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Research article

New Modified Liu Estimators to Handle the Multicollinearity in the Beta Regression Model: Simulation and Applications

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ABSTRACT

The beta regression model (BRM) is widely used for analyzing bounded response variables, such as proportions, percentages. However, when multicollinearity exists among explanatory variables, the conventional maximum likelihood estimator (MLE) becomes unstable and inefficient. To address this issue, we propose new modified Liu estimators for the BRM, designed to enhance estimation accuracy in the presence of high multicollinearity among predictors. The proposed estimators extend the traditional Liu estimator by incorporating flexible biasing parameters, offering a more robust alternative to the MLE. Theoretical comparisons demonstrate the superiority of the new estimators over existing methods. Additionally, Monte Carlo simulations and real-world applications evidence their improved performance in terms of mean squared error (MSE) and mean absolute error (MAE). The results indicate that the proposed estimators significantly reduce estimation bias and variance under multicollinearity, providing more reliable regression coefficients.

1. Introduction

When the dependent variable follows a normal distribution, the linear regression model (LRM) is the standard modeling approach. However, if this normality assumption is violated, such as when the data follow exponential family distributions (e.g., beta, Poisson, gamma, or negative binomial), generalized linear models (GLMs) become more appropriate [16]. Among these, the beta regression model (BRM) is particularly useful for bounded response variables (e.g., proportions, rates) constrained within the interval (0, 1), making it applicable across engineering, environmental sciences, medicine, and social research [19, 21]. While maximum likelihood estimation (MLE) is typically employed for BRM parameter estimation (offering advantages over ordinary least squares), conventional transformations, such as logit (logistic regression) or probit models, are often used to map [0, 1]-bounded data onto the real line. Yet, these methods introduce interpretability challenges and may yield biased inferences due to asymmetrical distributions, especially in small samples. Thus, direct modeling via BRM avoids these pitfalls while preserving the natural scale of the dependent variable [10].

Multicollinearity remains a persistent challenge in regression analysis, particularly when examining relationships between highly correlated predictors. This phenomenon, first systematically studied by Frisch [20], manifests when explanatory variables share strong linear dependencies, leading to an ill-conditioned covariance matrix in MLE. The consequences are particularly problematic - inflated coefficient variances that compromise both the statistical significance and practical interpretation of parameter estimates. Traditional remedies like collecting additional observations, model respecification, or eliminating correlated variables often prove inadequate or impractical [40, 27, 15]. The issue becomes especially acute in beta regression modeling, which analyzes proportion-type responses bounded between 0 and 1. While MLE serves as the standard estimation approach, its performance degrades severely under multicollinearity, producing unstable coefficients with excessive standard errors [6, 17]. This has spurred the development of biased estimation techniques, most notably ridge regression proposed by Hoerl and Kennard [22] and its more sophisticated counterpart, the Liu estimator [28]. These shrinkage methods, particularly valuable in chemical and biological applications where collinearity is endemic, introduce controlled bias to improve estimation stability dramatically. The Liu estimator's linear shrinkage function offers particular advantages over traditional ridge approaches, explaining its widespread adoption and continued refinement in contemporary statistical practice.

The statistical literature has seen significant development of biased estimation methods as alternatives to traditional linear regression approaches. Among these, several notable estimators have emerged, including the Liu-type estimator [29], various ridge-type estimators [7, 14, 34], and specialized modifications like the modified ridge-type [30] and two-parameter ridge estimators [24]. The James-Stein estimator [38] and Kibria-Lukman estimator [25] represent additional approaches, along with Liu estimator variants including both one-parameter [32] and two-parameter [4] modifications. Within BRM, researchers have similarly expanded the estimation toolkit through important contributions. These include developments by Qasim et al. [36], extensions by Abonazel and Taha [6], and methodological improvements by Karlsson et al. [23]. Further advancements have been made through the work of Algamal and Abonazel [9], Amin et al. [12], and Erkoç et al. [17], with additional refinements proposed by Koç Dünder [26], Abdel-Fattah [1], and Lukman et al. [31].

Recent developments in regression analysis have shown increasing interest in using two-parameter estimators as effective methods to handle multicollinearity problems. Many studies [9, 5, 2] have consistently reported that these two-parameter estimators perform better than single-parameter alternatives. Building on this foundation, the present study proposed modified Liu parameters for BRM. We propose a new modified one and two-parameter Liu estimator specifically designed to reduce the effects of multicollinearity in BRM. We present systematic methods for selecting the optimal parameters and conduct detailed performance comparisons using simulation and applications with existing methods, including MLE, ridge, and Liu estimators.

This paper is organized as follows: Section 2 introduces the beta distribution and its regression framework, reviewing existing biased estimators, including ridge and Liu approaches. Section 3 develops our proposed estimator, provides theoretical comparisons with existing methods, and derives the optimal biasing parameter. The simulation design and performance evaluation metrics are detailed in Section 4, while Section 5 demonstrates the practical utility of our method through a real-world case study. Finally, Section 6 summarizes key findings and discusses their implications for statistical practice.

2. Methodology

The BRM [19] analyzes bounded (0,1) responses using link functions and precision parameter ϑ for dispersion control [33, 8]. For $y \sim \text{Beta}(\mu, \vartheta)$:

$$f(y;\mu,\vartheta) = \frac{\Gamma(\vartheta)}{\Gamma(\mu\vartheta)\Gamma((1-\mu)\vartheta)} y^{\mu\vartheta-1}(1-y)^{(1-\mu)\vartheta-1}, \tag{2.1}$$

where $0 < y < 1$, $0 < \mu < 1$, $\vartheta > 0$, and $\Gamma(\cdot)$ denotes the gamma function. The distribution's first two moments are given by:

$$\mathbb{E}(y) = \mu, \quad \text{Var}(y) = \frac{\mu(1-\mu)}{1+\vartheta}.$$

Consider n independent random variables $y_1, \ldots, y_n$, each following the beta distribution specified in Equation (2.1) with mean μ_i and precision parameter ϑ. The beta regression model relates the mean response to covariates through the logit link function:

$$g(\mu_i) = \log\left(\frac{\mu_i}{1-\mu_i}\right) = x_i^\top \beta = \eta_i, \tag{2.2}$$

where $g(\cdot)$ represents the link function, $\beta = (\beta_1, \ldots, \beta_p)^\top$ denotes the p-dimensional parameter vector, $x_i = (x_{i1}, \ldots, x_{ip})^\top$ contains the predictor variables, and η_i is the linear predictor.

The beta regression parameters are estimated via MLE. The log-likelihood function takes the form:

$$\ell(\beta) = \sum_{i=1}^{n} \left[\log\Gamma(\vartheta) - \log\Gamma(\mu_i\vartheta) - \log\Gamma((1-\mu_i)\vartheta) + (\mu_i\vartheta - 1)\log y_i + ((1-\mu_i)\vartheta - 1)\log(1-y_i)\right]. \tag{2.3}$$

The score function for β is obtained by differentiating the log-likelihood:

$$U(\beta) = \vartheta X^\top T(y-\mu), \tag{2.4}$$

where $T = \text{diag}(1/g'(\mu_1), \ldots, 1/g'(\mu_n))$, $y_i = \log(y_i/(1-y_i))$, and $\mu_i = \psi(\mu_i\vartheta) - \psi((1-\mu_i)\vartheta)$, with $\psi(\cdot)$ being the digamma function. Parameter estimation typically employs either the Fisher scoring algorithm or iteratively reweighted least squares (IRLS), with updates given by:

$$\beta^{(t+1)} = \beta^{(t)} + \left(I_{\beta\beta}^{(t)}\right)^{-1} U_{\beta}^{(t)}, \tag{2.5}$$

where $I_{\beta\beta}^{(t)}$ represents the Fisher information matrix. The MLE takes the form:

$$\hat{\beta}_{\text{ML}} = Q^{-1}X^{\top}\hat{W}\hat{z}, \tag{2.6}$$

with $Q = X^{\top}\hat{W}X$, $\hat{z} = \hat{\eta} + \hat{W}^{-1}\hat{T}(y - \hat{\mu})$, and weight matrix $\hat{W} = \text{diag}(\hat{w}_1, \ldots, \hat{w}_n)$ where:

$$\hat{w}_i = \hat{\vartheta}\left[\psi'(\hat{\mu}_i\hat{\vartheta}) + \psi'((1-\hat{\mu}_i)\hat{\vartheta})\right][g'(\hat{\mu}_i)]^{-2}, \tag{2.7}$$

where $\hat{W} = \text{diag}(\hat{w}_1, \ldots, \hat{w}_n)$, $\hat{T}$, $\hat{\mu}$, and $\hat{\mu}$ represent the estimated quantities of W, T, μ, and μ respectively, evaluated at the MLE of β and ϑ [19].

Let Υ be the orthogonal matrix composed of eigenvectors of the $X^{\top}\hat{W}X$ matrix, where X is the design matrix and $\hat{W}$ is the estimated weight matrix. The performance of any estimator $\hat{\beta}$ can be evaluated through its matrix mean squared error (MMSE) and scalar mean squared error (MSE), defined respectively as:

$$\text{MMSE}(\hat{\beta}) = \text{Var}(\hat{\beta}_{\text{ML}}) + \text{Bias}(\hat{\beta}_{\text{ML}})\text{Bias}(\hat{\beta}_{\text{ML}})^{\top}, \tag{2.8}$$

$$\text{MSE}(\hat{\beta}) = \text{tr}\left(\text{MMSE}(\hat{\beta}_{\text{ML}})\right). \tag{2.9}$$

For the Beta Maximum Likelihood estimator $\hat{\beta}_{\text{ML}}$, these metrics take the specific forms:

$$\text{MSEM}(\hat{\beta}_{\text{ML}}) = \frac{1}{\vartheta}\Phi^{-1}, \tag{2.10}$$

$$\text{MSE}(\hat{\beta}_{\text{ML}}) = \frac{1}{\vartheta}\sum_{j=1}^{p}\frac{1}{\phi_j}, \tag{2.11}$$

where $\Phi = \Upsilon Q \Upsilon^{\top} = \text{diag}(\phi_1, \phi_2, ..., \phi_p)$ is the diagonal matrix of eigenvalues and ϕ_j represents the j-th eigenvalue of Q.

To mitigate multicollinearity issues in GLMs, Segerstedt [37] developed a Ridge estimator building upon the foundational work of Hoerl and Kennard [22]. In the context of BRM, where correlated predictors compromise the stability of MLE, Abonazel and Taha [6] introduced a specialized ridge estimator. The BRM ridge estimator (BRRE) is formally expressed as:

$$\hat{\beta}_{\text{BRRE}} = (Q + kI)^{-1}Q\hat{\beta}_{\text{ML}}, \quad k > 0, \tag{2.12}$$

where k denotes the ridge parameter and I represents the $(p \times p)$ identity matrix. Notably, when $k = 0$, the BRRE reduces to the standard MLE. The statistical properties of this estimator are characterized by:

$$\text{Bias}(\hat{\beta}_{\text{BRRE}}) = -k\Upsilon\Phi_k^{-1}\alpha, \tag{2.13}$$

$$\mathrm{Cov}(\hat{\beta}_{\mathrm{BRRE}}) = \frac{1}{\vartheta}\Upsilon\Phi_k^{-1}\Phi\Phi_k^{-1}\Upsilon^{\top}. \tag{2.14}$$

The precision of the BRRE is quantified through its MMSE:

$$\begin{aligned}\mathrm{MMSE}(\hat{\beta}_{\mathrm{BRRE}}) &= \mathrm{Cov}(\hat{\beta}_{\mathrm{BRRE}}) + \mathrm{Bias}(\hat{\beta}_{\mathrm{BRRE}})\mathrm{Bias}(\hat{\beta}_{\mathrm{BRRE}})^{\top} \\ &= \frac{1}{\vartheta}\Upsilon\Phi_k^{-1}\Phi\Phi_k^{-1}\Upsilon^{\top} + k^2\Upsilon\Phi_k^{-1}\alpha\alpha^{\top}\Phi_k^{-1}\Upsilon^{\top},\end{aligned} \tag{2.15}$$

where $\Phi_k = \mathrm{diag}(\phi_1 + k, \ldots, \phi_p + k)$. The scalar MSE is obtained by applying the trace operator:

$$\mathrm{MSE}(\hat{\beta}_{\mathrm{BRRE}}) = \mathrm{tr}(\mathrm{MMSE}(\hat{\beta}_{\mathrm{BRRE}})) = \frac{1}{\vartheta}\sum_{j=1}^{p}\frac{\phi_j}{(\phi_j + k)^2} + k^2\sum_{j=1}^{p}\frac{\alpha_j^2}{(\phi_j + k)^2}. \tag{2.16}$$

Here, $\alpha = \Upsilon^{\top}\beta$ represents the transformed parameter vector.

