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Research article

Statistical Inference for Alpha Power Burr-XII Distribution Under Progressive Type-II Censoring: Methodology and Application to Bladder Cancer Data

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ABSTRACT

In this paper we investigates the statistical inference and parameter estimation for the three-parameter Alpha Power Burr-XII (APB-XII) distribution under a Progressive Type-II censoring scheme. Both classical and Bayesian estimation methodologies are developed to estimate the unknown shape parameters (α,β, λ) as well as the corresponding reliability and hazard rate functions. Under the classical framework, maximum likelihood estimators (MLEs) are derived, and because they lack analytical solutions, the Newton-Raphson numerical technique is implemented. Furthermore, approximate asymptotic confidence intervals are constructed using the inverse Fisher information matrix alongside the delta method, complemented by parametric percentile (Boot-p) and studentized (Boot-t) bootstrap confidence intervals to address small sample limitations. Under the Bayesian framework, independent gamma prior distributions are utilized, and Bayes estimators are obtained under both symmetric Squared Error Loss (SEL) and asymmetric Linear Exponential (LINEX) loss functions. Due to the analytical complexity of the joint posterior distribution, the Markov Chain Monte Carlo (MCMC) algorithm specifically utilizing the Gibbs sampler within a Metropolis-Hastings approach is utilized to generate posterior samples and compute credible intervals. To demonstrate the real-world utility of the proposed methodology, a real lifetime dataset is modeled. The Kolmogorov-Smirnov test validates that the APB-XII distribution provides a highly robust fit for the empirical failure data. Finally, an extensive Monte Carlo simulation study is conducted under three distinct progressive censoring schemes.

1. Introduction

In lifetime tests and reliability, it is normal for experiments to end before all of the test units have failed. The removal of units prior to failure is frequently a process because of the experiment's financial and time constraints. In such circumstances, it is hard for experimenters to obtain all the sample information in a restricted period of time during a lifetime test. This leads to the concept of censoring, in which certain surviving units are removed from the experiment after the use of a specific censoring procedure. A life test is considered complete if every test item is followed until it fails. In practice, the available information is often incomplete. There are well-known censorship techniques such as type I and type II, refer to the works of Klein and Moeschber [13]. The progressive type-II censoring technique is described as follows, there are "n" units. The m^{th} unit (where $m < n$) denotes the unit that fails when the lifetime test is completed. At time $t_{(1)}$, the first failure is recorded and $R_{(1)}$ components are randomly selected from the remaining units $(n-1)$. When the second failure occurs at time $t_{(2)}$, $R_{(2)}$ components are chosen randomly from the remaining $n - R_{(1)} - 2$ surviving components. The process continues until the occurrence of the m^{th} failure at time t_m, where R_m is calculated as $R_m = n - m - \sum_{i=1}^{m-1} R_i$.

By employing this progressive type-II censoring approach, the testing procedure becomes more dynamic and closely mimics real-world scenarios, allowing for effective data analysis in the context of lifetime experiments.

Several authors have conducted extensive research on progressive type-II censoring, exploring various failure time breakdowns. Notably, Mann et al. [18], Eryilmaz and Bayramoglu [10], Meeker and Escobar [19], Balakrishnan [6] and [21] have analyzed the attributes of order statistics under progressive type-II censored samples, following a uniform or other continuous distributions, have been developed by Balakrishnan and Sandhu [7], and Aggarwala and Balakrishnan [3]. Maiti and Kayal [16] have developed methods to calculate the reliability, hazard, and unknown parameters of a log-logistic distribution for progressive type-II censored samples. Additionally, Maiti and Kayal [17] have investigated the generalized Rayleigh distribution in the presence of the second type of controlled stepwise sample. Saha and Yadav [24] have employed parametric (classical and Bayesian) point estimation algorithms to estimate reliability features for Exponentiated Xgamma distribution (XGD). Specifically, the reliability function and mean time to failure in progressive type-II censored cases have been estimated using four distinct techniques. These studies have contributed significantly to the understanding and application of progressive type-II censoring, providing insights into parameter estimation, hypothesis testing, and reliability analysis for various distributions and scenarios, see [22], [23], [4].

Bladder cancer, despite being one of the most common malignant tumors with high treatment costs per patient, often remains overlooked as a significant public health concern. It receives relatively limited funding and attention from the scientific community. Although it occurs frequently, it is associated with significant mortality and high management expenses. During recent observations, the 5-year survival rates for prostate and kidney cancers have shown improvement over the past three decades, bladder cancer lags behind and continues to face challenges in advancements in treatment. In order to address the needs of bladder cancer patients and improve their access to better testing, therapies, and healthcare services, it is crucial to increase funding and understanding of the condition. To carry out our study focused on bladder cancer patients, we will utilize the progressive type-II censoring ($Prog - II - C$) scheme for our analysis. This scheme effectively reduces the experimental time for patients and allows us to obtain accurate estimations while keeping some patients in the experiment to avoid premature failure. A suitable model

for characterizing bladder cancer patient component failure rates is the three-parameters of Alpha Power Burr-XII ($APB-XII$) distribution proposed by Nada et al. [1].

