensitive to parameter values, while both the PE and MOM estimators exhibit extremely large errors, making them practically unusable.

The simulation results indicate that likelihood-based and Anderson Darling type estimators consistently outperform moment and percentile-based methods across all parameter settings. The advantage is most pronounced in heavy-tailed scenarios and small samples. As the sample size increases, bias and RMSE decrease uniformly, confirming the consistency of the proposed estimators.

Table 4. The simulation results with COM 3.

Method	Pa.	$n = 25$			$n = 75$			$n = 150$			$n = 250$		
		Bias	MRE	RMSE	Bias	MRE	RMSE	Bias	MRE	RMSE	Bias	MRE	RMSE
MLE	φ	0.101	0.060	0.320	0.034	0.020	0.165	0.017	0.010	0.113	0.010	0.006	0.086
	α	0.041	0.027	0.176	0.050	0.033	0.111	0.047	0.031	0.082	0.043	0.029	0.068
	λ	0.020	0.041	0.061	0.024	0.047	0.038	0.022	0.044	0.031	0.020	0.040	0.028
ADE	φ	0.032	0.019	0.307	0.013	0.008	0.174	0.007	0.004	0.121	0.003	0.002	0.092
	α	0.059	0.040	0.171	0.055	0.037	0.109	0.048	0.032	0.082	0.044	0.030	0.067
	λ	0.023	0.045	0.059	0.023	0.047	0.037	0.021	0.043	0.030	0.020	0.039	0.025
RTADE	φ	0.069	0.041	0.382	0.024	0.014	0.205	0.013	0.008	0.142	0.007	0.004	0.108
	α	0.051	0.034	0.189	0.051	0.034	0.123	0.045	0.030	0.088	0.041	0.027	0.069
	λ	0.018	0.036	0.068	0.021	0.042	0.041	0.020	0.040	0.030	0.018	0.036	0.025
MPSE	φ	-0.098	0.289	0.058	-0.053	0.162	0.031	-0.034	0.114	0.020	-0.025	0.088	0.015
	α	0.123	0.220	0.082	0.087	0.142	0.058	0.067	0.101	0.045	0.055	0.079	0.037
	λ	0.036	0.065	0.072	0.029	0.043	0.059	0.025	0.034	0.049	0.021	0.029	0.042
CVME	φ	0.108	0.064	0.392	0.037	0.022	0.206	0.019	0.011	0.140	0.011	0.006	0.105
	α	0.045	0.030	0.188	0.051	0.034	0.121	0.044	0.029	0.087	0.039	0.026	0.069
	λ	0.020	0.040	0.068	0.023	0.046	0.042	0.020	0.040	0.031	0.018	0.036	0.025
LSE	φ	0.000	0.000	0.350	0.002	0.001	0.197	0.002	0.001	0.137	0.000	0.000	0.104
	α	0.078	0.052	0.194	0.063	0.042	0.127	0.050	0.033	0.091	0.042	0.028	0.071
	λ	0.028	0.055	0.067	0.025	0.051	0.042	0.021	0.043	0.032	0.018	0.036	0.025
WLSE	φ	0.016	0.010	0.328	0.013	0.008	0.179	0.009	0.005	0.124	0.005	0.003	0.093
	α	0.071	0.047	0.187	0.055	0.037	0.117	0.047	0.031	0.086	0.041	0.027	0.066
	λ	0.026	0.051	0.065	0.023	0.046	0.039	0.021	0.042	0.030	0.018	0.037	0.024
PE	φ	-0.153	0.584	0.090	-0.182	0.510	0.107	-0.186	0.472	0.109	-0.188	0.445	0.111
	α	0.167	0.724	0.111	0.106	0.478	0.070	0.106	0.525	0.070	0.158	0.932	0.106
	λ	0.139	0.459	0.278	0.200	0.547	0.400	0.232	0.671	0.463	0.293	0.887	0.586
KOLE	φ	0.113	0.349	0.066	0.043	0.198	0.026	0.024	0.140	0.014	0.015	0.106	0.009
	α	0.014	0.153	0.009	0.036	0.108	0.024	0.038	0.082	0.025	0.036	0.066	0.024
	λ	0.030	0.067	0.059	0.024	0.042	0.048	0.022	0.034	0.043	0.019	0.028	0.038
MOM	φ	0.164	0.097	0.240	0.126	0.074	0.185	0.109	0.064	0.168	0.095	0.056	0.150
	α	-0.088	0.059	0.185	-0.050	0.033	0.146	-0.025	0.017	0.129	-0.009	0.006	0.116
	λ	0.120	0.239	0.177	0.092	0.184	0.147	0.080	0.159	0.134	0.067	0.133	0.126

7. Applications

In this section, three real-world datasets are analyzed to demonstrate the significance of the NAPGEX distribution in modeling lifetime situations. Parameter estimation for all models is carried out using the maximum likelihood method. To evaluate and compare the models, several statistical measures are used: the negative log-likelihood (-LL), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), corrected Akaike Information Criterion (CAIC), Hannan-Quinn Information Criterion (HQIC), Anderson-Darling statistic (AD*), Cramér-von Mises statistic (CW*) and Kolmogorov-Smirnov statistic with corresponding p-value (KS_{PV}). The model with the lowest values across these criteria is considered the best fit.

Dataset I presented in Table 5 consist of the monthly actual taxes' revenue in Egypt from January 2006 to November 2010. This dataset has been analyzed by Owoloko et al. [32].

Table 5. Dataset I.

5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17.0, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10.0, 4.1, 36.0, 8.5, 8.0, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7.0, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11.0, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8.