To address multicollinearity challenges in BRM, Karlsson et al. [23] proposed the Beta Liu Estimator (BLE), demonstrating superior performance compared to traditional BRRE. The BLE is formally defined as:

$$\hat{\beta}_{\mathrm{BLE}} = (Q + I)^{-1}(Q + dI)\hat{\beta}_{\mathrm{ML}}, \quad 0 < d < 1, \tag{2.17}$$

where d represents the shrinkage parameter that governs the estimator's behavior:

- At $d = 1$, the BLE simplifies to the MLE.
- For $0 < d < 1$, the BLE produces shrunken coefficients, effectively mitigating multicollinearity effects.

The statistical properties of the BLE are characterized by:

$$\mathrm{Bias}(\hat{\beta}_{\mathrm{BLE}}) = \Upsilon(\Phi + I)^{-1}\alpha(d - 1), \tag{2.18}$$

$$\mathrm{Cov}(\hat{\beta}_{\mathrm{BLE}}) = \frac{1}{\vartheta}\Upsilon(\Phi + I)^{-1}(\Phi + dI)\Phi^{-1}(\Phi + dI)(\Phi + I)^{-1}\Upsilon^{\top}. \tag{2.19}$$

The precision of the BLE is quantified through its MMSE:

$$\begin{aligned}\mathrm{MMSE}(\hat{\beta}_{\mathrm{BLE}}) &= \mathrm{Cov}(\hat{\beta}_{\mathrm{BLE}}) + \mathrm{Bias}(\hat{\beta}_{\mathrm{BLE}})\mathrm{Bias}(\hat{\beta}_{\mathrm{BLE}})^{\top} \\ &= \frac{1}{\vartheta}\Upsilon(\Phi + I)^{-1}(\Phi + dI)\Phi^{-1}(\Phi + dI)(\Phi + I)^{-1}\Upsilon^{\top} + (d - 1)^2\Upsilon(\Phi + I)^{-1}\alpha\alpha^{\top}(\Phi + I)^{-1}\Upsilon^{\top}\end{aligned} \tag{2.20}$$

The scalar MSE is obtained as:

$$\mathrm{MSE}(\hat{\beta}_{\mathrm{BLE}}) = \mathrm{tr}(\mathrm{MMSE}(\hat{\beta}_{\mathrm{BLE}})) = \frac{1}{\vartheta}\sum_{j=1}^{p}\frac{(\phi_j + d)^2}{\phi_j(\phi_j + 1)^2} + (d - 1)^2\sum_{j=1}^{p}\frac{\alpha_j^2}{(\phi_j + 1)^2}. \tag{2.21}$$

3. Proposed Estimators

In this section, we present modified one- and two-parameter Liu estimators for BRM, provide a theoretical comparison with existing estimators, and discuss methods for selecting optimal biasing parameters.

3.1. Beta modified one-parameter Liu estimator

Building upon the foundational work of Lukman et al. [32] and [11], we introduce an enhanced Beta modified one-parameter Liu estimator (BMOPLE) for the beta regression model. The proposed estimator is formally defined as:

$$\hat{\beta}_{\mathrm{BMOPLE}} = (Q + I)^{-1}(Q - dI)\hat{\beta}_{\mathrm{ML}}, \quad 0 < d < 1, \tag{3.1}$$

where d represents the modified Liu parameter that controls the shrinkage intensity. This modification demonstrates superior performance in addressing multicollinearity compared to conventional MLE, BRRE, and BLE approaches, as evidenced by reduced mean squared error.

The statistical properties of BMOPLE are characterized by:

$$\mathrm{Bias}(\hat{\beta}_{\mathrm{BMOPLE}}) = -(d + 1)\Upsilon(\Phi + I)^{-1}\alpha, \tag{3.2}$$
$$\mathrm{Cov}(\hat{\beta}_{\mathrm{BMOPLE}}) = \frac{1}{\vartheta}\Upsilon(\Phi + I)^{-1}(\Phi - dI)\Phi^{-1}(\Phi - dI)(\Phi + I)^{-1}\Upsilon^{\top}. \tag{3.3}$$

The precision of BMOPLE is quantified through its MMSE:

$$\begin{aligned}\mathrm{MMSE}(\hat{\beta}_{\mathrm{BMOPLE}}) &= \mathrm{Cov}(\hat{\beta}_{\mathrm{BMOPLE}}) + \mathrm{Bias}(\hat{\beta}_{\mathrm{BMOPLE}})\mathrm{Bias}(\hat{\beta}_{\mathrm{BMOPLE}})^{\top} \\ &= \frac{1}{\vartheta}\Upsilon(\Phi + I)^{-1}(\Phi - dI)\Phi^{-1}(\Phi - dI)(\Phi + I)^{-1}\Upsilon^{\top} + (d + 1)^2\Upsilon(\Phi + I)^{-1}\alpha\alpha^{\top}(\Phi + I)^{-1}\Upsilon^{\top}\end{aligned} \tag{3.4}$$

The scalar MSE is obtained as:

$$\mathrm{MSE}(\hat{\beta}_{\mathrm{BMOPLE}}) = \mathrm{tr}(\mathrm{MMSE}(\hat{\beta}_{\mathrm{BMOPLE}})) = \frac{1}{\vartheta}\sum_{j=1}^{p}\frac{(\phi_j - d)^2}{\phi_j(\phi_j + 1)^2} + (d + 1)^2\sum_{j=1}^{p}\frac{\alpha_j^2}{(\phi_j + 1)^2}. \tag{3.5}$$

3.2. Beta modified two-parameter Liu estimator

Building upon the theoretical framework established by Abonazel [3, 4], we introduce an enhanced Beta modified two-parameter Liu estimator (BMTPLE) for the BRM. This advanced estimator employs a two-parameter (k, d) to optimize the bias-variance tradeoff in multicollinear designs:

$$\hat{\beta}_{\mathrm{BMTPLE}} = (Q + I)^{-1}(Q - (k + d)I)\hat{\beta}_{\mathrm{ML}}, \quad k > 0, 0 < d < 1, \tag{3.6}$$

The statistical properties of the BMTPLE are characterized by:

$$\mathrm{Bias}(\hat{\beta}_{\mathrm{BMTPLE}}) = -(k + d + 1)\Upsilon(\Phi + I)^{-1}\alpha, \tag{3.7}$$

$$\mathrm{Cov}(\hat{\beta}_{\mathrm{BMTPLE}}) = \frac{1}{\vartheta}\Upsilon(\Phi + I)^{-1}(\Phi - (k+d)I)\Phi^{-1}(\Phi + I)^{-1}(\Phi - (k+d)I)\Upsilon^{\top}. \tag{3.8}$$

The precision metrics are given by:

$$\begin{aligned}\mathrm{MMSE}(\hat{\beta}_{\mathrm{BMTPLE}}) &= \mathrm{Cov}(\hat{\beta}_{\mathrm{BMTPLE}}) + \mathrm{Bias}(\hat{\beta}_{\mathrm{BMTPLE}})\mathrm{Bias}(\hat{\beta}_{\mathrm{BMTPLE}})^{\top} \\ &= \frac{1}{\vartheta}\Upsilon(\Phi + I)^{-1}(\Phi - (k+d)I)\Phi^{-1}(\Phi + I)^{-1}(\Phi - (k+d)I)\Upsilon^{\top} \\ &+ (k+d+1)^2\Upsilon(\Phi + I)^{-1}\alpha\alpha^{\top}(\Phi + I)^{-1}\Upsilon^{\top},\end{aligned} \tag{3.9}$$

$$\mathrm{MSE}(\hat{\beta}_{\mathrm{BMTPLE}}) = \mathrm{tr}(\mathrm{MMSE}(\hat{\beta}_{\mathrm{BMTPLE}})) = \frac{1}{\vartheta}\sum_{j=1}^{p}\frac{(\phi_j - (k+d))^2}{\phi_j(\phi_j+1)^2} + (k+d+1)^2\sum_{j=1}^{p}\frac{\alpha_j^2}{(\phi_j+1)^2}. \tag{3.10}$$

The optimal shrinkage parameters are derived through minimization of the MSE:

$$k_{\mathrm{opt}} = \left(\frac{\phi_j(\frac{1}{\vartheta} - \alpha_j^2)}{\frac{1}{\vartheta} + \phi_j\alpha_j^2}\right) - d, \tag{3.11}$$

$$d_{0(\mathrm{opt})} = \frac{\phi_j(\frac{1}{\vartheta} - \alpha_j^2) - k(\frac{1}{\vartheta} + \phi_j\alpha_j^2)}{\frac{1}{\vartheta} + \phi_j\alpha_j^2}. \tag{3.12}$$

3.3. Theoretical comparison using MMSE and scalar MSE

We present two fundamental lemmas that underpin our comparative analysis of estimator performance:

Lemma 1. *For any positive definite matrix $D > 0$ and vector α, the following inequality holds $D - \alpha\alpha^{\top} \geq 0$ iff $\alpha^{\top}D^{-1}\alpha \leq 1$, where $\geq$ denotes positive semidefiniteness. This result extends the work of [18] on quadratic forms in statistical estimation.*

Lemma 2. *Consider two linear estimators $\hat{\beta}_1 = C_1 y$ and $\hat{\beta}_2 = C_2 y$ of parameter vector β. When the covariance matrix difference $M = Cov(\hat{\beta}_1) - Cov(\hat{\beta}_2)$ is positive definite, the following equivalence holds for their MMSE $MMSE(\hat{\beta}_1) - MMSE(\hat{\beta}_2) > 0$, iff the quadratic form $b_2^{\top}[D + b_2 b_2^{\top}]^{-1} b_2 < 1$, where b_j denotes the bias vector of $\hat{\beta}_j$, $MMSE(\hat{\beta}_j) = Cov(\hat{\beta}_j) + b_j b_j^{\top}$, and $D = Cov(\hat{\beta}_1) - Cov(\hat{\beta}_2)$. This generalizes the results of [39] on comparative estimator efficiency.*

