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Research article

Fundamental Properties of the Characteristic Function using the Compound Poisson Distribution as the Sum of the Gamma Model

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ABSTRACT

Probability distribution has proven its usefulness in almost every discipline of human endeavor. For this, the zero-truncated Poisson gamma (ZTP-G) model is widely recognized in probability theory and extensively used in various applied fields, specifically in survival, hydrology, insurance, and energy theory. The characterization of the zero-truncated Poisson sum of independent and identically distributed gamma random variables is proposed in this paper by applying the Laplace-Stieltjes transform technique. Further, the properties of continuity and quadratic form of the characteristic function are applied for definite positive properties from the ZTP-G model.

1. Introduction

The application of statistical distributions is widespread in almost all disciplines, reflecting their fundamental role in understanding and interpreting uncertainty in various contexts. Many families of distributions have been generated by applying different techniques to provide the best fit to the real data sets.

In recent decades, compound distribution has been provided as a new technique for creating different families of distributions. These techniques are important in various areas such as industrial, biological science, hydrology, economics, and finance. On may refer to Meraou et al. [11], Raqab et al. [13], and Revfeim [14]. The compound Poisson distribution is one of the most popular choices for developing new models. The compound Poisson distribution can be applied in many fields, particularly in insurance risk models, including Zhang et al. [18] and Hu et al. [6]. In this way, Meraou et al [10] provided the compound Poisson log-normal model by mixing zero-truncated Poisson and log-normal distributions and applied the obtained model in the precipitation field. Adamidis and Loukas [1] proposed a novel model named the exponential geometric distribution, obtaining several distributional and mathematical properties. Further, the compound Poisson exponential distribution considered by Cancho et al. [2]. They showed that the recommended distribution is more suitable among other well-known distributions to fit different kinds of data sets. Also. Rodrigues et al. [15] constructed a procedure of estimations for the Poisson exponential model, and Meraou and Raqab [9] defined the truncated Poisson Lindley.

Let $Z_1, Z_2, \ldots, Z_M$ be independent and identical distributed (i.i.d) gamma random variables with parameters α and β and M has zero truncated Poisson distribution with parameter θ, where M is independent of Z_i. In addition, let us define the random variable S as $S = Z_1 + Z_2 + \ldots Z_M$. We refer to the variable S as a zero-truncated Poisson gamma (ZTP-G) model.

Furthermore, various theories and properties of compound models have been considered by many authors. Devianto et al. [3] have provided the compound of a random variable from an exponential distribution with stabilizer constant by studying the properties of uniform continuity of the characteristic function. Serfozo [16] has introduced the approximate theory of compound Poisson for the sum of random variables, whereas Mailisa et al. [8] have considered the compound geometric as the sum of a gamma distribution, and they explained many theories and properties. The property of the compound Poisson Cauchy model on its infinite divisibility characteristic function has been studied by Devianto et al. [5], and Yurinanda has considered the same characteristic of the compound Poisson geometric distribution [17]. Petrauskienė and Čekanavičius [12] applied the compound Poisson approximations for sums of 1-dependent random variables. Koutras and Erylmaz [7] investigated compound geometric distributions of order k, and they studied the properties of uniform continuity of the characteristic function. Next, Devianto et al. [4] proposed the mixture negative Binomial-Exponential model, providing the characteristic function property of the proposed model.

The basic motivations for this work in practice are:

1. Providing the fundamental properties of the characterization function from the ZTP-G model using the Laplace-Stieltjes transform.
2. Another crucial motivation is the capability to demonstrate the definite positive property of the characteristic function for the ZTP-G distribution using the properties of continuity and quadratic form.

The paper is structured as follows: In section 2, some of the mathematical properties of the ZTP-G model, such as the mean, variance, and moment-generating function, are introduced. The continuity and positively definite characteristic function of the ZTP-G model are explained in Section 3. At the end, we concluded the paper in Section 4.