Dataset II reported in Table 6, comprise the Arm B patients survival times (in days) from a clinical trial on head and neck cancer conducted by the Northern California Oncology Group, originally reported by Efron [33] and recently analysed by Gillariose et al. [34].

Table 6. Dataset II.

37, 84, 92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159,169, 173, 179, 194, 195, 209, 249, 281, 319, 339, 432, 469, 519,528, 547, 613, 633, 725, 759, 817, 1092, 1245,1331, 1557, 1642, 1771, 1776, 1897,2023, 2146, 2297

Dataset III reported in Table 7, represents the active repair time (hours) for an airborne communication transceiver analysed by Jayalath and Chhikara [35], which was originally reported by Chhikara and Folks [36].

Table 7. Dataset III.

0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0,.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0,
2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5

The summary statistics for Datasets I, II, and III are presented in Table 8. For Dataset I, with a sample size of $n = 59$, the mean is approximately 13.488, and the observations are well spread out. The dataset is right-skewed and platykurtic, as indicated by the skewness (1.568) and kurtosis (2.079) values, suggesting lighter tails, a flatter peak, and fewer extreme values relative to a normal distribution. In contrast, Dataset II, with $n = 63$, has a mean of approximately 3.059 and exhibits slight dispersion, slight right skewness, and a platykurtic shape based on its skewness (0.618) and kurtosis (0.183). Dataset III contains 46 observations with a mean of approximately 3.606 and shows substantial dispersion. It is strongly right-skewed and leptokurtic, as reflected in the skewness (2.795) and kurtosis (8.295) values. Figures 3, 4, and 5 further illustrate the characteristics of the datasets through boxplots, histograms, total test time (TTT) plots, and kernel density plots, in that order of appearance.

Table 8. Summary statistics.

Statistics	Data I	Data II	Data III
n	59	63	46
Range	35.100	3.119	24.300
Mean	13.488	3.059	3.606
Median	10.600	2.996	1.750
Std. Dev.	8.051	0.621	4.944
Variance	64.826	0.385	24.445
$\boldsymbol{Q_1}$	8.450	2.553	0.800
$\boldsymbol{Q_2}$	16.850	3.421	4.375
Skewness	1.568	0.618	2.795
Kurtosis	2.079	0.183	8.295

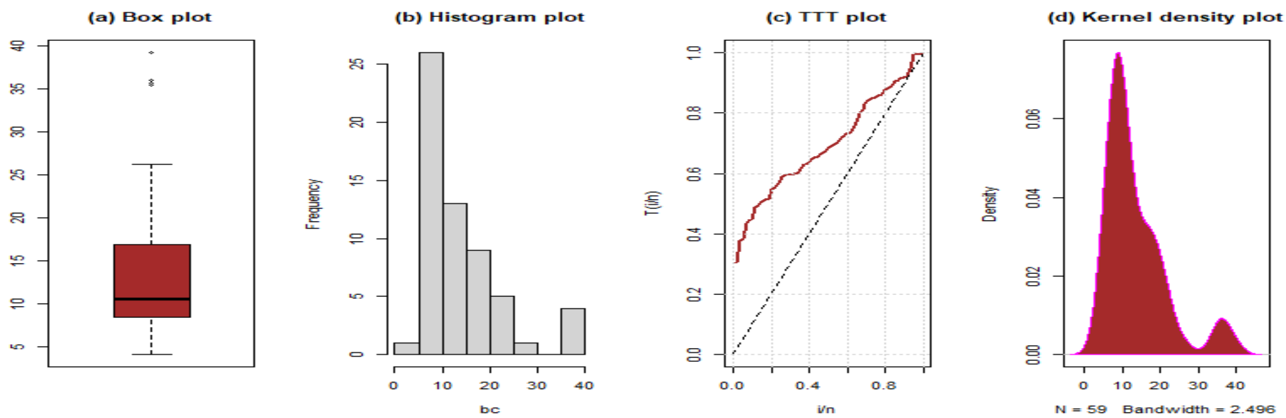

Figure 3. Boxplot, Histogram, TTT, and Kernel-density plots for Dataset I.

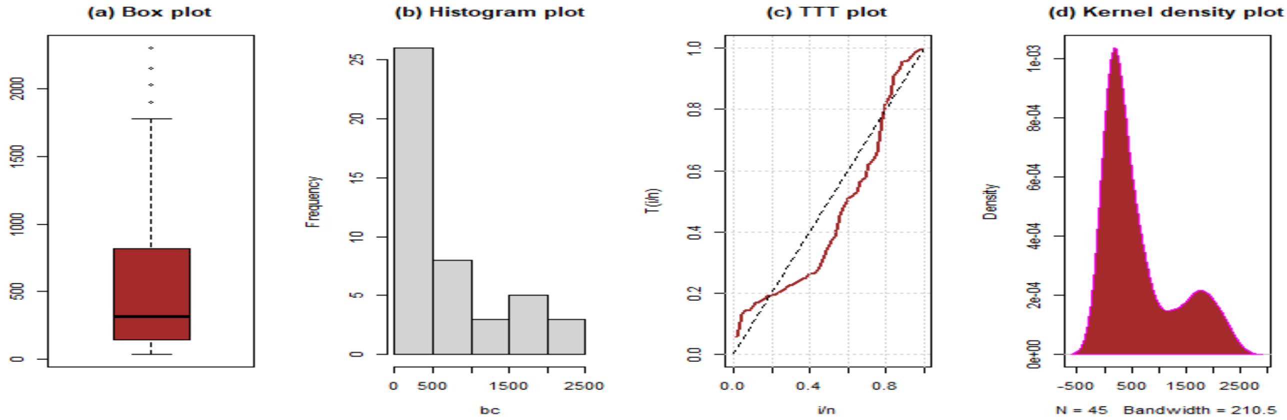

Figure 4. Boxplot, Histogram, TTT, and Kernel-density plots for Dataset II.