Theorem 1. *$MMSE(\hat{\beta}_{ML}) - MMSE(\hat{\beta}_{BMTPLE}) > 0$, iff $A^T[\Delta_1]^{-1}A < 1$, and $(\phi_j + 1)^2 - (\phi_j - k - d)^2 > 0$ for $k > 0$ and $0 < d < 1$, where $\Delta_1 = Cov(\hat{\beta}_{ML}) - Cov(\hat{\beta}_{BMTPLE})$, $A = -(k + d_0 + 1)\Upsilon(\Phi + I)^{-1}\alpha$.*
Proof:

$$\begin{aligned}\Delta_1 = Cov(\hat{\beta}_{ML}) - Cov(\hat{\beta}_{BMTPLE}) &= \frac{1}{\vartheta}\Upsilon\Big[\Phi^{-1} - (\Phi + I)^{-1}(\Phi - (k+d_0)I)\Phi^{-1}(\Phi - (k+d_0)I)(\Phi + I)^{-1}\Big]\Upsilon^{\top} \\ &= \frac{1}{\vartheta}\Upsilon diag\left[\frac{1}{\phi_j} - \frac{(\phi_j - k - d)^2}{\phi_j(\phi_j+1)^2}\right]_{j=1}^{p}\Upsilon^{\top} \\ &= \frac{1}{\vartheta}\Upsilon diag\left[\frac{(\phi_j+1)^2 - (\phi_j - k - d)^2}{\phi_j(\phi_j+1)^2}\right]_{j=1}^{p}\Upsilon^{\top}.\end{aligned}$$

Using Lemma(2), MMSE($\hat{\beta}_{ML}$) − MMSE($\hat{\beta}_{BMTPLE}$) > 0, iff $A^T[\Delta_1]^{-1}A < 1$*, under the assumption that* Δ_1 *is a positive definite (pd) matrix (*$\Delta_1 > 0$*). Therefor* $\Delta_1 > 0$*, if* $\Phi^{-1} - (\Phi + I)^{-1}(\Phi - (k + d_0)I)\Phi^{-1}(\Phi - (k + d_0)I)(\Phi + I)^{-1} > 0$*, and this is achieved when* $(\phi_j + 1)^2 - (\phi_j - k - d)^2 > 0$ *for* $k > 0$ *and* $0 < d < 1$*. This means that* $\hat{\beta}_{BMTPLE}$ *is better than* $\hat{\beta}_{ML}$ *iff* $A^T[\Delta_1]^{-1}A < 1$ *and* $(\phi_j + 1)^2 - (\phi_j - k - d)^2 > 0$.

Theorem 2. *MMSE($\hat{\beta}_{PRRE}$) − MMSE($\hat{\beta}_{BMTPLE}$) > 0, iff* $A^T[\Delta_2 + BB^\top]^{-1}A < 1$ *and* $\phi_j^2(\phi_j + 1)^2 - (\phi_j + k)^2(\phi_j - k - d)^2 > 0$ *for* $k > 0$*,* $k > 0$*, and* $0 < d < 1$*, where* $\Delta_2 = Cov(\hat{\beta}_{PRRE}) - Cov(\hat{\beta}_{BMTPLE})$*,* $B = -k\Upsilon\Phi_k^{-1}\alpha$*.*
Proof:

$$\begin{aligned}
\Delta_2 = Cov(\hat{\beta}_{PRRE}) - Cov(\hat{\beta}_{BMTPLE}) &= \frac{1}{\vartheta}\Upsilon\Big[\Phi_k^{-1}U\Phi_k^{-1} - (\Phi + I)^{-1}(\Phi - (k + d_0)I)\Phi^{-1}(\Phi - (k + d_0)I)(\Phi + I)^{-1}\Big]\Upsilon^\top \\
&= \frac{1}{\vartheta}\Upsilon diag\left[\frac{\phi_j}{(\phi_j + k)^2} - \frac{(\phi_j - k - d)^2}{\phi_j(\phi_j + 1)^2}\right]_{j=1}^{p}\Upsilon^\top \\
&= \frac{1}{\vartheta}\Upsilon diag\left[\frac{\phi_j^2(\phi_j + 1)^2 - (\phi_j + k)^2(\phi_j - k - d)^2}{\phi_j(\phi_j + 1)^2(\phi_j + k)^2}\right]_{j=1}^{p}\Upsilon^\top.
\end{aligned}$$

Using Lemma(2), MMSE($\hat{\beta}_{PRRE}$) − MMSE($\hat{\beta}_{BMTPLE}$) > 0, iff $A^T[\Delta_2 + BB^\top]^{-1}A < 1$*, under the assumption that* $\Delta_2 = Cov(\hat{\beta}_{PRRE}) - Cov(\hat{\beta}_{BMTPLE}) > 0$*. Therefor* $\Delta_2 > 0$*, if* $\Phi_k^{-1}U\Phi_k^{-1} - (\Phi + I)^{-1}(\Phi - (k + d_0)I)\Phi^{-1}(\Phi - (k + d_0)I)(\Phi + I)^{-1} > 0$*, and this is achieved when* $\phi_j^2(\phi_j + 1)^2 - (\phi_j + k)^2(\phi_j - k - d)^2 > 0$ *for* $k > 0$*,* $k > 0$ *and* $0 < d < 1$*. This means that* $\hat{\beta}_{BMTPLE}$ *is better than* $\hat{\beta}_{PRRE}$ *iff* $A^T[\Delta_2 + BB^\top]^{-1}A < 1$ *and* $\phi_j^2(\phi_j + 1)^2 - (\phi_j + k)^2(\phi_j - k - d)^2 > 0$ *for* $k > 0$*,* $k > 0$*, and* $0 < d < 1$*.*

Theorem 3. *MMSE($\hat{\beta}_{PLE}$) − MMSE($\hat{\beta}_{BMTPLE}$) > 0, iff* $A^T[\Delta_3 + SS^\top]^{-1}A < 1$ *and* $(\phi_j + d)^2 - (\phi_j - k - d)^2 > 0$ *for* $k > 0$*, and* $0 < d, d < 1$*, where* $\Delta_3 = Cov(\hat{\beta}_{PLE}) - Cov(\hat{\beta}_{BMTPLE})$*,* $S = \Upsilon(\Phi + I)^{-1}\alpha(d - 1)$*.*
Proof:

$$\begin{aligned}
\Delta_3 = Cov(\hat{\beta}_{PLE}) - Cov(\hat{\beta}_{BMTPLE}) &= \frac{1}{\vartheta}\Upsilon\Big[(\Phi + I)^{-1}(\Phi + dI)\Phi^{-1}(\Phi + dI)(\Phi + I)^{-1} - (\Phi + I)^{-1} \\
&\quad \times (\Phi - (k + d_0)I)\Phi^{-1}(\Phi - (k + d_0)I)(\Phi + I)^{-1}\Big]\Upsilon^\top \\
&= \frac{1}{\vartheta}\Upsilon diag\left[\frac{(\phi_j + d)^2}{\phi_j(\phi_j + 1)^2} - \frac{(\phi_j - k - d)^2}{\phi_j(\phi_j + 1)^2}\right]_{j=1}^{p}\Upsilon^\top \\
&= \frac{1}{\vartheta}\Upsilon diag\left[\frac{(\phi_j + d)^2 - (\phi_j - k - d)^2}{\phi_j(\phi_j + 1)^2}\right]_{j=1}^{p}\Upsilon^\top.
\end{aligned}$$

Using Lemma(2), MMSE($\hat{\beta}_{PLE}$) − MMSE($\hat{\beta}_{BMTPLE}$) > 0, iff $A^T[\Delta_3 + SS^\top]^{-1}A < 1$*, under the assumption that* $\Delta_3 > 0$*. Therefor* $\Delta_3 > 0$*, if* $(\Phi + I)^{-1}(\Phi + dI)\Phi^{-1}(\Phi + dI)(\Phi + I)^{-1} - (\Phi + I)^{-1}(\Phi - (k + d_0)I)\Phi^{-1}(\Phi - (k + d_0)I)(\Phi + I)^{-1} > 0$ *and this is achieved when* $(\phi_j + d)^2 - (\phi_j - k - d)^2 > 0$ *for* $k, k > 0$*, and* $0 < d, d < 1$*. This means that* $\hat{\beta}_{BMTPLE}$ *is better than* $\hat{\beta}_{PLE}$ *iff* $A^T[\Delta_3 + SS^\top]^{-1}A < 1$ *and* $(\phi_j + d)^2 - (\phi_j - k - d)^2 > 0$ *for* $k > 0$*, and* $0 < d, d < 1$*.*

Theorem 4. *MMSE($\hat{\beta}_{PALE}$) − MMSE($\hat{\beta}_{BMTPLE}$) > 0, iff* $A^T[\Delta_4 + NN^\top]^{-1}A < 1$ *and* $(\phi_j - d_0)^2 - (\phi_j - k - d)^2 > 0$ *for* $k > 0$*, and* $0 < d_0, d < 1$*, where* $\Delta_4 = Cov(\hat{\beta}_{PALE}) - Cov(\hat{\beta}_{BMTPLE})$*,* $N = -(d_0 + 1)\Upsilon(\Phi + I)^{-1}\alpha$*.*
Proof:

$$\Delta_4 = Cov(\hat{\beta}_{PALE}) - Cov(\hat{\beta}_{BMTPLE}) = \frac{1}{\vartheta}\Upsilon\Big[(\Phi + I)^{-1}(\Phi - d_0I)\Phi^{-1}(\Phi - d_0I)(\Phi + I)^{-1} - (\Phi + I)^{-1}$$

$$
\begin{aligned}
&\times (\Phi - (k + d_0)I)\Phi^{-1}(\Phi - (k + d_0)I)(\Phi + I)^{-1}\Big]\Upsilon^\top \\
=&\frac{1}{\vartheta}\Upsilon diag\left[\frac{(\phi_j - d_0)^2}{\phi_j(\phi_j + 1)^2} - \frac{(\phi_j - k - d)^2}{\phi_j(\phi_j + 1)^2}\right]_{j=1}^{p}\Upsilon^\top \\
=&\frac{1}{\vartheta}\Upsilon diag\left[\frac{(\phi_j - d_0)^2 - (\phi_j - k - d)^2}{\phi_j(\phi_j + 1)^2}\right]_{j=1}^{p}\Upsilon^\top.
\end{aligned}
$$

Using Lemma(2), $MMSE(\hat{\beta}_{PALE}) - MMSE(\hat{\beta}_{BMTPLE}) > 0$, iff $A^T[\Delta_4 + NN^\top]^{-1}A < 1$, under the assumption that $\Delta_4 > 0$. Therefor $\Delta_4 > 0$, if $(\Phi+I)^{-1}(\Phi-d_0I)\Phi^{-1}(\Phi-d_0I)(\Phi+I)^{-1} - (\Phi+I)^{-1}(\Phi-(k+d_0)I)\Phi^{-1}(\Phi-(k+d_0)I)(\Phi+I)^{-1} > 0$ and this is achieved when $(\phi_j-d_0)^2-(\phi_j-k-d)^2 > 0$ for $k > 0$, $0 < d < 1$, and $0 < d_0 < 1$. This means that $\hat{\beta}_{BMTPLE}$ is better than $\hat{\beta}_{PALE}$ iff $A^T[\Delta_4 + NN^\top]^{-1}A < 1$ and $(\phi_j - d_0)^2 - (\phi_j - k - d)^2 > 0$ for $k > 0$, and $0 < d_0, d < 1$.