2. Properties of Compound Poisson Gamma Model

Let Z be a random variable with a probability density function (PDF)

$$H(z;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} z^{\alpha-1} e^{-\beta z};\ z > 0,\ \alpha,\beta > 0.$$

The expected value and variance of Z are expressed as

$$\mathbb{E}(Z) = \frac{\alpha}{\beta}, \quad \text{and} \quad V(Z) = \frac{\alpha}{\beta^2}.$$

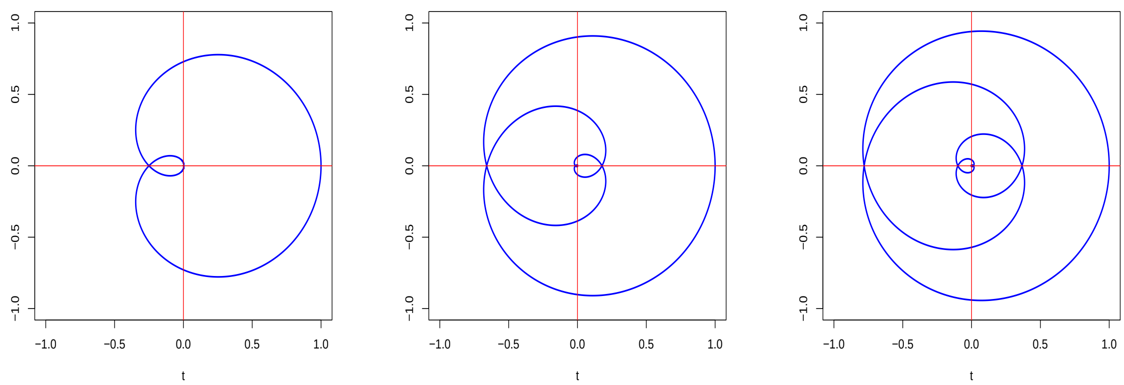
Further, the moment-generating function (MGF) can be written as

$$M_Z(t) = \mathbb{E}(e^{tZ}) = \left(1 + \frac{t}{\beta - t}\right)^{\alpha}.$$

The characteristic function (CF) of Z is

$$\phi_Z(t) = E(e^{itZ}) = \left(1 + \frac{it}{\beta - it}\right)^{\alpha}.$$

Figure 1 represents the graphs of the characteristic function for Z with different parameter value choices in the complex plane. The graphs describe a smooth line and indicate that the characteristic function of Z is continuous and never vanishes on the complex plane. Now, suppose M follows a zero-truncated Poisson



(a) $\alpha = 4, \beta = 1$ (b) $\alpha = 12, \beta = 1$ (c) $\alpha = 20, \beta = 1$

Figure 1. CF plots of Z with different parameter values.

(ZTP) distribution with probability mass function (PMF)

$$P(M = m) = \frac{\theta^m}{m!(e^{\theta} - 1)}; \; m = 1, 2, 3, ..., \; \theta > 0.$$

The expectation and variance of M are defined as

$$\mathbb{E}(M) = \frac{\theta e^{\theta}}{e^{\theta} - 1}, \quad \text{and} \quad V(M) = \frac{\theta e^{\theta}}{e^{\theta} - 1}\left(\theta + 1 - \frac{\theta}{1 - e^{-\theta}}\right).$$

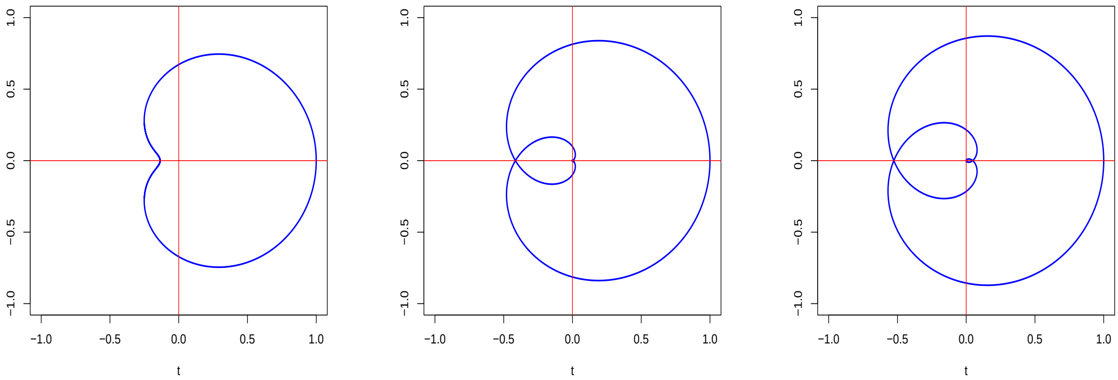
Consequently, the MGF and CGF of M are

$$M_M(t) = E(e^{tm}) = \frac{e^{\theta e^t} - 1}{e^{\theta} - 1},$$

and

$$\phi_M(t) = E(e^{itm}) = \frac{e^{\theta e^{it}} - 1}{e^\theta - 1}.$$

Figure 2 represents the graphs of CF of M with various choices of θ in the complex plane. The graphs describe a smooth line and indicate that the characteristic function of M is continuous and never vanishes on the complex plane.