The random variable X have the Alpha Power Burr-XII distribution denoted by APB-XII (α,β,λ). The cdf of a random variable X that have three parameters for $APB-XII$ distribution can be described as

$$F(x;\alpha,\beta,\lambda)=\begin{cases}\frac{\alpha^{1-(1+x^{\beta})^{-\lambda}}-1}{\alpha-1} & if\ \alpha>0,\alpha\neq 1\\ 1-(1+x^{\beta})^{-\lambda} & if\ \alpha=1,\end{cases} \tag{1.1}$$

where $\alpha,\beta,\lambda>0$ and $x>0$. The corresponding pdf is given as

$$f(x;\alpha,\beta,\lambda)=\begin{cases}\frac{\log(\alpha)}{\alpha-1}\lambda\beta x^{\beta-1}(1+x^{\beta})^{-(\lambda+1)}\alpha^{1-(1+x^{\beta})^{-\lambda}} & if\ \alpha>0,\alpha\neq 1\\ \lambda\beta x^{\beta-1}(1+x^{\beta})^{-(\lambda+1)}\alpha^{1-(1+x^{\beta})^{-\lambda}} & if\ \ \alpha=1,\end{cases} \tag{1.2}$$

where $\alpha,\beta,\lambda>0\ and\ x>0$. The survival function $S(x)$, of the $APB-XII$ distribution is obtained as

$$S(x;\alpha,\beta,\lambda)=\frac{\alpha-\alpha^{1-(1+x^{\beta})^{-\lambda}}}{\alpha-1}\ ;x>0,\ \alpha,\beta,\lambda>0. \tag{1.3}$$

The hazard rate function $h(x)$, of the APB-XII distribution is expressed as

$$h(x;\alpha,\beta,\lambda)=\frac{\log(\alpha)\lambda\beta x^{\beta-1}(1+x^{\beta})^{-(1+\lambda)}\alpha^{1-(1+x^{\beta})^{-\lambda}}}{\alpha-\alpha^{1-(1+x^{\beta})^{-\lambda}}}\ ;x>0,\ \alpha,\beta,\lambda>0. \tag{1.4}$$

In reliability engineering, survival analysis, and quality control monitoring, analyzing lifetime data accurately is crucial for making informed decisions. However, practitioners often face two major challenges that hinder standard statistical analysis: the insufficiency of classical probability models to fit complex empirical data, and the presence of severe time and cost constraints during experiments. These challenges serve as the primary motivations for this research:

1. The study introduces an advanced inference framework by integration of the highly flexible Alpha Power Burr-XII (APB-XII) distribution within a Progressive Type-II Censoring scheme. This combination bridges the gap between sophisticated probability modeling and practical experimental limitations, ensuring accurate statistical decisions under strict time and budget constraints.

2. While traditional lifetime models (like the baseline Burr-XII) often fail to fit skewed or non-monotonic failure behaviors, the APB-XII distribution introduces a crucial alpha-power shape parameter. This mathematical modification significantly enhances the model's flexibility, allowing it to easily adapt to unimodal and complex bathtub-shaped hazard rate functions commonly found in nature.

3. Unlike rigid traditional censoring plans, the Progressive Type-II scheme allows for the removal of active, surviving units at intermediate stages of the experiment. This provides a highly realistic and cost-effective framework for long-term clinical trials or expensive reliability stress tests where units might drop out unexpectedly or due to sudden budget limits.

4. A major motivation of this paper is its direct contribution to biostatistics and survival analysis. By successfully fitting the APB-XII distribution to a real clinical dataset of bladder cancer patients, the study demonstrates that this model offers a superior, robust fit compared to traditional alternatives. This accuracy gives medical researchers a highly precise predictive tool to evaluate patient survival rates and optimize healthcare planning.

5. Since parameter estimation under progressive censoring lacks simple closed-form solutions, this research is driven by the need to equip practitioners with robust computational tools. It successfully develops and compares classical numerical methods (MLE with Newton-Raphson), advanced small-sample techniques (Parametric Boot-p and Boot-t), and a comprehensive Bayesian framework (MCMC) under both symmetric and asymmetric loss functions.