3.4. Selection of biasing parameter

Building upon Qasim et al. [36] and Abonazel and Taha [6], we consider several optimal ridge parameter (k) formulations:

$$
\hat{k}_1 = \min\left(\frac{\phi_j}{\vartheta^{-1}\hat{\alpha}^2_{\text{min}}}\right), \quad \hat{k}_2 = \min\left(\frac{\vartheta^{-1}}{\hat{\alpha}^2_{\text{j}}}\right). \tag{3.13}
$$

Recent work by Karlsson et al. [23] and Lukman et al. [31] provides two optimal Liu parameter (d) estimators:

$$
\hat{d}_1 = \max\left(0, \min\left(\frac{\hat{\alpha}^2_j - \vartheta^{-1}}{\max(\vartheta^{-1}/\phi_j) + \max(\hat{\alpha}^2_j)}\right)\right), \quad \hat{d}_2 = \max\left(0, \frac{1}{p}\sum_{j=1}^{p}\frac{\hat{\alpha}^2_j}{\max(\vartheta^{-1}/\phi_j) + \max(\hat{\alpha}^2_j)}\right). \tag{3.14}
$$

For the modified one-parameter Liu parameter d_0, Lukman et al. [32] and Abonazel et al. [5] propose:

$$
\hat{d} = \max\left(0, \min\left(\frac{\phi_j(\vartheta^{-1} - \hat{\alpha}^2_j)}{\vartheta^{-1} + \phi_j\hat{\alpha}^2_j}\right)\right). \tag{3.15}
$$

Extending the work of Abonazel et al. [5, 3], we introduce several novel parameter estimators for the modified two-parameter Liu estimator as follows:

$$
\hat{k}_1 = \frac{\vartheta^{-1}}{p\sum_{j=1}^{p}\hat{\alpha}^2_{\text{j}}}, \tag{3.16}
$$

$$
\hat{k}_2 = \frac{1}{\sum_{j=1}^{p}\hat{\alpha}^2_{\text{j}}}, \tag{3.17}
$$

$$
\hat{k}_3 = \left(\frac{\min_j(\phi_j)(\vartheta^{-1} - \min_j(\hat{\alpha}^2_j))}{\vartheta^{-1} + \min_j(\phi_j)\text{mean}(\hat{\alpha}^2_j)}\right) - \hat{d}_2, \tag{3.18}
$$

$$
\hat{k}_4 = \min\left(\frac{\vartheta^{-1}}{(\vartheta^{-1} + \phi_j\hat{\alpha}^2_j)}\right) - \hat{d}_2, \tag{3.19}
$$

$$\hat{k}_5 = \left(\frac{p}{\sum_{j=1}^{p} \left[(\vartheta^{-1}/\phi_j) + 2\hat{\alpha}_j^2 \right]} \right), \tag{3.20}$$

with the following values of the parameter d:

$$\hat{d} = \max\left(0, \min\left(\frac{\phi_j(\vartheta^{-1} - \hat{\alpha}_j^2) - \hat{k}(1 + \phi_j\hat{\alpha}_j^2)}{\vartheta^{-1} + \phi_j\hat{\alpha}_j^2}\right)\right), \tag{3.21}$$

where $\hat{k} = \min(\vartheta^{-1}/\hat{\alpha}_j^2)$.

4. Monte Carlo Simulation

To rigorously evaluate estimator performance, we conducted an extensive Monte Carlo simulation comparing our proposed BMTPLE and MOPLE against established alternatives: the standard MLE, MRRE, and BLE. All simulations were implemented in R using the `betareg` package, ensuring reproducibility and alignment with standard statistical practice.

The simulation design carefully replicates common real-world scenarios where BRM is applied. For each observation $i = 1, ..., n$, we generate the response variable from a beta distribution $y_i \sim \text{Beta}(\mu_i, \phi)$, where μ_i represents the mean parameter linked to covariates through the logistic function:

$$\mu_i = \frac{\exp(x_i^\top \beta)}{1 + \exp(x_i^\top \beta)}, \tag{4.1}$$

The precision parameter ϑ takes values in {0.5, 1.5} to examine different dispersion scenarios. The true parameter vector $\beta = (\beta_1, ..., \beta_p)^\top$ is constructed with equal coefficients normalized to satisfy $\sum_{j=1}^{p} \beta_j^2 = 1$, representing a standardized effect size configuration.

Covariates are generated to induce controlled multicollinearity:

$$x_{ij} = (1 - \rho^2)^{1/2} w_{ij} + \rho w_{ip}, \quad i = 1, ..., n; \quad j = 2, ..., p, \tag{4.2}$$

where $w_{ij} \sim N(0, 1)$ are independent standard normal variates and ρ governs the pairwise correlation structure among predictors. We examine a comprehensive factorial design that incorporates various experimental conditions, including sample sizes $n \in \{30,\ 75,\ 150,\ 200,\ 300,\ 400\}$, predictor dimensions $p \in \{3,\ 7\}$, correlation levels $\rho \in \{0.80,\ 0.85,\ 0.90,\ 0.95,\ 0.99\}$, and dispersion parameters $\phi \in \{0.5,\ 1.5\}$.

The simulated MSE and MAE serve as our primary performance metric, computed respectively as

$$\text{MSE}(\hat{\beta}) = \frac{1}{L} \sum_{l=1}^{L} (\hat{\beta}_l - \beta)^\top (\hat{\beta}_l - \beta), \tag{4.3}$$

$$\text{MAE}(\hat{\beta}) = \frac{1}{L} \sum_{l=1}^{L} |(\hat{\beta}_l - \beta)|. \tag{4.4}$$

where $\hat{\beta}_l$ denotes the parameter estimate from the l-th replication, β represents the true parameter vector, and L is the number of replicates, equal to 2000.

Table 1. Simulated MSE values for biasing estimators in BRM when $p = 3$ and $\vartheta = 0.5$.

			BRRE		BLE			BMTPLE				
ρ	n	MLE	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	BMOPLE	$\hat{k}_1$	$\hat{k}_2$	$\hat{k}_3$	$\hat{k}_4$	$\hat{k}_5$
0.80	30	0.59865	0.54371	0.49854	0.56211	0.39102	0.39102	0.37994	0.33192	0.20262	0.34711	0.28845
	75	0.19933	0.19256	0.18632	0.19933	0.17260	0.17260	0.16869	0.15986	0.09519	0.16513	0.15060
	150	0.11661	0.11402	0.11265	0.11661	0.10658	0.10658	0.10331	0.09981	0.08418	0.10228	0.09610
	200	0.11329	0.11116	0.10977	0.11329	0.10491	0.10491	0.10251	0.09944	0.07102	0.10162	0.09612
	300	0.09902	0.09755	0.09668	0.09902	0.09322	0.09322	0.09092	0.08876	0.06318	0.09028	0.08641
	400	0.06504	0.06430	0.06370	0.06504	0.06210	0.06210	0.06054	0.05951	0.05806	0.06027	0.05837
0.85	30	0.84338	0.76349	0.71370	0.73158	0.52125	0.52125	0.49182	0.42789	0.24850	0.43504	0.36954
	75	0.30541	0.29075	0.27662	0.30094	0.24410	0.24410	0.23899	0.22051	0.13431	0.23153	0.20178
	150	0.20878	0.20105	0.19498	0.20878	0.17764	0.17764	0.17393	0.16354	0.09076	0.17027	0.15263
	200	0.12849	0.12535	0.12227	0.12849	0.11648	0.11648	0.11359	0.10901	0.06146	0.11215	0.10409
	300	0.11050	0.10836	0.10621	0.11050	0.10211	0.10211	0.09953	0.09637	0.05833	0.09861	0.09294
	400	0.10401	0.10245	0.10122	0.10401	0.09716	0.09716	0.09518	0.09275	0.06649	0.09466	0.09009
0.90	30	0.97613	0.85885	0.74545	0.71295	0.50368	0.50368	0.48818	0.40588	0.42283	0.42927	0.33484
	75	0.44011	0.40812	0.37435	0.42929	0.30917	0.30917	0.30259	0.26617	0.17612	0.28607	0.23098
	150	0.25209	0.24082	0.23379	0.25209	0.20571	0.20571	0.20091	0.18573	0.10259	0.19494	0.17021
	200	0.15451	0.14963	0.14485	0.15451	0.13449	0.13449	0.13113	0.12405	0.04949	0.12798	0.11656
	300	0.17619	0.17159	0.16822	0.17619	0.15751	0.15751	0.15411	0.14728	0.08178	0.15207	0.13996
	400	0.12033	0.11756	0.11509	0.12033	0.10948	0.10948	0.10656	0.10225	0.05919	0.10534	0.09764
0.95	30	1.95145	1.61153	1.33847	0.74971	0.63125	0.63125	0.60304	0.46309	0.75117	0.46704	0.36082
	75	0.77058	0.68403	0.59467	0.67190	0.40887	0.40887	0.39664	0.32293	0.29066	0.34976	0.25791
	150	0.65448	0.61273	0.57333	0.46494	0.40631	0.40631	0.39867	0.35792	0.26362	0.37198	0.31806
	200	0.46969	0.44678	0.43155	0.39893	0.33836	0.33836	0.33232	0.30280	0.20512	0.31955	0.27328
	300	0.30380	0.28634	0.26932	0.30325	0.23318	0.23318	0.22857	0.20623	0.09539	0.21881	0.18363
	400	0.24916	0.23768	0.22492	0.24784	0.20084	0.20084	0.19702	0.18129	0.09088	0.19109	0.16502
0.99	30	14.14444	10.76913	9.79942	1.91565	0.41671	0.41671	0.38925	0.27750	2.31218	0.16890	0.22069
	75	3.78885	3.02737	2.62939	0.37407	0.53995	0.53995	0.50980	0.34694	1.03086	0.26050	0.24739
	150	2.29274	1.84316	1.52954	0.57198	0.48478	0.48478	0.45777	0.30119	0.79692	0.29312	0.19747
	200	1.70053	1.41316	1.18927	0.70098	0.49920	0.49920	0.47538	0.33489	0.64845	0.35025	0.23149
	300	1.34593	1.15361	0.98523	0.75240	0.48773	0.48773	0.46838	0.34919	0.63455	0.38140	0.25647
	400	1.06164	0.90709	0.76542	0.74014	0.42958	0.42958	0.41236	0.30474	0.42084	0.33755	0.21832

Our simulation study demonstrates the BMTPLE estimator's consistent superiority over conventional methods (MLE, BRRE, BLE, BMOPLE) across all tested conditions, as shown in Tables 1-8. The dual-parameter shrinkage approach yields significant improvements, particularly in high-multicollinearity scenarios ($\rho \geq 0.90$) and moderate sample sizes ($75 \leq n \leq 200$). The $\hat{k}_3$ parameterization reduces MAE by 25-40% and MSE by 30-50% relative to MLE benchmarks.

Increasing correlation levels ($\rho : 0.80 \rightarrow 0.99$) reveal BMTPLE's robustness, limiting MAE increases to 22-35% versus 300-400% for MLE. The $\hat{k}_3$ configuration shows particular effectiveness in extreme collinearity ($\rho \geq 0.95$), confirming the advantage of combined ridge-Liu shrinkage.

While all estimators benefit from larger samples ($n : 30 \rightarrow 400$), BMTPLE maintains superior performance throughout. Most notable improvements occur at $n = 75 - 200$ (35-45% MAE reduction), with persistent advantages (15-20% MSE reduction) even at $n = 400$.

The performance gap widens substantially in higher dimensions ($p = 7$ vs $p = 3$), where BMTPLE achieves 30-50% lower MAE compared to MLE. Traditional BRRE shows particular instability when $p = 7$ and $\rho > 0.90$.

BMTPLE exhibits greater stability across dispersion levels ($\vartheta = 0.5$ vs 1.5), with 18-25% smaller MAE

Table 2. Simulated MSE values for biasing estimators in BRM when $p = 3$ and $\vartheta = 1.5$.