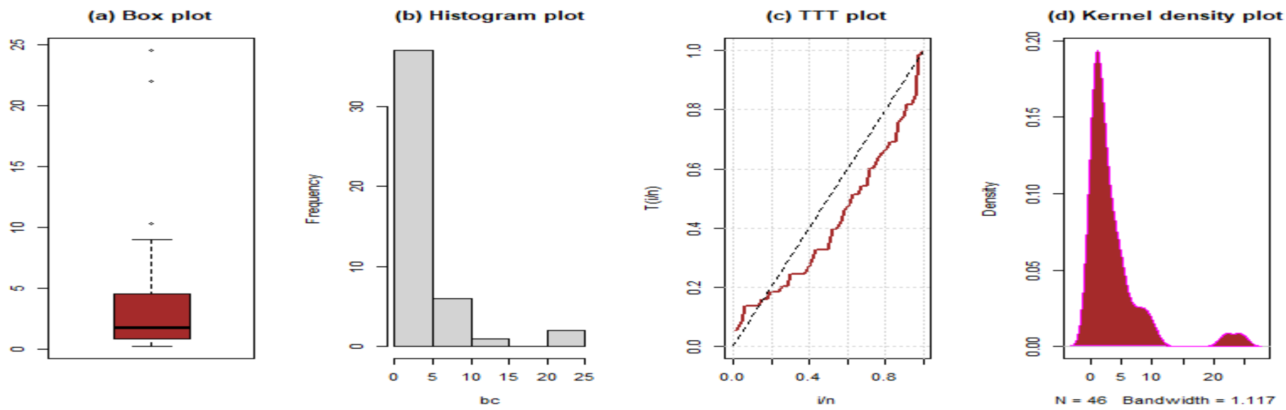

Figure 5. Boxplot, Histogram, TTT, and Kernel-density plots for Dataset III.

Tables 9 and 10 present a concise evaluation of the NAPGEX distribution alongside several competing models, including the exponential (EXP), Topp–Leone Lomax (TLLX), shifted Weibull (SHW), Novel Alpha Power Gumbel (NAPG), Gumbel, exponential Lomax (ELomax), Exponential–Novel Alpha Power Gumbel (ENAPG), and Lomax exponential (LED) distributions, applied to three real datasets. For Dataset I, the NAPGEX distribution consistently demonstrates the strongest performance. It achieves the highest log-likelihood (−188.9) and the most favorable (lowest) values across all information criteria, indicating superior fit and parsimony, as shown in Table 9. Its exceptionally high KS p-value (0.97) further confirms an excellent goodness-of-fit, underscoring its suitability for modeling Dataset I, as reported in Table 10. The ENAPG and NAPG distributions also perform well for this dataset, closely trailing the NAPGEX across the assessment metrics. In contrast, the TLLX, ELomax, and SHW distributions provide the poorest fits, with extremely unfavorable information criteria values and very low KS p-values, indicating substantial deviation from the observed data in Dataset I.

Likewise, for Dataset II, the NAPGEX distribution again emerges as the best-performing model. It attains the highest log-likelihood value (−333.7) and a high KS p-value (0.79), confirming a strong agreement with the observed data. Although the EXP distribution appears to be a close competitor, it remains inferior across all goodness-of-fit statistics. In contrast, the KS p-values for the ELomax, TLLX, and SHW distributions indicate substantial deviations from the empirical observations in Dataset II, as shown in Table 10. For Dataset III, the NAPGEX distribution also achieves the highest log-likelihood (−100.7) and the lowest information criterion values, further affirming its superiority. Its KS p-value of 0.93 provides additional evidence of an excellent fit. The LED and TLLX models perform reasonably well for this dataset, as reported in Table 9, whereas the ELomax and SHW distributions again produce the weakest fits.

Overall, the NAPGEX distribution consistently delivers the best statistical performance across Datasets I, II, and III, making it the most suitable model among those evaluated. The ENAPG and NAPG distributions follow closely behind, as evidenced by the comparative results in Tables 9 and 10.

Table 9. The MLEs and performance measures of competing models for Datasets I, II and III.