The remainder of this paper is structured as follows: Section 2 formulates the maximum likelihood estimators (MLEs) for the unknown parameters α, β, λ, as well as the survival $S(t)$ and hazard $h(t)$ functions, followed by the construction of their approximate asymptotic confidence intervals. In Section 3, parametric bootstrap confidence intervals are developed to handle small sample sizes. Section 4 presents the Bayesian estimation framework, where Bayes estimators for the parameters and reliability functions are derived under both symmetric squared error loss (SEL) and asymmetric LINEX loss functions. Section 5 demonstrates the practical utility of the proposed model through an application to a real lifetime dataset. Finally, Section 6 provides an extensive Monte Carlo simulation study to numerically evaluate and compare the performance of the various classical and Bayesian estimators. The manuscript ends with the conclusions Section.

2. Maximum-Likelihood Estimation

In this section, we develop the estimation of parameters in the $APB-XII$ distribution, specifically the $Prog-II-C$ data. Let $x_{1:m:n}, x_{2:m:n}, ... x_{m:m:n}$ represent a sample from the $Prog-II-C$, which is characterized by parameters α, β and λ. The likelihood function associated with the $Prog-II-C$ data exhibits the following characteristics:

$$f(x_{1:m:n}, x_{2:m:n}, ...x_{m:m:n}) = C\prod_{i=1}^{m} f(x_{i:m:n;\alpha,\beta,\lambda}).[1-F(x_{i:m:n;\alpha,\beta,\lambda})]^{R_i}, m < n \tag{2.1}$$

as $C = [n(n-R_1-1), ..., (n-\sum_{i=1}^{m-1}(R_i-1))]$.
From $Eqs.$ (1.1) and (1.2) into $Eq.$ (2.1), then the log-likelihood function is define as

$$L(\alpha,\beta,\lambda) = C\prod_{i=1}^{m}\frac{\ln[\alpha]}{\alpha-1}\lambda\beta x_i^{\beta-1}(1+x_i^{\beta})^{-(\lambda+1)}\alpha^{1-(1+x_i^{\beta})^{-\lambda}}\left[1-\frac{\alpha^{(1-(1+x_i^{\beta})^{-\lambda})}-1}{\alpha-1}\right]^{R_i}. \tag{2.2}$$

The log-likelihood function results for samples from the $Prog-II-C$ in terms of the parameters (α, β, λ) can be obtained as

$$l = m \ln\left[\frac{\ln[\alpha]}{\alpha - 1}\right] + m\ln(\lambda) + m\ln(\beta) + (\beta - 1)\sum_{i=1}^{m}\ln(x_i) - (\lambda + 1)\sum_{i=1}^{m}\ln(1 + x_i^{\beta})$$
$$+ \ln(\alpha)\sum_{i=1}^{m}(1 - (1 + x_i^{\beta})^{-\lambda}) + \sum_{i=1}^{m}R_i \ln\left[\alpha - \alpha^{(1-(1+x_i^{\beta})^{-\lambda})}\right] - \sum_{i=1}^{m}R_i \ln(\alpha - 1).$$

To derive the likelihood equations, we calculate the initial partial derivatives of the likelihood function, denoted as l, with respect to the parameters α, β, λ. Then, we set each derivative equal to zero.

$$\frac{\partial l}{\partial \alpha} = m\left(\frac{(\alpha - 1) - \alpha\ln(\alpha)}{\alpha(\alpha - 1)\ln(\alpha)}\right) - (\alpha)^{-1}\sum_{i=1}^{m}(1 + x_i^{\beta})^{-\lambda}$$
$$+ \sum_{i=1}^{m}R_i\left[\frac{\alpha - \alpha^{1-(1+x_i^{\beta})^{-\lambda}}(1 - (1 + x_i^{\beta})^{-\lambda})}{\alpha(\alpha - \alpha^{(1-(1+x_i^{\beta})^{-\lambda})})}\right] - \sum_{i=1}^{m}R_i(\alpha - 1)^{-1}, \tag{2.3}$$

$$\frac{\partial l}{\partial \beta} = \frac{m}{\beta} + \sum_{i=1}^{m}\ln(x_i) - \sum_{i=1}^{m}\frac{x_i^{\beta}\ln(x_i)(\lambda + 1)}{1 + x_i^{\beta}} + \ln(\alpha)\sum_{i=1}^{m}\frac{\lambda x_i^{\beta}\ln(x_i)(1 + x_i^{\beta})^{-\lambda}}{1 + x_i^{\beta}}$$
$$- \sum_{i=1}^{m}R_i\left(\frac{\lambda x_i^{\beta}\ln(x_i)\ln(\alpha)(1 + x_i^{\beta})^{-\lambda-1}\alpha^{(1-(1+x_i^{\beta})^{-\lambda})}}{(\alpha - \alpha^{(1-(1+x_i^{\beta})^{-\lambda})})}\right) \tag{2.4}$$