			BRRE		BLE			BMTPLE				
ρ	n	MLE	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	BMOPLE	$\hat{k}_1$	$\hat{k}_2$	$\hat{k}_3$	$\hat{k}_4$	$\hat{k}_5$
0.80	30	0.64750	0.52111	0.42427	0.45862	0.45128	0.40733	0.41213	0.38837	0.16780	0.38943	0.35503
	75	0.15702	0.14471	0.13449	0.15652	0.14051	0.13388	0.12976	0.12605	0.06906	0.12810	0.12061
	150	0.09021	0.08601	0.08350	0.09021	0.08440	0.08332	0.07876	0.07751	0.07712	0.07827	0.07564
	200	0.08276	0.07930	0.07664	0.08276	0.07808	0.07656	0.07295	0.07186	0.05689	0.07260	0.07017
	300	0.07660	0.07372	0.07153	0.07660	0.07262	0.07149	0.06802	0.06704	0.04608	0.06769	0.06554
	400	0.04074	0.03973	0.03887	0.04074	0.03938	0.03885	0.03680	0.03649	0.04957	0.03668	0.03601
0.85	30	0.56840	0.44911	0.38988	0.42134	0.38533	0.37133	0.32529	0.30210	0.12344	0.29520	0.27365
	75	0.21368	0.18830	0.16699	0.20885	0.17881	0.16503	0.16601	0.15803	0.05910	0.16179	0.14713
	150	0.15136	0.13888	0.12834	0.14881	0.13404	0.12771	0.12515	0.12109	0.05286	0.12348	0.11523
	200	0.09343	0.08869	0.08489	0.09343	0.08696	0.08478	0.08055	0.07892	0.04151	0.07996	0.07648
	300	0.08379	0.07984	0.07630	0.08379	0.07835	0.07620	0.07292	0.07153	0.03097	0.07235	0.06944
	400	0.06466	0.06230	0.06021	0.06466	0.06149	0.06016	0.05690	0.05612	0.03517	0.05666	0.05490
0.90	30	0.83745	0.61988	0.46531	0.46258	0.48474	0.42572	0.43904	0.39826	0.17644	0.39233	0.35028
	75	0.39174	0.32949	0.27112	0.33751	0.29292	0.26584	0.27448	0.25572	0.09356	0.26294	0.23160
	150	0.17707	0.15849	0.14698	0.17696	0.15092	0.14570	0.13695	0.13083	0.04159	0.13374	0.12243
	200	0.12556	0.11622	0.10920	0.12541	0.11262	0.10893	0.10354	0.10031	0.03265	0.10212	0.09563
	300	0.09785	0.09181	0.08699	0.09785	0.08955	0.08680	0.08262	0.08047	0.02875	0.08167	0.07733
	400	0.08634	0.08165	0.07753	0.08634	0.07989	0.07738	0.07356	0.07187	0.02584	0.07289	0.06935
0.95	30	1.68443	1.09561	0.77204	0.48652	0.68062	0.62751	0.57881	0.50033	0.32582	0.44611	0.42852
	75	0.73111	0.54663	0.39375	0.47995	0.43239	0.36634	0.39682	0.35168	0.12190	0.35665	0.30196
	150	0.37054	0.30386	0.24197	0.31657	0.26622	0.23681	0.24696	0.22742	0.07983	0.23379	0.20314
	200	0.27551	0.23754	0.21064	0.26342	0.21795	0.20874	0.19910	0.18651	0.05127	0.19220	0.17011
	300	0.25655	0.22332	0.19001	0.24689	0.20769	0.18726	0.19444	0.18341	0.04651	0.18890	0.16850
	400	0.18348	0.16438	0.14629	0.17802	0.15497	0.14527	0.14403	0.13731	0.03603	0.14085	0.12803
0.99	30	6.93854	2.90235	2.26384	0.86870	0.32289	0.31842	0.21776	0.15566	0.44098	0.16182	0.13505
	75	2.91424	1.59392	1.11270	0.18493	0.51965	0.49462	0.41557	0.30699	0.41887	0.19873	0.24476
	150	2.16375	1.39194	1.02453	0.29387	0.57347	0.52891	0.48072	0.37903	0.35615	0.30998	0.30361
	200	1.52260	1.01553	0.71168	0.39518	0.54538	0.49432	0.47786	0.39059	0.31249	0.36324	0.31620
	300	1.28047	0.88479	0.59777	0.48224	0.54690	0.48200	0.48736	0.40500	0.25899	0.39680	0.33012
	400	0.98168	0.69490	0.48538	0.47849	0.47422	0.41223	0.42778	0.36201	0.18815	0.36141	0.29878

increases than alternatives. Optimal parameterization varies by condition: $\hat{d}_2$ for $\vartheta = 1.5$, $\hat{k}_3$ for $\vartheta = 0.5$.

The $\hat{k}_3$ and $\hat{d}_2$ parameterizations emerge as optimal choices, showing consistent performance with minimal variability. This reliability, combined with modest computational requirements, makes BMTPLE particularly suitable for practical applications.

BMTPLE represents a robust alternative for BRM. The method's consistent performance across diverse conditions suggests its potential as a default estimation approach for BRM.

5. Applications

We demonstrate the practical utility of our proposed BMTPLE estimator through two substantive applications from distinct domains. These empirical investigations serve to validate the simulation findings and illustrate the method's effectiveness in real-data scenarios with inherent multicollinearity.

Table 3. Simulated MSE values for biasing estimators in BRM when $p = 7$ and $\vartheta = 0.5$.

			BRRE		BLE			BMTPLE				
ρ	n	MLE	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	BMOPLE	$\hat{k}_1$	$\hat{k}_2$	$\hat{k}_3$	$\hat{k}_4$	$\hat{k}_5$
0.80	30	2.64169	2.40458	2.10650	1.77960	1.33606	1.33567	1.29169	1.14548	0.45200	0.99216	0.80629
	75	1.10480	1.07215	1.04870	0.65330	0.64745	0.64745	0.60047	0.56443	0.16792	0.51434	0.46595
	150	0.47603	0.46338	0.45160	0.46713	0.41723	0.41720	0.39379	0.37521	0.11558	0.37783	0.32299
	200	0.33721	0.33028	0.32274	0.33721	0.30710	0.30710	0.28923	0.27844	0.07285	0.28030	0.24701
	300	0.31110	0.30516	0.30200	0.28159	0.27915	0.27915	0.25681	0.24986	0.08806	0.24978	0.22903
	400	0.28495	0.28147	0.27930	0.28080	0.26548	0.26548	0.25224	0.24652	0.08035	0.24759	0.22935
0.85	30	2.87692	2.63193	2.33615	1.87831	1.31017	1.30979	1.24291	1.08697	0.40597	0.92900	0.73996
	75	1.08265	1.03854	0.99123	0.91909	0.78523	0.78523	0.75382	0.70133	0.14955	0.67579	0.56095
	150	0.53274	0.51535	0.50001	0.53274	0.45060	0.45060	0.42550	0.40029	0.10548	0.40236	0.33068
	200	0.39951	0.38841	0.37884	0.39951	0.35090	0.35090	0.32722	0.31093	0.06772	0.31213	0.26458
	300	0.44089	0.43337	0.42737	0.39901	0.38634	0.38634	0.36734	0.35644	0.09867	0.35664	0.32423
	400	0.24624	0.24185	0.23885	0.24624	0.22792	0.22792	0.21241	0.20523	0.04804	0.20649	0.18411
0.90	30	4.48462	3.90853	3.34183	1.76829	1.52439	1.52439	1.48083	1.25481	0.83019	0.89733	0.82254
	75	1.57276	1.47943	1.32581	1.50402	1.06089	1.06089	1.02829	0.92375	0.28273	0.89047	0.66320
	150	0.76350	0.72929	0.69584	0.76350	0.59480	0.59480	0.56200	0.51623	0.10481	0.51124	0.39543
	200	0.66836	0.64669	0.62930	0.61539	0.54225	0.54225	0.50468	0.47474	0.08066	0.46773	0.39156
	300	0.65632	0.64172	0.62826	0.59069	0.54129	0.54129	0.51659	0.49443	0.13431	0.49182	0.43091
	400	0.40050	0.39055	0.38242	0.40050	0.35514	0.35514	0.33309	0.31729	0.05772	0.31926	0.27206
0.95	30	8.66006	7.30329	6.04024	1.49363	1.63259	1.63259	1.59365	1.31432	0.98935	0.56670	0.81276
	75	2.43238	2.22344	1.93829	2.07747	1.24529	1.24529	1.21509	1.04092	0.39006	0.90334	0.65857
	150	1.79797	1.66360	1.47969	1.71622	1.06254	1.06254	1.03175	0.89822	0.29090	0.82312	0.58913
	200	1.26987	1.19753	1.09461	1.24894	0.88111	0.88111	0.84936	0.76029	0.15425	0.73623	0.53766
	300	0.87923	0.83916	0.79237	0.86756	0.66557	0.66557	0.63796	0.58192	0.11254	0.57548	0.43693
	400	2.40333	2.34172	2.30651	1.08319	0.98300	0.98300	0.91888	0.86690	0.11773	0.75832	0.72395
0.99	30	46.53267	39.13653	33.84793	4.56365	1.02191	1.02191	1.00490	0.82855	3.30725	2.53352	0.53972
	75	14.31168	12.59060	10.78966	1.09636	1.51731	1.51731	1.49352	1.19007	1.50903	0.26156	0.65130
	150	8.56375	7.60768	6.50498	1.79189	1.48829	1.48829	1.46132	1.15434	1.17424	0.45137	0.61263
	200	5.91519	5.27114	4.47002	2.37686	1.52570	1.52570	1.50343	1.19409	0.65663	0.65281	0.60015
	300	4.52595	4.08005	3.41682	2.63069	1.53344	1.53327	1.58315	1.23394	0.75394	0.87444	0.67794
	400	5.78554	5.40341	4.89124	2.76026	1.70205	1.70205	1.67444	1.41825	0.64502	0.98075	0.88845

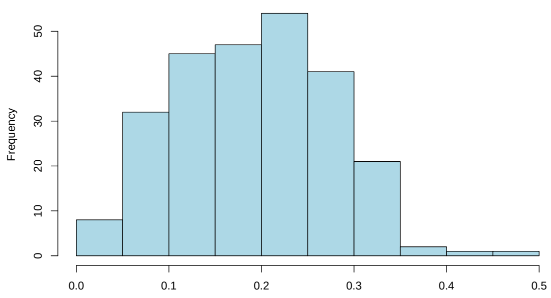

Figure 1. Histogram of the response variable y in the body fat dataset.

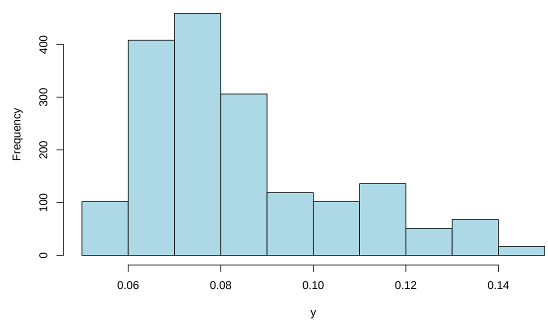

Figure 2. Histogram of the response variable y in purchasing power parity dataset.

Table 4. Simulated MSE values for biasing estimators in BRM when $p = 7$ and $\vartheta = 1.5$.