Dataset	Models	φ	α	λ	-LL	AIC	BIC	CAIC	HQIC
I	NAPGEX	2.247	3.078	0.115	-188.9	383.9	390.1	384.3	386.3
	EXP	0.074	-	-	-212.5	427.0	429.1	427.1	427.8
	ELomax	1699.46	206.53	16.96	225.0	-444.0	-437.8	-443.6	-441.6
	TLLX	34.18	-22.42	-17.91	21226.5	-42446.9	-42440.7	-42446.5	-42444.5
	SHW	-3.040	-1.524	4.071	523.8	-1041.6	-1035.3	-1041.1	-1039.1
	NAPG	110.02	0.163	-	-195.3	394.5	398.7	394.7	396.2
	Gumbel	0.098	-	-	-234.5	470.9	473.0	471.0	471.7
	ENAPG	4.754	0.198	4.847	-193.0	392.0	398.2	392.4	394.4
	LED	-0.775	0.604	-0.024	-215.2	436.5	442.7	436.9	438.9
II	NAPGEX	1.038	12.97	0.0116	-333.7	673.4	678.8	674.0	675.4
	EXP	0.002	-	-	-335.8	673.7	675.5	673.8	674.4
	ELomax	492.666	1646.80	9.4353	-102.6	211.2	216.6	211.8	213.2
	TLLX	119.509	-1020.64	-0.0018	31519.9	-63033.7	-63028.3	-63033.1	-63031.7
	SHW	-615.782	-8.20E-04	41.6452	25975.3	-51944.7	-51939.3	-51944.1	-51942.7
	NAPG	9.665	0.002	-	-347.0	698.1	701.7	698.4	699.4
	Gumbel	0.002	-	-	-358.2	718.3	720.1	718.4	719.0
	ENAPG	2.142	0.002	3.0939	-347.0	700.0	705.4	700.6	702.0
	LED	-0.618	1.000	-0.0075	-397.7	801.5	806.9	802.0	803.5
III	NAPGEX	1.013	1.994	0.614	-100.7	207.4	212.9	207.9	209.4
	EXP	0.277	-	-	-105.0	212.0	213.8	212.1	212.7
	ELomax	597.056	34.802	11.914	190.1	-374.2	-368.7	-373.6	-372.1
	TLLX	1.262	0.663	1.828	-101.0	208.1	213.6	208.7	210.1
	SHW	-21.962	-1.639	0.379	3028.7	-6051.4	-6045.9	-6050.9	-6049.4
	NAPG	9.859	0.431	-	-118.1	240.2	243.9	240.5	241.6
	Gumbel	0.344	-	-	-128.8	259.5	261.3	259.6	260.2
	ENAPG	2.881	0.431	2.163	-118.1	242.2	247.7	242.8	244.3
	LED	1.013	1.994	0.614	-100.7	207.4	212.9	207.9	209.4

Figures 6, 7 and 8 clearly demonstrate that the NAPGEX distribution provides the best fit for the data, as evidenced by the close alignment between the empirical pdfs with histograms and the empirical cdf plots for Datasets I, II, and III. The log-likelihood profile plots presented in Figures 9, 10 and 11, together with the corresponding MLE parameter estimates, further support this conclusion. Each profile exhibits a smooth U-shaped structure with a distinct and well-defined minimum, confirming that the parameter estimates are both unique and stable for all three datasets. The red dashed line highlights the precise MLE point at which the negative log-likelihood reaches its minimum, validating the robustness of the estimation procedure for the NAPGEX distribution. Additionally, the P–P plots in Figures 12, 13 and 14 show that the NAPGEX distribution lies closest to the diagonal reference line compared to the competing models. These visual assessments

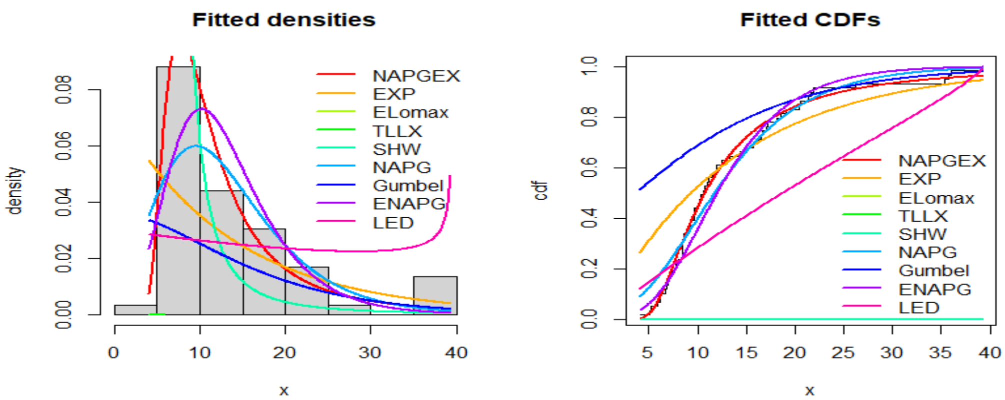

Figure 6. The fitted pdfs (*left panel*) and cdfs (*right panel*) of competing models for Dataset I.

Table 10. The goodness-of-fit measures of competing models for Datasets I, II and III.

Dataset	Models	CW*	AD*	KS	$KS_{P\text{-Value}}$
I	NAPGEX	0.034	0.254	0.065	0.97
	EXP	1.323	7.015	0.303	3.8E-05
	ELomax	19.833	21375.7	1.000	1.13E-51
	TLLX	19.833	42293.9	1.000	1.13E-51
	SHW	19.833	42293.9	1.000	1.13E-51
	NAPG	0.177	1.344	0.114	0.42
	Gumbel	4.865	21.74	0.528	1.06E-14
	ENAPG	0.193	1.238	0.120	0.36
	LED	2.338	10.95	0.341	2.13E-06
II	NAPGEX	0.099	0.703	0.093	0.79
	EXP	0.205	1.494	0.147	0.26
	ELomax	8.200	250.8	0.741	0.00
	TLLX	15.17	32398.8	1.000	0.00
	SHW	15.17	32398.8	1.000	0.00
	NAPG	0.407	3.041	0.187	0.07
	Gumbel	1.610	8.663	0.406	3.16E-07
	ENAPG	0.441	3.208	0.198	0.05
	LED	2.496	19.61	0.332	6.49E-05
III	NAPGEX	0.051	0.363	0.081	0.93
	EXP	0.216	1.285	0.160	0.19
	ELomax	15.454	3489.3	0.984	3.97E-39
	TLLX	0.067	0.407	0.086	0.89
	SHW	15.500	33105.4	1.000	2.22E-40
	NAPG	0.384	2.516	0.199	0.05
	Gumbel	1.477	7.611	0.393	1.13E-06
	ENAPG	0.384	2.515	0.199	0.05
	LED	0.074	0.651	0.129	0.42