(a) $\theta = 2$ (b) $\theta = 6$ (c) $\theta = 8$

Figure 2. CF plots of M with different parameter values.

Definition 1. *We say that the random variable S follows the ZTP-G model if it can be represented*

$$S = \sum_{i=1}^{M} Z_i. \tag{2.1}$$

Proposition 2.1. *The expectation of S is given by*

$$\mathbb{E}(S) = \frac{\alpha\theta e^\theta}{\beta(e^\theta - 1)}. \tag{2.2}$$

Proof. We have

$$\mathbb{E}(S) = \mathbb{E}\left(\sum_{i=1}^{M} Z_i\right) = \mathbb{E}\left(\mathbb{E}\left(\sum_{i=1}^{M} Z_i \mid M\right)\right) = \mathbb{E}(M\mathbb{E}(Z)) = \mathbb{E}(M)\mathbb{E}(Z) = \frac{\alpha\theta e^\theta}{\beta(e^\theta - 1)}.$$

□

Proposition 2.2. *The variance of S is obtained by*

$$V(S) = \frac{\alpha\theta e^\theta}{\beta^2(e^\theta - 1)^2} + \frac{\alpha^2\theta e^\theta}{\beta^2(e^\theta - 1)}\left(\theta + 1 - \frac{\theta}{1 - e^{-\theta}}\right). \tag{2.3}$$

Proof. We have

$$\begin{aligned} V(S) &= V\left(\sum_{i=1}^{M} Z_i\right) = \mathbb{E}\left(V\left(\sum_{i=1}^{M} Z_i \mid M\right)\right) + V\left(\mathbb{E}\left(\sum_{i=1}^{M} Z_i \mid M\right)\right) \\ &= \mathbb{E}(M)V(Z) + V(M)(\mathbb{E}(Z))^2 = \frac{\alpha\theta e^{\theta}}{\beta^2(e^{\theta}-1)^2} + \frac{\alpha^2\theta e^{\theta}}{\beta^2(e^{\theta}-1)}\left(\theta + 1 - \frac{\theta}{1-e^{-\theta}}\right). \end{aligned}$$

□

Proposition 2.3. *The MGF of S is expressed as follows*

$$M_S(t) = \frac{1}{e^{\theta}-1}\left\{\exp\left\{\theta\left(\frac{\beta}{\beta-t}\right)^{\alpha}\right\} - 1\right\}. \tag{2.4}$$

Proof. We have

$$M_S(t) = \mathbb{E}\left(\exp\left(t\sum_{i=1}^{M} Z_i\right)\right) = \mathbb{E}\left(\mathbb{E}\left(\exp\left(t\sum_{i=1}^{M} Z_i\right) \mid M\right)\right) = \mathbb{E}(M_Z(t)^M) = M_M(\ln(M_Z(t))).$$

Finally,

$$M_S(t) = M_M\left(\ln\left[\left(\frac{\beta}{\beta-t}\right)^{\alpha}\right]\right) = \frac{1}{e^{\theta}-1}\left\{\exp\left\{\theta\left(\frac{\beta}{\beta-t}\right)^{\alpha}\right\} - 1\right\}.$$

□

Proposition 2.4. *The CF of S is expressed as follows*

$$\phi_S(t) = \frac{1}{e^{\theta}-1}\left\{\exp\left\{\theta\left(\frac{\beta}{\beta-it}\right)^{\alpha}\right\} - 1\right\}. \tag{2.5}$$