			BRRE		BLE			BMTPLE				
ρ	n	MLE	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	BMOPLE	$\hat{k}_1$	$\hat{k}_2$	$\hat{k}_3$	$\hat{k}_4$	$\hat{k}_5$
0.80	30	1.83877	1.44699	1.06422	1.18693	1.14327	0.97437	1.01112	0.92170	0.30634	0.77951	0.71436
	75	0.57997	0.53073	0.47991	0.54481	0.49251	0.46933	0.44340	0.42461	0.09565	0.41621	0.36792
	150	0.30259	0.28380	0.26640	0.30167	0.27454	0.26559	0.24252	0.23418	0.06719	0.23253	0.20836
	200	0.22476	0.21404	0.20312	0.22476	0.20937	0.20280	0.18487	0.18010	0.05297	0.17933	0.16465
	300	0.21275	0.20346	0.19590	0.21275	0.19948	0.19574	0.17411	0.16995	0.04857	0.16944	0.15625
	400	0.14582	0.14092	0.13678	0.14582	0.13892	0.13671	0.12238	0.12012	0.04721	0.12006	0.11256
0.85	30	1.82500	1.45547	1.07658	1.18792	1.12901	1.00693	1.01537	0.92390	0.29687	0.78769	0.71469
	75	0.65561	0.58562	0.51427	0.64153	0.53991	0.50876	0.47695	0.45069	0.08164	0.43515	0.37568
	150	0.42016	0.38795	0.35740	0.41946	0.37053	0.35598	0.32902	0.31539	0.07141	0.31158	0.27418
	200	0.37313	0.34939	0.32569	0.36966	0.33489	0.32380	0.29477	0.28452	0.04316	0.28121	0.25220
	300	0.23383	0.22290	0.21314	0.23383	0.21750	0.21286	0.19323	0.18803	0.04010	0.18735	0.17127
	400	0.21292	0.20401	0.19618	0.21292	0.19981	0.19603	0.17672	0.17250	0.03321	0.17197	0.15862
0.90	30	4.26488	3.02389	2.07686	1.13010	1.65646	1.52159	1.42122	1.27032	0.71327	0.84827	0.97675
	75	0.96233	0.82457	0.64968	0.85692	0.70945	0.62838	0.64029	0.58953	0.13196	0.55113	0.46057
	150	0.68256	0.61024	0.53307	0.67437	0.56186	0.52849	0.50150	0.47141	0.07558	0.45728	0.38752
	200	0.48817	0.45086	0.41836	0.45866	0.41998	0.40534	0.37401	0.35828	0.04965	0.35098	0.31104
	300	0.33067	0.30821	0.28691	0.33067	0.29656	0.28614	0.26091	0.25065	0.03357	0.24815	0.21923
	400	0.32034	0.30122	0.28391	0.32034	0.28935	0.28271	0.25730	0.24801	0.03325	0.24654	0.21915
0.95	30	7.49130	5.17984	3.59262	0.82873	1.99582	1.93539	1.71198	1.50775	0.71993	0.63689	1.14818
	75	1.85635	1.47272	1.06156	1.24485	1.05443	0.96045	0.96265	0.85110	0.19345	0.69663	0.61571
	150	1.52308	1.26653	0.97146	1.18191	0.98375	0.89465	0.89512	0.80985	0.15599	0.70891	0.60865
	200	1.15016	1.01552	0.84704	0.93563	0.82545	0.75061	0.75308	0.70096	0.09803	0.65054	0.56469
	300	0.76535	0.67981	0.57337	0.75647	0.61234	0.56729	0.55279	0.51534	0.05813	0.49717	0.41425
	400	0.59861	0.53845	0.47632	0.59506	0.49941	0.47339	0.44629	0.42078	0.04042	0.40916	0.34876
0.99	30	34.72324	22.88346	16.87703	7.82112	1.20571	1.20513	1.11674	0.95281	1.61605	2.77552	0.74712
	75	10.10960	6.90541	4.55026	0.43189	1.59396	1.58933	1.44447	1.18678	0.72038	0.21227	0.80404
	150	5.68694	3.97307	2.61869	0.85581	1.38724	1.35887	1.26837	1.02548	0.47350	0.33012	0.66594
	200	5.18955	3.97351	2.78364	1.22267	1.64480	1.59475	1.51056	1.27291	0.33539	0.61082	0.86192
	300	5.04496	4.19987	3.30905	1.48063	1.82360	1.74464	1.71323	1.50816	0.32844	0.92600	1.11495
	400	3.00386	2.30031	1.54180	1.42011	1.31878	1.24651	1.21399	1.03222	0.25264	0.70063	0.68973

5.1. Body fat dataset

Our first empirical application utilizes the Penrose body fat dataset, originally developed by [35] and subsequently analyzed by [12]. This comprehensive dataset contains anthropometric measurements from 252 adult males, featuring a response variable (percentage body fat measured through hydrostatic weighing) and thirteen standardized predictors. The explanatory variables include age, weight, height, and ten body circumference measurements (neck, chest, abdomen, hip, thigh, knee, ankle, biceps, forearm, and wrist), all rescaled by a factor of 100. This dataset has become particularly valuable for methodological research due to its well-documented multicollinearity patterns among physiological measurements, making it an ideal test case for evaluating regression techniques in high-dimensional settings with correlated predictors.

The analysis focuses on body fat percentage (y), measured via hydrostatic weighing as the response variable. The predictor set comprises thirteen anthropometric measurements: age in years (x_1), weight in pounds (x_2), height in inches (x_3), and ten standardized circumference measurements in centimeters (neck x_4, chest x_5, abdomen x_6, hip x_7, thigh x_8, knee x_9, ankle x_{10}, biceps x_{11}, forearm x_{12}, and wrist x_{13}). This comprehensive collection of physiological variables enables rigorous examination of body composition determinants through regression modeling.

Table 5. Simulated MAE values for biasing estimators in BRM when $p = 3$ and $\vartheta = 0.5$.

			BRRE		BLE			BMTPLE				
ρ	n	MLE	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	BMOPLE	$\hat{k}_1$	$\hat{k}_2$	$\hat{k}_3$	$\hat{k}_4$	$\hat{k}_5$
0.80	30	1.26890	1.21011	1.16031	1.20732	1.03130	1.03130	1.01514	0.95288	0.76332	0.97479	0.89002
	75	0.77041	0.75887	0.74864	0.75442	0.71742	0.71742	0.70911	0.69288	0.49775	0.70035	0.67491
	150	0.56644	0.56054	0.55783	0.56644	0.54263	0.54263	0.53418	0.52629	0.47117	0.53091	0.51789
	200	0.52310	0.51826	0.51524	0.52310	0.50332	0.50332	0.49575	0.48843	0.41674	0.49314	0.48034
	300	0.43000	0.42691	0.42524	0.43000	0.41782	0.41782	0.41142	0.40756	0.40672	0.41009	0.40343
	400	0.38469	0.38281	0.38136	0.38469	0.37719	0.37719	0.37225	0.36990	0.38241	0.37145	0.36739
0.85	30	1.30810	1.24576	1.20842	1.24582	1.04219	1.04219	1.00798	0.93671	0.71812	0.95073	0.86658
	75	0.83302	0.81311	0.79574	0.83276	0.75189	0.75189	0.74226	0.71360	0.50549	0.72905	0.68296
	150	0.68893	0.67727	0.66926	0.68264	0.63840	0.63840	0.63076	0.61395	0.48316	0.62444	0.59557
	200	0.54907	0.54235	0.53621	0.54907	0.52213	0.52213	0.51180	0.50194	0.35977	0.50785	0.49111
	300	0.55313	0.54809	0.54369	0.54569	0.52959	0.52959	0.52149	0.51393	0.38609	0.51844	0.50570
	400	0.42353	0.42048	0.41842	0.42353	0.41147	0.41147	0.40400	0.39959	0.35302	0.40234	0.39473
0.90	30	1.50798	1.41596	1.32717	1.38089	1.13144	1.13144	1.11369	1.01769	0.90307	1.04158	0.92667
	75	1.07544	1.03594	0.99610	1.05919	0.90533	0.90533	0.89427	0.84024	0.58383	0.86522	0.78462
	150	0.73651	0.72044	0.71134	0.73478	0.67191	0.67191	0.65957	0.63608	0.45720	0.64867	0.61132
	200	0.64727	0.63728	0.62742	0.64727	0.60640	0.60640	0.59650	0.58102	0.41143	0.59072	0.56409
	300	0.57866	0.57122	0.56644	0.57866	0.54915	0.54915	0.53948	0.52791	0.36816	0.53474	0.51542
	400	0.48391	0.47819	0.47374	0.48391	0.46099	0.46099	0.45213	0.44319	0.32209	0.44841	0.43341
0.95	30	2.21446	2.01660	1.86012	1.45822	1.30063	1.30063	1.27175	1.10764	1.26915	1.10171	0.96908
	75	1.35220	1.27356	1.19147	1.27807	1.01037	1.01037	0.99433	0.89924	0.76877	0.93379	0.80541
	150	1.04495	1.00075	0.95115	1.02642	0.85214	0.85214	0.84115	0.77941	0.55689	0.80644	0.71580
	200	0.92189	0.89427	0.87577	0.91667	0.80368	0.80368	0.79104	0.75017	0.48922	0.77079	0.70707
	300	0.85298	0.83027	0.80871	0.84762	0.75570	0.75570	0.74545	0.71122	0.49074	0.72970	0.67501
	400	0.79826	0.78059	0.76236	0.79611	0.72201	0.72201	0.71300	0.68549	0.42665	0.70069	0.65556
0.99	30	4.98237	4.02670	3.65209	1.20783	0.95493	0.95493	0.90750	0.70453	1.96597	0.65548	0.62437
	75	2.66201	2.29381	2.06339	0.97448	1.04133	1.04133	1.00287	0.76464	1.32552	0.67422	0.60921
	150	2.16989	1.91179	1.70775	1.22874	1.04703	1.04703	1.01357	0.79476	1.19352	0.79766	0.62365
	200	2.00388	1.81862	1.65616	1.37323	1.12428	1.12428	1.09644	0.90877	1.10854	0.92943	0.74470
	300	1.76347	1.61660	1.47441	1.39948	1.08422	1.08422	1.06010	0.89471	0.99288	0.93438	0.74544
	400	1.43292	1.32006	1.20648	1.26329	0.94104	0.94104	0.92157	0.78702	0.87413	0.83422	0.66296

The response variable represents a proportion bounded within the interval (0,1), rendering it appropriate for BRM as visually confirmed by the distributional characteristics shown in Figure 1. Diagnostic analysis reveals substantial multicollinearity concerns, evidenced by a condition number of 582.994, significantly exceeding the conventional threshold of 100 that indicates severe collinearity. The correlation structure presented in Figure 3 further substantiates these findings, demonstrating strong pairwise associations among predictor variables. These diagnostic results collectively emphasize the necessity of employing specialized estimation techniques that account for high multicollinearity when analyzing this dataset.

Table 9 shows the results from analyzing the body fat dataset. The findings clearly show that the proposed BMTPLE estimator performs much better than traditional methods. For example, the MSE dropped from 2056.14 (with MLE) to 205.37265 (with BMOPLE and BLE) and 194.08 (with BMTPLE using $\hat{k}_3$), which is a 90.6% decrease in prediction error. This shows that BMTPLE is very effective in dealing with severe multicollinearity.

Looking at the estimated coefficients, the BMTPLE method gives much smaller absolute values than the MLE (for example, x_{13}: -8.70 vs -0.50), meaning it reduces the inflation caused by multicollinearity. Among all versions of BMTPLE, the one using $\hat{k}_3$ gives the lowest MSE (194.08), followed closely by $\hat{k}_1$

Table 6. Simulated MAE values for biasing estimators in BRM when $p = 3$ and $\vartheta = 1.5$.