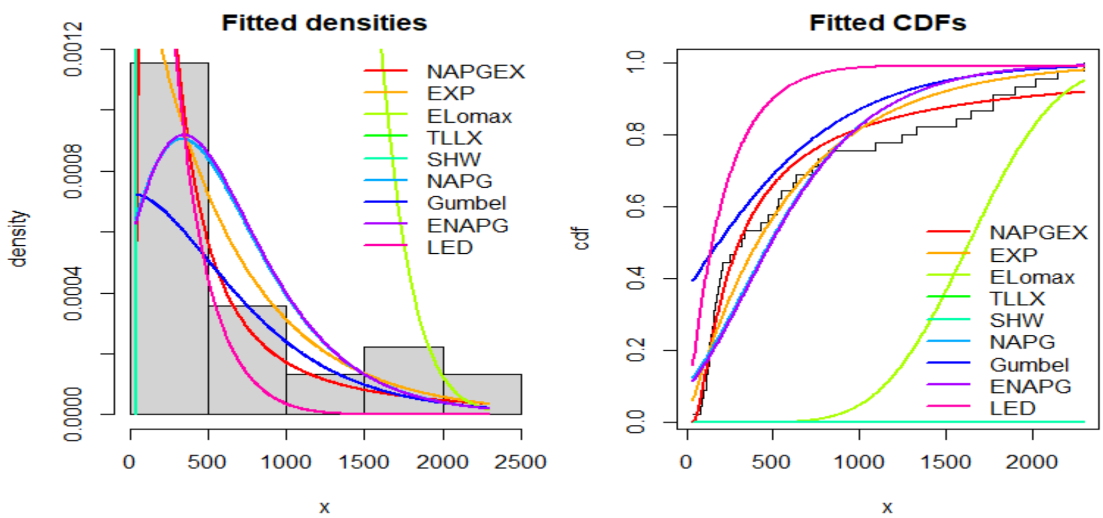

Figure 7. The fitted pdfs (*left panel*) and cdfs (*right panel*) of competing models for Dataset II.

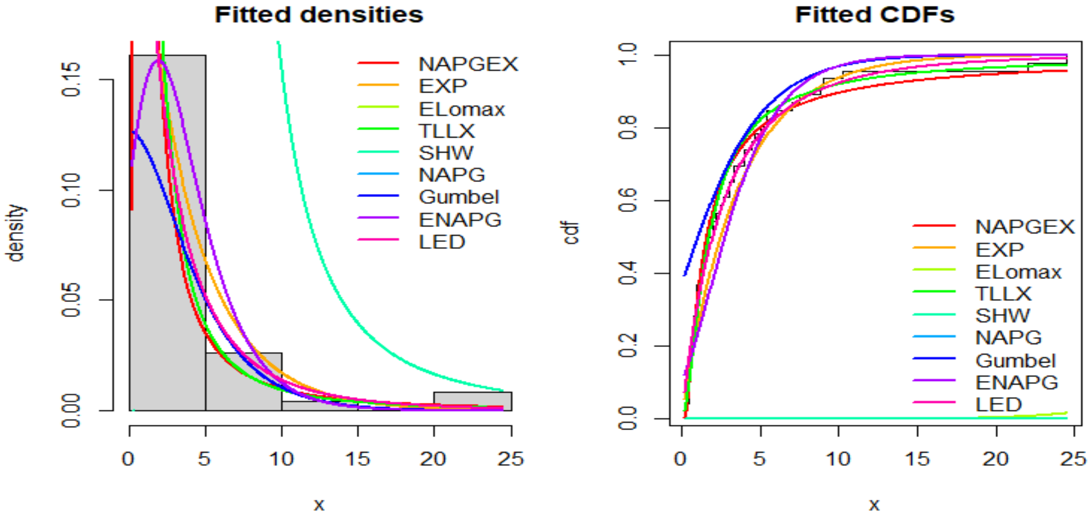

Figure 8. The fitted pdfs (*left panel*) and cdfs (*right panel*) of competing models for Dataset III.

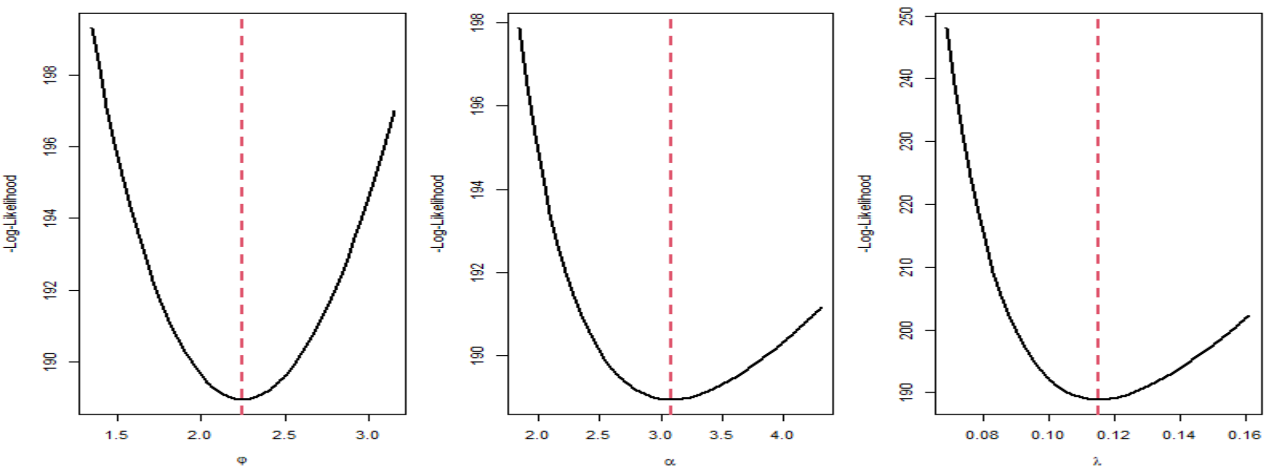

Figure 9. Negative log-likelihood profile of NAPGEX model for Dataset I.