and

$$\frac{\partial l}{\partial \lambda} = \frac{m}{\lambda} - \sum_{i=1}^{m}\ln(1 + x_i^{\beta}) + \ln(\alpha)\sum_{i=1}^{m}(1 + x_i^{\beta})^{-\lambda}\ln(1 + x_i^{\beta})$$
$$- \sum_{i=1}^{m}R_i\left(\frac{\ln(\alpha)\ln(1 + x_i^{\beta})(1 + x_i^{\beta})^{-\lambda}\alpha^{(1-(1+x_i^{\beta})^{-\lambda})}}{(\alpha - \alpha^{(1-(1+x_i^{\beta})^{-\lambda})})}\right). \tag{2.5}$$

Since *Eqs.* (2.3)-(2.5) do not have analytical solutions, a numerical approach is required to obtain estimates. The Newton-Raphson method is a commonly used numerical technique for solving such equations. The algorithm for applying the Newton-Raphson method to derive the estimates can be found in the works of [8]-[11].

2.1. Asymptotic confidence intervals

The Fisher's information matrix, denoted as I, is derived from the inverse of the Hessian matrix of the negative log-likelihood function and serves as an estimation of the variances and covariances for the parameters α, β *and* λ based on the maximum likelihood estimators (*MLEs*) obtained from the data. The individual components of the Fisher's information matrix can be computed as follows $I_{ij} = E\{-[\partial^2 l(\psi)/\partial_{\Psi_i}\partial_{\Psi_j}]\}$

where i *and* j take values $(1-3)$ and $\Psi = (\Psi_1, \Psi_2, \Psi_3) = (\alpha, \beta, \lambda)$. Obtaining the asymptotic forms of these equations can be challenging due to their complexity and precision. Once we have the Fisher's information matrix, we can utilize it to construct confidence intervals (CIs) for the parameters through inference.

$$\hat{I}_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} = \left[\begin{array}{ccc} -\frac{\partial^2 l}{\partial \alpha^2} & -\frac{\partial^2 l}{\partial \alpha \partial \beta} & -\frac{\partial^2 l}{\partial \alpha \partial \lambda} \\ -\frac{\partial^2 l}{\partial \beta \partial \alpha} & -\frac{\partial^2 l}{\partial \beta^2} & -\frac{\partial^2 l}{\partial \beta \partial \lambda} \\ -\frac{\partial^2 l}{\partial \lambda \partial \alpha} & -\frac{\partial^2 l}{\partial \lambda \partial \beta} & -\frac{\partial^2 l}{\partial \lambda^2} \end{array} \right]_{(\alpha=\hat{\alpha},\beta=\hat{\beta},\lambda=\hat{\lambda})}.$$

Thus, by invers information matrix $\hat{I}_{(\hat{\alpha},\hat{\beta},\hat{\lambda})}$ into something that resembled the asymptotic variance-covariance matrix. The asymptotic variance-covariance matrix, denoted as $\hat{V}$, can be calculated for the maximum likelihood estimators ($MLEs$). This matrix provides an approximation of the variances and covariances of the $MLEs$, was created as follows:

$$\hat{V} = \hat{I}^{-1}_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} = \left[\begin{array}{ccc} var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\beta}) & cov(\hat{\alpha},\hat{\lambda}) \\ cov(\hat{\beta},\hat{\alpha}) & var(\hat{\beta}) & cov(\hat{\beta},\hat{\lambda}) \\ cov(\hat{\lambda},\hat{\alpha}) & cov(\hat{\lambda},\hat{\beta}) & var(\hat{\lambda}) \end{array} \right]_{(\hat{\alpha},\hat{\beta},\hat{\lambda})}.$$

Based on the required symmetry conditions outlined by Lawless [14], it can be inferred that the $MLEs$ (α, β, λ) approximate a multivariate normal distribution with a mean (α, β, λ) and a covariance matrix $I^{-1}_{(\alpha,\beta,\lambda)}$. Consequently, to construct $(1-\gamma)100\%$ asymptotic confidence intervals ($ACIs$) for α ,β and λ, the following approach can be adopted:

$$\hat{\alpha} \pm z_{\gamma/2}\sqrt{var(\hat{\alpha})},\ \hat{\beta} \pm z_{\gamma/2}\sqrt{var(\hat{\beta})} \text{ and } \hat{\lambda} \pm z_{\gamma/2}\sqrt{var(\hat{\lambda})},$$

where $z_{\gamma/2}$ denote the percentile of the standard Normal distribution with righttail probability $\gamma/2$.