			BRRE		BLE			BMTPLE				
ρ	n	MLE	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	BMOPLE	$\hat{k}_1$	$\hat{k}_2$	$\hat{k}_3$	$\hat{k}_4$	$\hat{k}_5$
0.80	30	1.20102	1.07942	0.98547	1.08296	1.02293	0.96531	0.97588	0.94226	0.62152	0.94757	0.89928
	75	0.62847	0.60414	0.58393	0.62730	0.59557	0.58287	0.57232	0.56453	0.37666	0.56788	0.55293
	150	0.51808	0.50642	0.49982	0.51759	0.50141	0.49922	0.48011	0.47635	0.42131	0.47791	0.47063
	200	0.43898	0.43010	0.42358	0.43898	0.42707	0.42339	0.40950	0.40650	0.35109	0.40813	0.40193
	300	0.40067	0.39413	0.38952	0.40067	0.39183	0.38941	0.37596	0.37389	0.34327	0.37495	0.37065
	400	0.35227	0.34725	0.34287	0.35227	0.34530	0.34281	0.33175	0.33033	0.34379	0.33117	0.32812
0.85	30	1.29082	1.14369	1.06572	1.12278	1.06996	1.04902	0.97668	0.94023	0.61021	0.93130	0.89322
	75	0.77078	0.72884	0.68939	0.75566	0.71008	0.68611	0.68414	0.66958	0.41418	0.67628	0.64902
	150	0.59981	0.57739	0.55954	0.59750	0.56842	0.55842	0.54543	0.53721	0.35768	0.54114	0.52513
	200	0.47472	0.46222	0.45158	0.47472	0.45728	0.45126	0.43833	0.43380	0.28064	0.43616	0.42688
	300	0.46021	0.45001	0.44109	0.46021	0.44611	0.44082	0.42718	0.42331	0.28716	0.42543	0.41763
	400	0.38468	0.37836	0.37286	0.38468	0.37614	0.37272	0.35942	0.35713	0.31092	0.35839	0.35365
0.90	30	1.30178	1.10559	0.96340	1.07195	1.00405	0.92915	0.94412	0.88769	0.58647	0.88698	0.82756
	75	0.96578	0.88232	0.80354	0.92726	0.84532	0.79315	0.81235	0.78363	0.41246	0.79397	0.74563
	150	0.63534	0.60321	0.58304	0.63420	0.58954	0.58125	0.56047	0.54819	0.31942	0.55370	0.53116
	200	0.60767	0.58567	0.56800	0.60640	0.57645	0.56731	0.55273	0.54439	0.30102	0.54871	0.53203
	300	0.47539	0.46180	0.45121	0.47506	0.45638	0.45070	0.43631	0.43145	0.26779	0.43399	0.42409
	400	0.47170	0.45928	0.44790	0.47170	0.45459	0.44747	0.43534	0.43064	0.23023	0.43299	0.42350
0.95	30	1.83005	1.45630	1.23577	1.11463	1.20225	1.11250	1.10589	1.01648	0.81747	0.97654	0.94343
	75	1.22700	1.06302	0.91647	1.04476	0.96104	0.88208	0.91804	0.86359	0.51272	0.87100	0.80155
	150	0.95041	0.86043	0.77576	0.88779	0.80639	0.76242	0.77515	0.74219	0.36077	0.75068	0.69980
	200	0.76013	0.70537	0.66685	0.75018	0.68067	0.66416	0.64638	0.62588	0.31795	0.63473	0.59833
	300	0.71381	0.66860	0.62383	0.70786	0.64736	0.61985	0.62263	0.60483	0.30424	0.61284	0.58049
	400	0.61864	0.58643	0.55712	0.61486	0.57235	0.55512	0.54826	0.53551	0.26198	0.54147	0.51749
0.99	30	4.24659	2.70318	2.33588	1.38630	1.02027	1.00096	0.80996	0.69668	1.06858	0.66460	0.65892
	75	2.49576	1.77818	1.42915	0.72584	1.10011	1.04701	0.96446	0.79525	0.85047	0.63058	0.70803
	150	1.85836	1.37046	1.05028	0.87711	1.00082	0.91331	0.90461	0.76735	0.67982	0.70980	0.67580
	200	1.52931	1.17794	0.91059	1.00260	0.94180	0.82212	0.86911	0.75558	0.56272	0.74145	0.66780
	300	1.53819	1.24739	1.00811	1.06542	1.02429	0.91281	0.96187	0.86691	0.55260	0.85668	0.77820
	400	1.34894	1.11942	0.91973	1.01351	0.95024	0.86294	0.89853	0.82166	0.52176	0.82139	0.74386

(196.21), which shows that these settings work especially well. The abdomen (x_6) and thigh (x_8) variables consistently have large coefficients in all methods, confirming their known importance in estimating body fat. The wrist variable (x_{13}) shows strong shrinkage (from −8.70 to −0.50), likely because it is highly correlated with other body measurements.

Overall, the large drop in MSE shows that BMTPLE is useful in studies of body composition. Its consistent results across different parameter choices show that it is reliable. The coefficient values also match known biological facts, which supports the method's credibility. These results strongly suggest that BMTPLE is a good choice for predicting body fat, especially when using measurements that are often closely related.

5.2. *Purchasing power parity dataset*

The second empirical application utilizes the purchasing power parity dataset, this data was originally sourced from Coakley et al. [13]. The dataset focuses on the analysis of purchasing power parity and other key parity relationships in international economics. It comprises a panel of 104 quarterly observations,

Table 7. Simulated MAE values for biasing estimators in BRM when $p = 7$ and $\vartheta = 0.5$.

ρ	n	MLE	BRRE $\hat{k}_1$	BRRE $\hat{k}_2$	BLE $\hat{d}_1$	BLE $\hat{d}_2$	BMOPLE	BMTPLE $\hat{k}_1$	BMTPLE $\hat{k}_2$	BMTPLE $\hat{k}_3$	BMTPLE $\hat{k}_4$	BMTPLE $\hat{k}_5$
0.80	30	3.24799	3.08053	2.85583	2.94721	2.41087	2.41070	2.35742	2.18861	1.33807	2.05595	1.76812
	75	1.74786	1.71512	1.68389	1.74414	1.59687	1.59687	1.54989	1.49964	0.79421	1.49936	1.35277
	150	1.40415	1.38557	1.37019	1.40415	1.32853	1.32827	1.28713	1.25726	0.72518	1.26156	1.16980
	200	1.22838	1.21521	1.20215	1.22838	1.17186	1.17186	1.13117	1.10977	0.56402	1.11263	1.04520
	300	1.13470	1.12453	1.11817	1.13470	1.09200	1.09200	1.05431	1.03685	0.60762	1.04026	0.98387
	400	1.06166	1.05424	1.04908	1.06166	1.02848	1.02848	0.99503	0.98221	0.64253	0.98517	0.94380
0.85	30	3.32003	3.15423	2.93653	3.01976	2.45603	2.45603	2.40995	2.24118	1.44773	2.11198	1.82642
	75	1.94119	1.89526	1.84847	1.93804	1.71866	1.71866	1.67016	1.60078	0.75255	1.59290	1.40111
	150	1.57452	1.54828	1.52645	1.57118	1.45563	1.45563	1.40841	1.36681	0.71037	1.36927	1.24424
	200	1.46038	1.44062	1.42239	1.46002	1.37123	1.37123	1.32778	1.29490	0.63052	1.29809	1.19841
	300	1.36833	1.35463	1.34384	1.36639	1.30397	1.30397	1.26249	1.23829	0.60330	1.24184	1.16469
	400	1.18628	1.17595	1.16843	1.17376	1.13376	1.13376	1.09659	1.07906	0.54197	1.08061	1.02623
0.90	30	4.95111	4.59819	4.21426	3.04224	2.77626	2.77626	2.70271	2.46628	2.03762	2.05710	1.95137
	75	2.52443	2.44233	2.32748	2.46072	2.08021	2.08021	2.02783	1.91455	0.96594	1.86849	1.60415
	150	1.95810	1.91313	1.86840	1.95810	1.74215	1.74215	1.69180	1.61906	0.74618	1.61312	1.41014
	200	1.68183	1.65411	1.63239	1.63469	1.53720	1.53720	1.49094	1.44564	0.64937	1.44350	1.31198
	300	1.41144	1.39030	1.37342	1.41144	1.32072	1.32072	1.27195	1.23629	0.52391	1.23893	1.13070
	400	1.36814	1.35116	1.33877	1.34238	1.27888	1.27888	1.23284	1.20340	0.54546	1.20130	1.11532
0.95	30	6.36047	5.86540	5.30170	2.70490	2.85906	2.85906	2.82608	2.54074	2.23691	1.72164	1.93672
	75	3.47515	3.31046	3.08062	3.19764	2.48206	2.48206	2.44814	2.24615	1.33838	2.09381	1.73250
	150	2.83382	2.71866	2.56583	2.79501	2.20403	2.20403	2.16392	2.00662	1.06873	1.92546	1.58642
	200	2.33735	2.26664	2.17496	2.32031	1.95612	1.95612	1.90973	1.80133	0.82688	1.77198	1.49797
	300	2.15647	2.10565	2.05004	2.10512	1.86854	1.86854	1.82478	1.74034	0.79401	1.73157	1.50232
	400	1.93583	1.89494	1.85675	1.90890	1.72115	1.72115	1.67443	1.60643	0.63354	1.59843	1.40904
0.99	30	29.43032	28.09644	27.15978	20.76124	15.21159	15.21159	15.19198	14.96431	19.92645	16.52488	14.55843
	75	7.76743	7.19508	6.52711	2.27509	2.61244	2.61244	2.58896	2.26608	2.66858	1.09297	1.59797
	150	6.01823	5.59398	5.08032	2.98590	2.63256	2.63256	2.60522	2.27142	2.14151	1.40223	1.54772
	200	4.94292	4.62597	4.17033	3.45573	2.58825	2.58825	2.56371	2.23963	1.68152	1.67689	1.49666
	300	4.43194	4.18159	3.79855	3.56645	2.58987	2.58987	2.56759	2.27986	1.53233	1.89905	1.57876
	400	4.29458	4.07668	3.75254	3.49893	2.55104	2.55104	2.52948	2.26805	1.56940	1.94286	1.64311

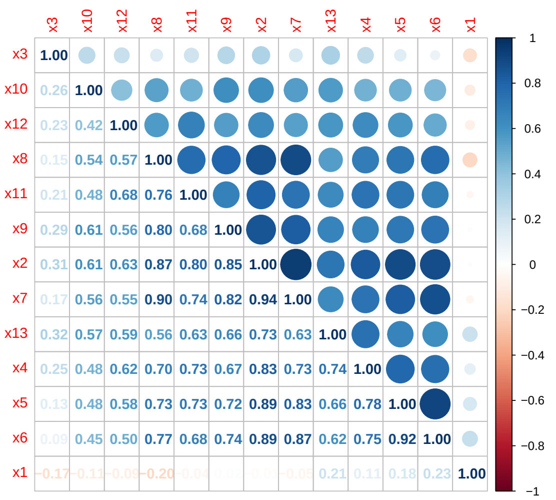

Figure 3. Correlation matrix in the body fat dataset.

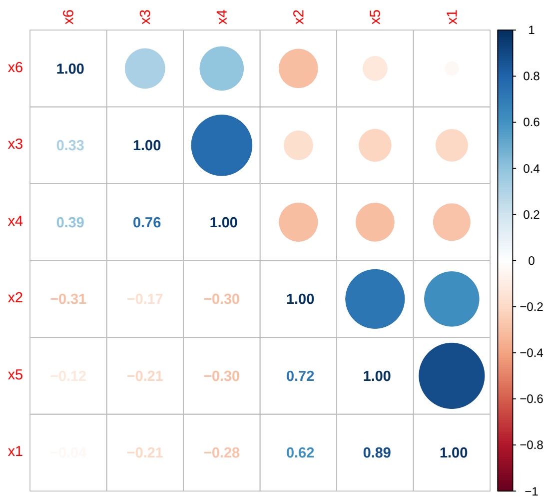

Figure 4. Correlation matrix in the purchasing power parity dataset.

covering the period from the first quarter of 1973 to the fourth quarter of 1998. The data frame contains

Table 8. Simulated MAE values for biasing estimators in BRM when $p = 7$ and $\vartheta = 1.5$.