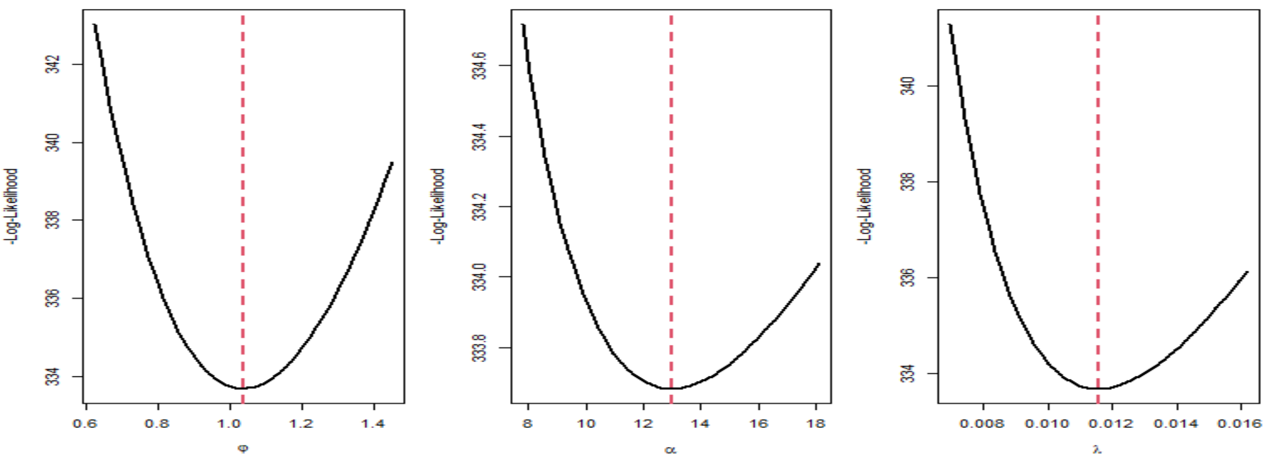

Figure 10. Negative log-likelihood profile of NAPGEX model for Dataset II.

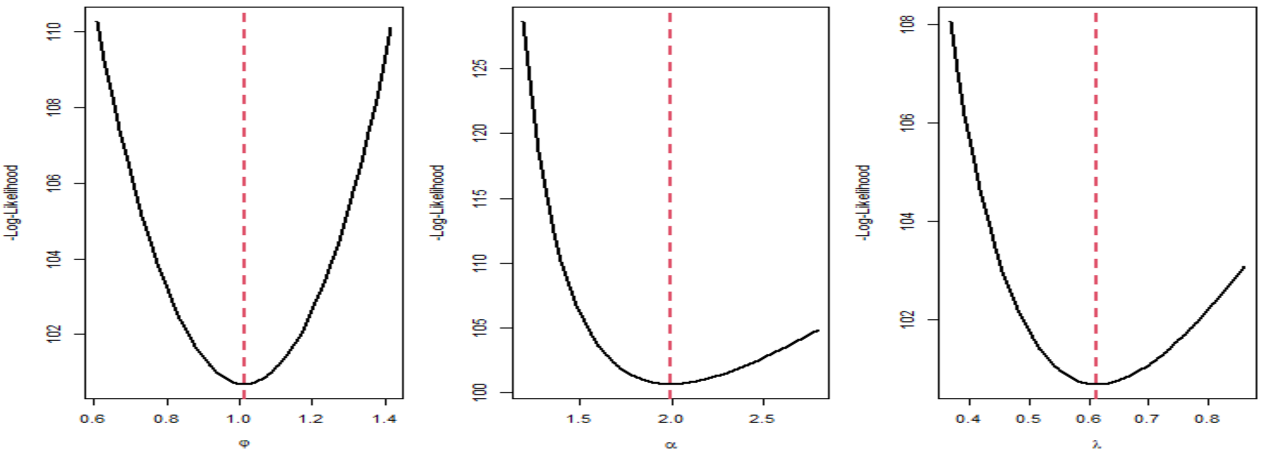

Figure 11. Negative log-likelihood profile of NAPGEX model for Dataset III.

corroborate the statistical results, confirming the strong goodness-of-fit of the NAPGEX model for Datasets I, II, and III.