In order to create asymptotic confidence intervals for the reliability and hazard functions, which depend on the parameters (α, β, λ), it is essential to determine their variances. To accomplish this, we can utilize the delta method (∇), allows us to approximate the variances of the $\hat{S}(t)$ and $\hat{h}(t)$. By employing this method, we can approximate the variances of $\hat{S}(t)$ and $\hat{h}(t)$ as follows:

$$\hat{\sigma}^2_{\hat{S}(t)} = [\nabla \hat{S}(t)]^T [\hat{V}][\nabla \hat{S}(t)] \text{ and } \hat{\sigma}^2_{\hat{h}(t)} = [\nabla \hat{h}(t)]^T [\hat{V}][\nabla \hat{h}(t)].$$

The $(1-\gamma)100\%$ two-sided confidence intervals for $\hat{S}(t)$ and $\hat{h}(t)$ in terms of α, β and λ, we need to consider the gradient of $S(t)$ and $h(t)$ with respect to α, β and λ denoted as $\nabla \hat{S}(t)$ and $\nabla \hat{h}(t)$, respectively. Using these gradients, we can construct the confidence intervals for $S(t)$ and $h(t)$ within the $(1-\gamma)100\%$ range.

$$\hat{S}(t) \pm z_{\gamma/2}\sqrt{\hat{\sigma}^2{}_{\hat{S}(t)}} \text{ and } \hat{h}(t) \pm z_{\gamma/2}\sqrt{\hat{\sigma}^2{}_{\hat{h}(t)}}.$$

3. Bootstrap Confidence Intervals

To compute bootstrap confidence intervals for the parameters α, β and λ, as well as for the reliability function $S(t)$ and the hazard function $h(t)$, two parametric bootstrap techniques are presented below. Efron [9] was the first to propose the percentile bootstrap ($Boot-p$) confidence interval. The second is the bootstrap-t ($Boot-t$) confidence interval. In the creation of Boot-t method developed by Hall [12], a studentized "pivot" was utilised, which necessitates a variance estimator for the MLE of $\alpha, \beta, \lambda, S(t)$ and $h(t)$.

3.1. Parameteric Boot-p

(1) Based on $x = x_{1:m:n}, x_{2:m:n}, ..., x_{m:m:n,}\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ are obtained by maximizing $Eqs.$ (2.3)-(2.5).

(2) Given the progressive censoring schema $(R_1, R_2, ..., R_m)$, the $APB-XII$ distribution generates $Prog-II-C$ sample with parameters $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ applying the algorithm outlined in Balakrishnan and Sandhu [7].

(3) The bootstrap sample is used to determine the $MLEs$ and the bootstrap estimate is indicated by $\hat{\Psi}^*$where ($\Psi = \alpha, \beta, \lambda, S(t)$ and $h(t)$).

(4) Repeat steps (1) and (2) N $boot$ times, and $\hat{\Psi}^*_1, \hat{\Psi}^*_2, ..., \hat{\Psi}^*_{N\ boot}$ is obtained.

(5) $\hat{\Psi}^*_{(1)}, \hat{\Psi}^*_{(2)}, ..., \hat{\Psi}^*_{(N\ boot)}$ is obtained after $\hat{\Psi}^*_i, i = 1, 2, 3, ..., N$, ascending order is used.

$$[\hat{\Psi}_{boot-p}(\frac{\eta}{2}), \hat{\Psi}_{boot-p}(1-\frac{\eta}{2})],$$

gives the unknow parameters α, β and λ and a two-sided $100(1-\eta)\%$ bootstrap-p.

3.2. Parameteric Boot-t

(1)-(3) Similar to the initial three phases in parametric bootstrap-p.

(4) The matrix of the asymptotic variance and covariance is utilized to calculate the approximate estimates of the variance-covariance matrix $I^{-1*}(\partial^2 l/\partial\Psi_{\ i}\partial\Psi_{\ j})$,

$i, j = 1, 2, 3$ and the variance $S(t)$, $h(t)$ and get the t-statistic of Ψ as $T^{*\Psi} = (\Psi^* - \Psi)/\sqrt{var(\hat{\Psi}^*)}$.

(5) Repeat steps 2 and 3 N $Boot$ times to get $T^{*\Psi}_1, T^{*\Psi}_2, ..., T^{*\Psi}_{NBoot}$.

(6) $T^{*\Psi}_1, T^{*\Psi}_2, ..., T^{*\Psi}_{NBoot}$the ordered sequences are ranked from highest to lowest.

$$[\hat{\Psi}_{boot-t}(\frac{\eta}{2}), \hat{\Psi}_{boot_t}(1-\frac{\eta}{2})],$$

gives the unknowable parameters α, β and λ and a two-sided $100(1-\eta)\%$ bootstrap-t.