Proof. We have

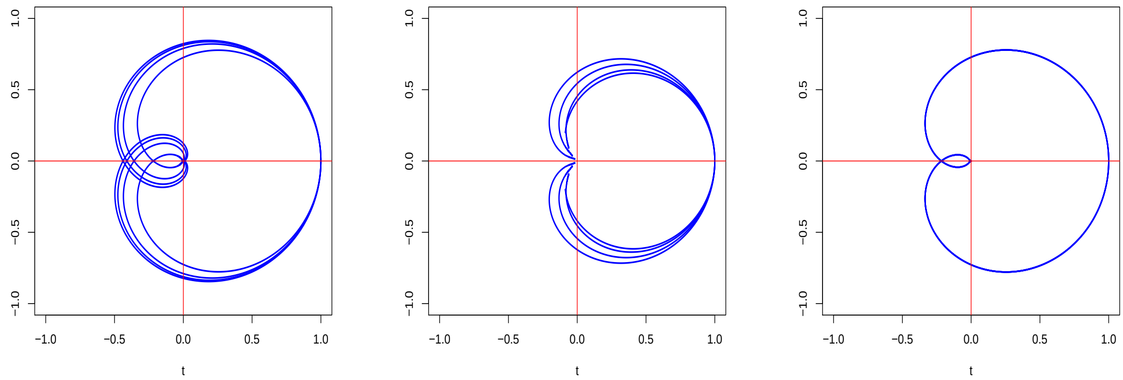
$$\phi_S(t) = \mathbb{E}\left(\exp\left(it\sum_{i=1}^{M} Z_i\right)\right) = \mathbb{E}\left(\mathbb{E}\left(\exp\left(it\sum_{i=1}^{M} Z_i\right) \mid M\right)\right) = \mathbb{E}(\phi_Z(t)^M) = M_M(\ln(\phi_Z(t))).$$

At the end,

$$M_S(t) = M_M\left(\ln\left[\left(\frac{\beta}{\beta-it}\right)^{\alpha}\right]\right) = \frac{1}{e^{\theta}-1}\left\{\exp\left\{\theta\left(\frac{\beta}{\beta-it}\right)^{\alpha}\right\} - 1\right\}.$$

□

Figure 3 represents the CF plots of the ZTP-G model. It can be deduced from 3 that the characteristic function of the ZTP-G model is continuous since the graph is a smooth line. In Figure 3, parts (a) and (b), the graphs form four smooth lines for various parameter values of α and θ when β is kept fixed. While in part(c) and for different choices of β, the graph forms the same smooth line when α and θ are kept fixed. This is to ensure that the characteristic function of the ZTP-G model is continuous and never vanishes on the complex plane.



(a) $\alpha = 1(1)4, \beta = 4, \theta = 8$ (b) $\alpha = 2, \beta = 4, \theta = 1(1)4$ (c) $\alpha = 2, \beta = 0.5(0.5)2, \theta = 6$

Figure 3. CF plots of ZTPG model with different parameter values.

3. The Property of Characteristic Function of Compound Poisson Gamma Model

Lemma 3.1. *Consider f a function. We say that f is uniform continuity if and only if for every $\varepsilon > 0$, $\exists$ $\delta > 0 \setminus \ \mid f(l_1) - f(l_2) \mid < \varepsilon$ where $\mid l_1 - l_2 \mid < \delta$, where δ depends only on ε.*

Proposition 3.2. *Let S follow the ZTP-G model, then the CF of S, $\phi_S(t)$, is continuous.*

Proof. By using Lemma 3.1, we have

$$\mid \phi_S(l_1) - \phi_S(l_2) \mid = \left| \frac{\exp\left\{\theta\left(\frac{\beta}{\beta - il_1}\right)^{\alpha}\right\} - 1}{e^{\theta} - 1} - \frac{\exp\left\{\theta\left(\frac{\beta}{\beta - il_2}\right)^{\alpha}\right\} - 1}{e^{\theta} - 1} \right|.$$

By setting $h = l_1 - l_2$, we immediately have

$$\lim_{h \to 0} \left| \frac{\exp\left\{\theta\left(\frac{\beta}{\beta - i(h + l_2)}\right)^{\alpha}\right\} - 1}{e^{\theta} - 1} - \frac{\exp\left\{\theta\left(\frac{\beta}{\beta - il_2}\right)^{\alpha}\right\} - 1}{e^{\theta} - 1} \right| = 0.$$

As consequence $\phi_S(t)$ satisfies the condition of uniformly continuous, so $\phi_S(t)$ is continuous. □

Proposition 3.3. *Let S follows ZTP-G model, then CF of S $\phi_S(t)$ is positively defined function with quadratic form*

$$\sum_{1 \le j \le n} \sum_{1 \le l \le n} c_j \overline{c_l} \phi_S(t_j - t_l) \ge 0,$$

for any complex number c_j and c_l.