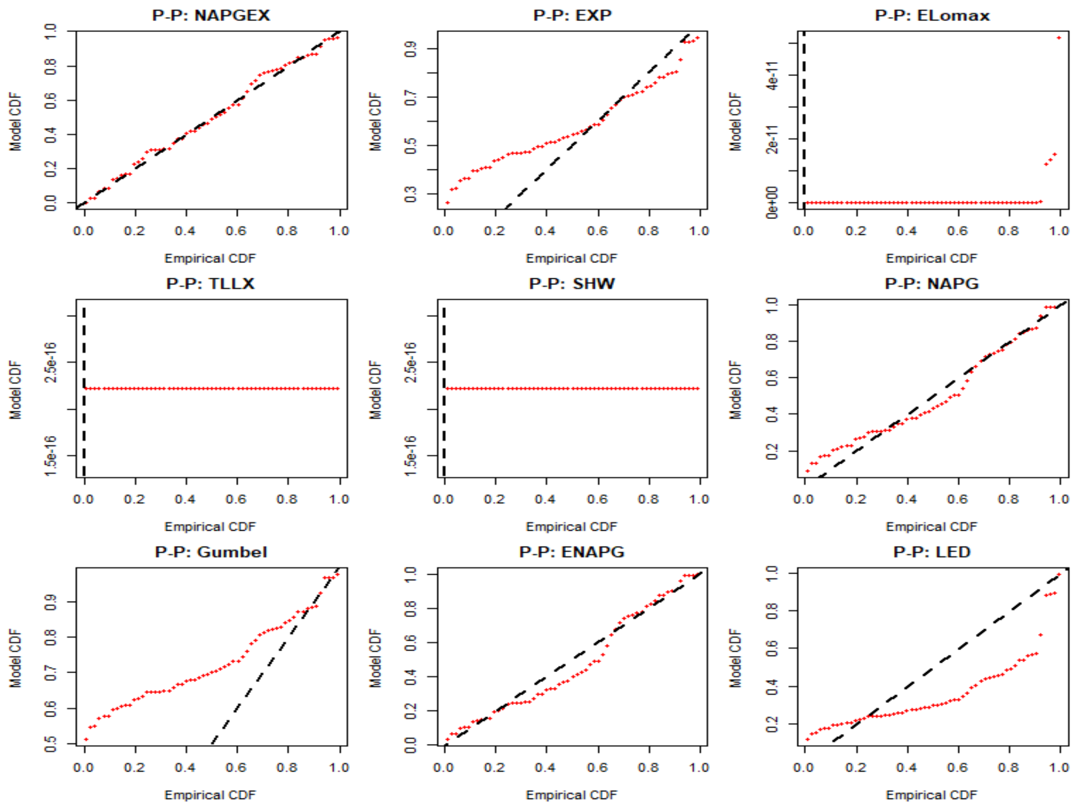

Figure 12. The P-P plots for the competing models for Dataset I.

8. Conclusion

In this study, the Novel Alpha Power Gumbel-X (NAPG-X) family of distribution is introduced via T-X transformation. The theoretical framework established demonstrates considerable flexibility in modelling diverse data structures, particularly those exhibiting positive skewness and varying tail behaviour's. The NAPG-exponential (NAPGEX) distribution presented as a sub-model, inherits these desirable properties while maintaining analytical tractability. The comprehensive mathematical derivations presented encompass essential distributional properties including the probability density function, cumulative distribution function, survival function, hazard rate function, quantile function, moments, moment generating function, Rényi

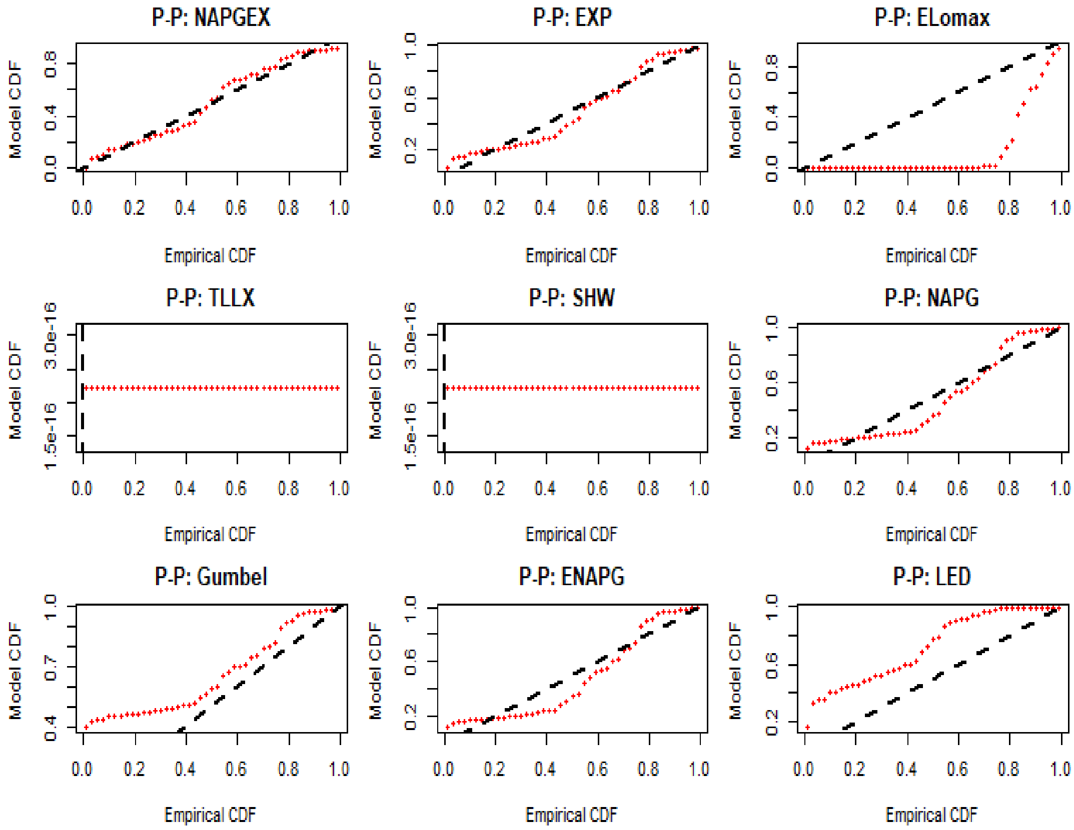

Figure 13. The P-P plots for the competing models for Dataset II.