4. Bayes Estimation

In this section, we explore the Bayesian estimation of $APB-XII$ using $Prog-II-C$. Our focus lies on generating Bayes estimators utilizing the SEL function and $LINEX$ loss functions. It's important to highlight that the parameters α, β and λ are each independently distributed by gamma distributions.

$$\pi_1(\alpha) \propto \alpha^{k_1-1} \exp(-h_1\alpha), \alpha > 0,$$
$$\pi_2(\beta) \propto \beta^{k_2-1} \exp(-h_2\beta), \beta > 0$$
$$\pi_3(\lambda) \propto \lambda^{k_3-1} \exp(-h_3\lambda), \lambda > 0, \tag{4.1}$$

where the assumption that the hyperparameters k_i and h_i for $i = 1, 2, 3$, are known and carefully selected to represent our prior distribution assumptions regarding the unknown parameters. By using the likelihood function equation, we derive the posterior distribution as well as the prior distribution, resulting in the posterior distribution for α, β and λ, which is denoted as $\pi^*(\alpha, \beta, \lambda|x)$.

$$\pi^*(\alpha, \beta, \lambda|x) = \frac{L(\alpha, \beta, \lambda|x)\pi_1(\alpha)\pi_2(\beta)\pi_3(\lambda)}{\int_0^\infty \int_0^\infty \int_0^\infty L(\alpha, \beta, \lambda|x)\pi_1(\alpha)\pi_2(\beta)\pi_3(\lambda)d\alpha d\beta d\lambda}. \tag{4.2}$$

Loss function

To improve statistical Bayesian inference's applicability and efficacy, an asymmetric loss function must be developed. This loss function is a real-valued function that attempts to reasonably handle a broad variety of estimates and parameters. Through the use of an asymmetric loss function, we can enhance the overall usefulness of statistical Bayesian inference and obtain important insights. This enables us to satisfy all possible estimates and parameters.

(I) Squared error loss function

Squared error loss function (SEL) is a loss function that can be applied in a learning context where we are forecasting a real-valued variable, SEL function is described as:

$$L(\Psi, \hat{\Psi}) = (\hat{\Psi} - \Psi)^2.$$

The Bayes estimate under the SEL for any function of α, β and λ, denoted as g, is represented as

$$\hat{g}_{BS}(\alpha, \beta, \lambda|x) = E_{\alpha,\beta,\lambda|x}(g(\alpha, \beta, \lambda)),$$

where

$$E_{\alpha,\beta,\lambda|x}(g(\alpha, \beta, \lambda)) = \frac{\int_0^\infty \int_0^\infty \int_0^\infty g(\alpha, \beta, \lambda)\pi_1(\alpha)\pi_2(\beta)\pi_3(\lambda)L(\alpha, \beta, \lambda|x)d\alpha d\beta d\lambda}{\int_0^\infty \int_0^\infty \int_0^\infty \pi_1(\alpha)\pi_2(\beta)\pi_3(\lambda)L(\alpha, \beta, \lambda|x)d\alpha d\beta d\lambda}. \tag{4.3}$$

(II) Liner Exponential loss function

By Hall [12] , the equation below represents the suggested *LINEX* loss function $L(\Delta)$ for a parameter Ψ.

$$L(\Delta) = (e^{c\Delta} - c\Delta - 1), c \neq 0, \Delta = \hat{\Psi} - \Psi.$$

Consequently, the following equation that Zellner [25] created, according to the *LINEX* loss function, represents the bayes estimate of a function $g(\alpha, \beta, \lambda)$ as follows:

$$\hat{g}_{BL}(\alpha, \beta, \lambda|x) = -\frac{1}{c}\log[E(e^{-g(\alpha,\beta,\lambda)}|x)], c \neq 0.$$

$$E(e^{-g(\alpha,\beta,\lambda)}) = \frac{\int_0^\infty \int_0^\infty \int_0^\infty e^{-g(\alpha,\beta,\lambda)}\pi_1(\alpha)\pi_2(\beta)\pi_3(\lambda)L(\alpha,\beta,\lambda|x)d\alpha d\beta d\lambda}{\int_0^\infty \int_0^\infty \int_0^\infty \pi_1(\alpha)\pi_2(\beta)\pi_3(\lambda)L(\alpha,\beta,\lambda|x)d\alpha d\beta d\lambda}. \tag{4.4}$$