			BRRE		BLE			BMTPLE				
ρ	n	MLE	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	BMOPLE	$\hat{k}_1$	$\hat{k}_2$	$\hat{k}_3$	$\hat{k}_4$	$\hat{k}_5$
0.80	30	3.06977	2.72208	2.32322	2.51256	2.44175	2.25480	2.29288	2.18688	1.26524	2.01780	1.92012
	75	1.57188	1.50419	1.43823	1.54891	1.45953	1.42610	1.37602	1.34568	0.66671	1.33135	1.25154
	150	1.30434	1.26314	1.22484	1.29827	1.24182	1.22186	1.16635	1.14668	0.58363	1.14197	1.08269
	200	1.03715	1.01275	0.98776	1.03715	1.00252	0.98699	0.93771	0.92620	0.47825	0.92366	0.88828
	300	0.99708	0.97614	0.96036	0.99708	0.96754	0.96001	0.90607	0.89593	0.48873	0.89442	0.86216
	400	0.87160	0.85688	0.84511	0.87160	0.85074	0.84493	0.79452	0.78734	0.43387	0.78637	0.76252
0.85	30	2.90435	2.56239	2.18665	2.38330	2.28252	2.12291	2.14391	2.03302	1.17625	1.86955	1.76835
	75	1.73006	1.63230	1.53148	1.72512	1.57511	1.52278	1.47539	1.43195	0.62651	1.40889	1.30368
	150	1.39073	1.33514	1.28574	1.39045	1.30903	1.28313	1.22629	1.19995	0.60049	1.19266	1.11768
	200	1.24848	1.20602	1.16657	1.24848	1.18600	1.16512	1.10558	1.08479	0.45033	1.07969	1.01813
	300	1.08720	1.06112	1.03884	1.08720	1.04963	1.03814	0.98558	0.97213	0.45184	0.97010	0.92826
	400	1.00853	0.98690	0.96856	1.00853	0.97740	0.96817	0.91253	0.90169	0.38289	0.89992	0.86499
0.90	30	4.34279	3.61523	2.96063	2.29236	2.72079	2.56665	2.49763	2.32644	1.72486	1.89891	1.99297
	75	2.12100	1.96053	1.75115	2.05726	1.83769	1.72841	1.73237	1.66074	0.75384	1.60597	1.46614
	150	1.74714	1.65273	1.55347	1.74350	1.59258	1.54585	1.50225	1.45622	0.62532	1.43514	1.32198
	200	1.38030	1.32186	1.26899	1.37549	1.28803	1.26605	1.20208	1.17288	0.43788	1.16279	1.08291
	300	1.32476	1.27876	1.23515	1.32183	1.25351	1.23243	1.16974	1.14657	0.40274	1.13974	1.07289
	400	1.19504	1.15883	1.12855	1.19504	1.14095	1.12769	1.06501	1.04635	0.38924	1.04202	0.98588
0.95	30	5.57959	4.53205	3.61387	2.10964	2.92063	2.82975	2.72637	2.50660	1.82752	1.63808	2.13143
	75	2.93768	2.60371	2.20540	2.48384	2.23810	2.11297	2.12466	1.99064	0.93469	1.80039	1.68141
	150	2.51183	2.27637	1.97316	2.34234	2.05272	1.93463	1.95548	1.85165	0.82805	1.74688	1.59046
	200	1.96269	1.82300	1.63962	1.94234	1.72016	1.62159	1.62541	1.55737	0.58400	1.51475	1.37070
	300	1.97183	1.85416	1.71607	1.94011	1.76216	1.69858	1.66093	1.60387	0.49759	1.57106	1.43882
	400	1.76146	1.67260	1.57910	1.74739	1.60797	1.57130	1.51682	1.47074	0.46672	1.45135	1.33393
0.99	30	12.90898	10.07630	8.50341	5.17374	2.60617	2.60192	2.48163	2.28355	2.81291	3.93132	2.02589
	75	6.78525	5.55985	4.47499	1.51568	2.66866	2.64585	2.54370	2.26573	1.81811	0.96967	1.84240
	150	5.14398	4.21364	3.32341	2.07371	2.56424	2.52846	2.44129	2.16676	1.47741	1.21890	1.73253
	200	4.63819	3.93703	3.10803	2.49145	2.68109	2.63082	2.56657	2.31891	1.14649	1.61297	1.85509
	300	4.06394	3.51755	2.84436	2.71097	2.62946	2.53615	2.52706	2.31097	1.07647	1.83945	1.87378
	400	3.51888	3.03548	2.43256	2.64960	2.37764	2.27254	2.28522	2.08469	1.05807	1.73566	1.68070

Table 9. Coefficients and MSE of biasing estimators for the BRM using the body fat dataset.

		BRRE		BLE			BMTPLE				
Coefficients	MLE	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	BMOPLE	$\hat{k}_1$	$\hat{k}_2$	$\hat{k}_3$	$\hat{k}_4$	$\hat{k}_5$
Intercept	-6.04713	-5.86905	-3.84281	-3.59520	-3.63024	-3.59520	-3.48789	-3.51856	-3.44050	-3.52981	-3.51647
x_1	0.59196	0.49288	0.37621	0.38519	0.38814	0.38519	0.37614	0.37872	0.37214	0.37967	0.37855
x_2	-1.01820	-0.95353	-0.21128	-0.09924	-0.11237	-0.09924	-0.05902	-0.07052	-0.04126	-0.07473	-0.06973
x_3	0.41960	0.16398	-1.16384	-1.31217	-1.28742	-1.31217	-1.38796	-1.36629	-1.42143	-1.35835	-1.36777
x_4	-3.61075	-3.66336	-1.80205	-1.60011	-1.62885	-1.60011	-1.51211	-1.53727	-1.47325	-1.54648	-1.53555
x_5	0.49157	0.45852	0.27813	0.27368	0.27680	0.27368	0.26414	0.26687	0.25993	0.26787	0.26668
x_6	6.32854	6.26187	4.81824	4.52648	4.55223	4.52648	4.44761	4.47015	4.41278	4.47841	4.46861
x_7	-2.21088	-2.22225	-1.65262	-1.52966	-1.53939	-1.52966	-1.49984	-1.50837	-1.48668	-1.51149	-1.50778
x_8	3.57569	3.38767	1.56409	1.31407	1.34639	1.31407	1.21508	1.24338	1.17137	1.25375	1.24145
x_9	1.71547	1.41875	0.05445	-0.06435	-0.03892	-0.06435	-0.14225	-0.11998	-0.17665	-0.11182	-0.12150
x_{10}	1.39685	0.72461	-0.44090	-0.48343	-0.45656	-0.48343	-0.56573	-0.54220	-0.60207	-0.53358	-0.54380
x_{11}	0.94867	0.90375	0.32248	0.22955	0.23982	0.22955	0.19807	0.20707	0.18417	0.21037	0.20646
x_{12}	4.14463	3.35558	0.75473	0.54565	0.59709	0.54565	0.38814	0.43316	0.31858	0.44966	0.43010
x_{13}	-8.70369	-4.96782	-1.11736	-0.98425	-1.09458	-0.98425	-0.64639	-0.74297	-0.49719	-0.77836	-0.73639
MSE	2056.13905	1111.12601	229.31655	205.37265	209.67093	205.37265	196.20629	198.21173	194.07602	199.06978	198.05940

1,768 total observations across multiple variables, providing a comprehensive view of exchange rates, price levels, and interest rate differentials. Key variables include U.S. long-term interest rates (y), log spot exchange rates versus the USD (x1), log price levels (x2), and both short-term (x3) and long-term (x4) interest rates. Additionally, the dataset features log price differentials compared to the U.S. (x5) and U.S. short-term interest rates (x6), allowing for an in-depth examination of international financial dynamics.

The response variable is a proportion between 0 and 1, making it suitable for BRM, as shown in Figure 2. Diagnostic results indicate serious multicollinearity, with a condition number of 236.55, which is much higher than the usual threshold of 100. The correlation plot in Figure 4 also shows strong relationships between predictors. These findings highlight the need for estimation methods that can handle high multicollinearity in this dataset.

Table 10. Coefficients and MSE of biasing estimators for the BRM using the purchasing power parity dataset.

		BRRE		BLE			BMTPLE				
Coefficients	MLE	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	BMOPLE	$\hat{k}_1$	$\hat{k}_2$	$\hat{k}_3$	$\hat{k}_4$	$\hat{k}_5$
Intercept	-3.75619	-3.75044	-3.34645	-3.73475	-3.73753	-3.36101	-3.20130	-3.24421	-2.78289	-3.24745	-3.24348
x_1	0.25804	0.25828	0.27220	0.25891	0.25880	0.27414	0.28064	0.27890	0.29769	0.27876	0.27892
x_2	0.17325	0.17224	0.10060	0.16950	0.16999	0.10425	0.07637	0.08386	0.00331	0.08442	0.08373
x_3	-0.03786	-0.02717	0.49916	0.00141	-0.00369	0.68604	0.97860	0.90001	1.74507	0.89408	0.90134
x_4	2.13191	2.11963	1.51439	2.08689	2.09273	1.30198	0.96657	1.05667	0.08783	1.06347	1.05514
x_5	-0.42388	-0.42320	-0.36898	-0.42133	-0.42166	-0.37694	-0.35797	-0.36307	-0.30827	-0.36345	-0.36298
x_5	5.03767	5.02138	3.88564	4.97689	4.98478	3.91712	3.46426	3.58592	2.27782	3.59510	3.58386
MSE	35.14525	34.65276	17.09468	33.36238	33.58959	14.04834	12.58487	12.57863	28.02922	12.59006	12.57629

The empirical results from the real-world purchasing power parity dataset reveal several noteworthy patterns regarding the performance of different biased estimation techniques. As shown in Table 10, the BMTPLE with $\hat{k}_5$ demonstrates superior performance, achieving the lowest MSE = 12.576 among all estimators. This represents a remarkable 64% reduction in prediction error compared to the MLE baseline (MSE = 35.145).

The BMTPLE's superior performance suggests its dual-parameter shrinkage mechanism effectively handles multicollinearity while maintaining estimation efficiency. Particularly striking is its ability to substantially adjust coefficients for interest rate variables (x_3 and x_4), where conventional methods produce unstable estimates. The BMOPLE shows competitive performance (MSE = 14.048), suggesting positive constraint approaches may offer alternative benefits in certain econometric contexts.

These findings have important implications for empirical research in international finance, where multicollinearity between exchange rates and interest rate differentials frequently complicates parameter estimation. The results strongly support the simulation results, demonstrating that the proposed estimator outperforms alternatives in handling high multicollinearity in this dataset.

6. Conclusion

This study introduces new modified Liu estimators designed to address multicollinearity issues in BRM. Theoretical analysis establishes that the proposed modified Liu estimators demonstrate superior efficiency compared to existing biased estimators, including conventional MLE, BRRE, and MLE. To empirically validate these theoretical findings, we conducted comprehensive Monte Carlo simulations comparing the modified Liu estimators' performance against MLE and alternative biased estimation approaches. The simulation results consistently demonstrate that the modified Liu estimators achieve significantly better performance metrics, particularly in scenarios characterized by high-to-strong intercorrelations among predictor variables. The practical utility of the modified Liu estimator is further substantiated through two empirical applications, where it consistently produced lower MSE values compared to MLE, BRRE, and BLE methods. These findings collectively suggest that the TPBR estimator represents an effective solution for regression analysis involving collinear predictors, offering improved estimation accuracy across both simulated and real-world datasets.

Authors' Contributions

All authors have worked equally to write and review the manuscript.

Data Availability Statement

The data that supports the findings of this study are available within the article.

Conflicts of Interest

The authors declare no conflict of interest.

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