Proof. We have the following equation

$$
\begin{aligned}
\sum_{1\le j\le n}\sum_{1\le l\le n} c_j\overline{c_l}\phi_S(t_j-t_l) &= \sum_{1\le j\le n}\sum_{1\le l\le n} c_j\overline{c_l}\left(\frac{\exp\left\{\theta\left(\frac{\beta}{\beta-i(t_j-t_l)}\right)^{\alpha}\right\}-1}{e^{\theta}-1}\right)\\
&= \sum_{1\le j\le n}\sum_{1\le l\le n} c_j\overline{c_l}\left(\frac{\exp\left\{\theta E(\exp(i(t_j-t_l)X))\right\}-1}{e^{\theta}-1}\right)\\
&= \frac{1}{e^{\theta}-1}\sum_{1\le j\le n}\sum_{1\le l\le n} c_j\overline{c_l}\left(\sum_{k=1}^{\infty}\frac{\left[\theta E(\exp(i(t_j-t_l)Z))\right]^k}{k!}\right).
\end{aligned}
$$

By using the definition of complex conjugate on the characteristic function, we rewrite the above equation in the following form

$$
\begin{aligned}
\sum_{1\le j\le n}\sum_{1\le l\le n} c_j\overline{c_l}\phi_S(t_j-t_l) &= \frac{1}{e^{\theta}-1}\left(\sum_{1\le j\le n} c_j\left(\sum_{k=1}^{\infty}\frac{\left[\theta E(\exp(it_jZ))\right]^k}{k!}\right)\right)\\
&\quad\times\overline{\left(\sum_{1\le l\le n} c_l\left(\sum_{k=1}^{\infty}\frac{\left[\theta E(\exp(it_lZ))\right]^k}{k!}\right)\right)}.
\end{aligned}
\tag{3.1}
$$

Next, by appling the property of quadratic form we get

$$
\begin{aligned}
\sum_{1\le j\le n}\sum_{1\le l\le n} c_j\overline{c_l}\phi_S(t_j-t_l) &= \frac{1}{e^{\theta}-1}\left|\sum_{1\le j\le n} c_j\left(\sum_{k=1}^{\infty}\frac{\left[\theta E(\exp(it_jZ))\right]^k}{k!}\right)\right|^2\\
&= \frac{1}{e^{\theta}-1}\left|\sum_{1\le j\le n} c_j\left(\sum_{k=1}^{\infty}\frac{\left[\theta\left(\frac{\beta}{\beta-it}\right)^{\alpha}\right]^k}{k!}\right)\right|^2 \ge 0.
\end{aligned}
\tag{3.2}
$$

At the end, it is shown that the CF of ZTP-G model $\phi_S(t)$ is a positively defined function where the quadratic form has non-negative values. □

4. Conclusion

The compound zero truncated Poisson gamma (ZTP-G) distribution is the mixture truncated Poisson and gamma distributions. We introduced different characterizations of the ZTP-G model in this study,

which represents the zero-truncated Poisson sum of i.i.d. gamma random variables. The characterization of the ZTP-G model is provided by employing the property of the characteristic function. This characteristic function $\phi_S(t)$ is obtained by using the Fourier-Stieltjes transform. In addition, the so obtained characteristic function $\phi_S(t)$ of ZTP-G model α, β and θ is positively definite and continuous and it is also demonstrated graphically by the graph of CF with smooth line and never vanish on the complex plane.
In future work, this study may attract wider sets of applications, such as the suggested CZTP-G model, which may contribute to incorporating classical and Bayesian estimation methods based on complete and random sampling data. Further, the application of the proposed model in the censored sample estimation using numerous types.

Authors' Contributions

All authors have worked equally to write and review the manuscript.

Data Availability Statement

The data that supports the findings of this study are available within the article.

Conflicts of Interest

The authors declare no conflict of interest.

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