It is important to highlight that directly calculating the ratio of the multiple integrals *Eqs*. (4.3) and (4.4) is not feasible. To obtain samples from the joint posterior density function described in *Eq*. (11), we employ the Markov Chain Monte Carlo (*MCMC*) method. Specifically, we employ the Gibbs within the Metropolis-Hastings samplers approach, see Ahmed [5], Panahi and Asadi [20], and Abushal [2]. The joint posterior distribution can be expressed as follows

$$\pi^*(\alpha, \beta, \lambda|x) = \alpha^{a_1-1}\beta^{a_2+m-1}\lambda^{a_3+m-1}e^{-b_1\alpha-b_2\beta-b_3\lambda}\left[\frac{\log[\alpha]}{\alpha-1}\right]^m \prod_{i=1}^m x_i^{\beta-1}(1+x_i^\beta)^{-(\lambda+1)}\alpha^{1-(1+x_i^\beta)^{-\lambda}}\left[\frac{\alpha-\alpha^{(1-(1+x_i^\beta)^{-\lambda})}}{\alpha-1}\right]^{R_i}.$$

The conditional posterior densities of α, β and λ represent as

$$\pi_1^*(\alpha|\beta, \lambda, x) = \alpha^{a_1-1}e^{-b_1\alpha}\left[\frac{\log[\alpha]}{\alpha-1}\right]^m \prod_{i=1}^m \alpha^{1-(1+x_i^\beta)^{-\lambda}}\left[\frac{\alpha-\alpha^{(1-(1+x_i^\beta)^{-\lambda})}}{\alpha-1}\right]^{R_i}, \tag{4.5}$$

$$\pi_2^*(\beta|\alpha, \lambda, x) = \beta^{a_2+m-1}e^{-b_2\beta}\prod_{i=1}^m x_i^{\beta-1}(1+x_i^\beta)^{-(\lambda+1)}\alpha^{1-(1+x_i^\beta)^{-\lambda}}\left[\frac{\alpha-\alpha^{(1-(1+x_i^\beta)^{-\lambda})}}{\alpha-1}\right]^{R_i} \tag{4.6}$$

and

$$\pi_3^*(\lambda|\alpha,\beta,x) = \lambda^{a_3+m-1}e^{-b_3\lambda}\prod_{i=1}^{m}(1+x_i^{\beta})^{-(\lambda+1)}\alpha^{1-(1+x_i^{\beta})^{-\lambda}}\left[\frac{\alpha-\alpha^{(1-(1+x_i^{\beta})^{-\lambda})}}{\alpha-1}\right]^{R_i}. \tag{4.7}$$

The conditional posterior distributions of α, β and λ in the preceding equations do not conform to any specific established distribution, which requires the adoption of the Metropolis-Hastings sampler for the *MCMC* approach.

The Metropolis-Hasting approach, produces the posterior sample as follows using Gibbs sampling:

(1) Following the initial suggestion $(\alpha^{(0)}, \beta^{(0)}, \lambda^{(0)})$.

(2) Indicate $j = 0$.

(3) The Metropolis-Hastings $(M-H)$ approach is employed to generate $\alpha^{(j)}, \beta^{(j)}$ *and* $\lambda^{(j)}$ using *Eqs.* (4.5)-(4.9) and the normal suggested distribution

$$N(\alpha^{j-1}, var(\alpha)), N(\beta^{j-1}, var(\beta)) \text{ and } N(\lambda^{j-1}, var(\lambda)). \tag{4.8}$$

The variances of α, β and λ can be computed by using the inverse Fisher information matrix.

(4) Calculate α^*, β^* *and* λ^* by using *Eq.* (4.9).

(i) Analyze the chances of acceptance probability,

$$\eta_\alpha = \min\left[1, \frac{\pi_1^*(\alpha^*|\beta^{(j-1)}, \lambda^{(j-1)}, x)}{\pi_1^*(\alpha^{(j-1)}|\beta^{(j-1)}, \lambda^{(j-1)}, x)}\right],$$
$$\eta_\beta = \min\left[1, \frac{\pi_2^*(\beta^*|\alpha^{(j)}, \lambda^{(j-1)}, x)}{\pi_2^*(\beta^{(j-1)}|\alpha^{(j)}, \lambda^{(j-1)}, x)}\right],$$

and

$$\eta_\lambda = \min\left[1, \frac{\pi_3^*(\lambda^*|\alpha^{(j)}, \beta^{(j)}, x)}{\pi_3^*(\lambda^{(j-1)}|\alpha^{(j)}, \beta^{(j)}, x)}\right].$$

(ii) Create a $u_1, u_2$and u_3, using a uniform $(0, 1)$ distribution.

(iii) If $u_1 < \eta_\alpha$, give your approval to the proposal and set $\alpha^{(j)} = \alpha^*$.
Otherwise, if $u_1 \geq \eta_\alpha$, set $\alpha^{(j)} = \alpha^{(j-1)}$.