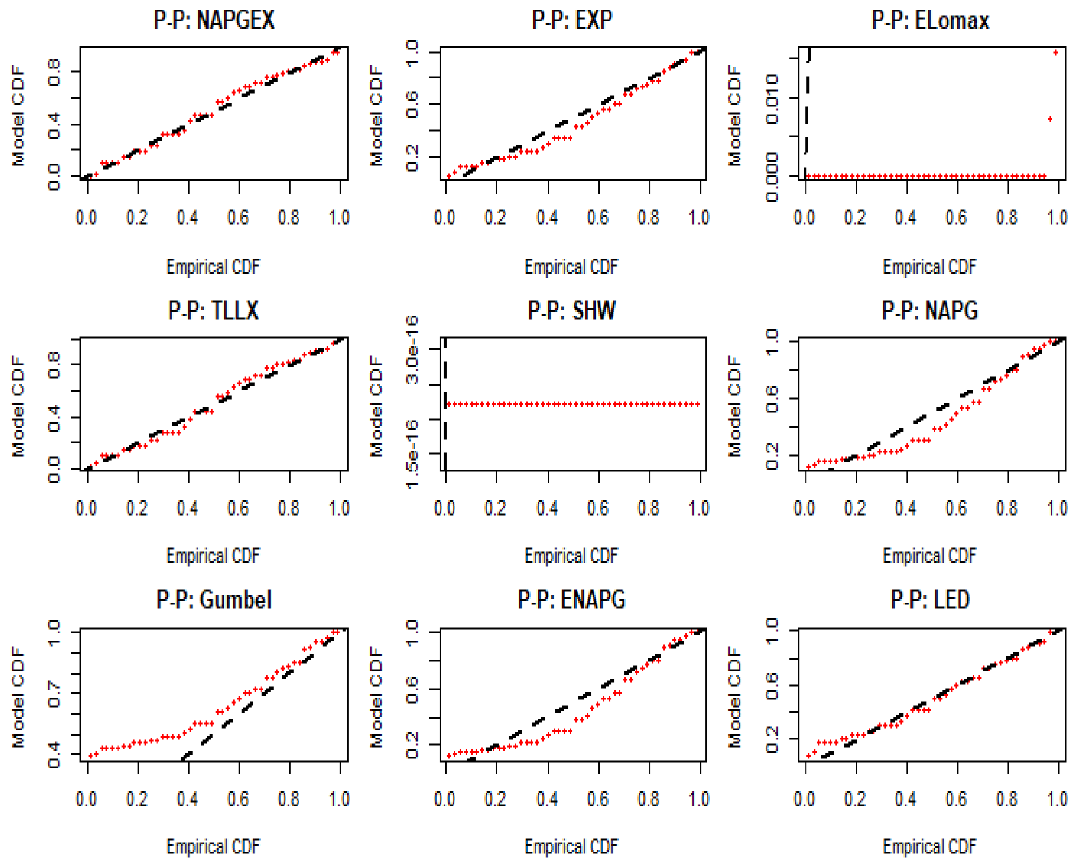

Figure 14. The P-P plots for the competing models for Dataset III.

entropy, and order statistics. The hazard rate function exhibits versatile shapes including increasing-decreasing, reversed-J, and L-shaped patterns making the distribution particularly suitable for reliability analysis and lifetime data modelling.

An extensive simulation study evaluated ten classical estimation methods across varying sample sizes and three distinct parameter combinations. The results consistently demonstrated that the maximum likelihood estimation (MLE) and Anderson-Darling estimation (ADE) methods provide superior performance, characterized by minimal bias, low mean relative errors, and reduced root mean square errors. These estimators maintain their robustness even under challenging parameter configurations, with estimation accuracy systematically improving as sample size increases. In contrast, percentile estimation and method of moments exhibited poor performance across all scenarios, rendering them unsuitable for NAPGEX parameter estimation. The practical utility of the NAPGEX distribution was validated through three real-world applications involving Egyptian monthly tax revenue data, head and neck cancer survival times, and airborne communication transceiver repair times. Across all datasets, the NAPGEX distribution consistently outperformed competing models. The superior fit was evidenced by the lowest values of negative log-likelihood, Akaike information criterion, Bayesian information criterion, and other goodness-of-fit measures, alongside exceptionally high Kolmogorov-Smirnov p-values for the three datasets, respectively. Visual diagnostics including probability-probability plots and empirical distribution comparisons further confirmed the excellent fit of the NAPGEX model.

The findings establish the NAPG-X family, particularly the NAPGEX distribution as a valuable addition to statistical models for modeling lifetime datasets, reliability studies, and asymmetric phenomena. The flexibility, mathematical tractability, and superior empirical performance make the NAPGEX model, a compelling alternative to existing models. Future research directions include extending this framework to incorporate additional baseline distributions, developing regression models based on the NAPG-X family, and exploring Bayesian inference methodologies for parameter estimation.

Authors' Contributions:

Conceptualization, A.O.D. and I.P.R.; Methodology, A.O.D. and I.P.R.; Software, A.O.D. and I.P.R.; Validation, A.O.D. and I.P.R.; Formal Analysis, A.O.D., I.P.R and B.B.B..; Investigation, A.O.D. and I.P.R.; Resources, A.O.D. and I.P.R.; Data Curation, A.O.D. and I.P.R.; Writing – Original Draft Preparation, I.P.R; Writing – Review and Editing, A.O.D. and I.P.R.; Visualization, A.O.D. and I.P.R.; Supervision, A.O.D. and D.I.J.; Project Administration, A.O.D.; Funding Acquisition, A.O.D., I.P.R. and B.B.B.

Data Availability Statement:

The data that supports the findings of this study are available within the article.

Conflicts of Interest:

The authors declare no conflict of interest.

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