(iv) If $u_2 < \eta_\beta$,give your approval to the proposal and set $\beta^{(j)} = \beta^*$.
Otherwise, if $u_2 \geq \eta_\beta$, set $\beta^{(j)} = \beta^{(j-1)}$.

(v) If $u_3 < \eta_\lambda$, give your approval to the proposal set $\lambda^{(j)} = \lambda^*$. Otherwise, if $u_3 \geq \eta_\lambda$, set $\lambda^{(j)} = \lambda^{(j-1)}$.

(5) Calculate the hazard and reliability functions as

$$S^{(j)}(x;\alpha,\beta,\lambda) = \frac{\alpha^{(j)} - \alpha^{(j)[1-(1+x^{\beta^{(j)}})^{-\lambda^{(j)}}]}}{\alpha^{(j)}-1}, t \geq 0,$$

and

$$h^{(j)}(x;\alpha,\beta,\lambda) = \frac{\log(\alpha^{(j)})\lambda^{(j)}\beta^{(j)}x^{\beta^{(j)}-1}(1+x^{\beta^{(j)}})^{-(1+\lambda^{(j)})}\alpha^{(j)[1-(1+x^{\beta^{(j)}})^{-\lambda^{(j)}}]}}{\alpha^{(j)}-\alpha^{(j)[1-(1+x^{\beta^{(j)}})^{-\lambda^{(j)}}]}} \quad ,t \geq 0.$$

(6) Put $j = j + 1$.

(7) Repeat steps (3) and (5) N times to obtain $\alpha^{(i)}, \beta^{(i)}$and $\lambda^{(i)}$, where $i = 1, 2, 3, ..., N$.

(8) To evaluate credible intervals ($CRIs$) of α, β *and* λ for Ψ_k^i, where $(\Psi_1, \Psi_2, \Psi_3) = (\alpha, \beta, \lambda)$ then the $(1-\gamma)100\%$ $CRIs$ of Ψ_k is

$$\left(\Psi_k\left(\frac{\gamma}{2}(N-M)\right), \Psi_k\left(\left(1-\frac{\gamma}{2}\right)(N-M)\right)\right),$$

where N is large enough, the selected samples are denoted as $\Psi_k^{(i)}$, $j = M + 1, ..., N$. The SEL function is typically used to obtain the following approximations of the Bayes estimations of $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ as follows,

$$\hat{\alpha}_{BL} = \left[\frac{1}{N-M}\sum_{i=M+1}^{N}\alpha^{(j)}\right],$$
$$\hat{\beta}_{BL} = \left[\frac{1}{N-M}\sum_{i=M+1}^{N}\beta^{(j)}\right],$$

and

$$\hat{\lambda}_{BL} = \left[\frac{1}{N-M}\sum_{i=M+1}^{N}\lambda^{(j)}\right].$$

With the $LINEX$ loss function, the estimates for the parameters are derived as,

$$\alpha_{BL} = \frac{-1}{c}\log\left[\frac{1}{N-M}\sum_{i=M+1}^{N}e^{-c\alpha^{(i)}}\right],$$
$$\beta_{BL} = \frac{-1}{c}\log\left[\frac{1}{N-M}\sum_{i=M+1}^{N}e^{-c\beta^{(i)}}\right],$$

and

$$\lambda_{BL} = \frac{-1}{c}\log\left[\frac{1}{N-M}\sum_{i=M+1}^{N}e^{-c\lambda^{(i)}}\right].$$

5. Application of Real Data

In this section, we employ the suggested estimation techniques to examine failure statistics in a group of 30 bladder cancer patients. Bladder cancer arises when there is an abnormal growth of tissue, leading to the development of a tumor in the bladder lining. See Lee and Wang [15], the tumor may spread into the bladder muscle. The Kolmogorov-Smirnov ($K-S$) distance between the cumulative distribution function (cdf) of the $APB-XII$ distribution and the empirical failure data distribution is 0.0713385 and the $p-value$ is 0.509777. These results indicate a close match between the $APB-XII$ distribution and the provided data. Randomly selecting a sample of size $m = 20$ from the 30 observed failures using the $Prog-II-C$, we obtained the progressive schema, denoted as $R = (5,5,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$. Table 2 presents the maximum likelihood estimates ($MLEs$) for $\hat{\alpha}$, $\hat{\beta}$, $\hat{\lambda}$, $\hat{S}(t)$ and $\hat{h}(t)$, which were determined using $Prog-II-C$ data from Table 1. The Table 1 also includes the mean values of 1000 $Boot-p$ and $Boot-t$ samples for the parameters lifetime, obtained using the techniques described in Section 3. Furthermore, Table 2 displays the Bayes estimates for the SEL and $LINEX$ functions, generated for various shape parameter values (c) of the $LINEX$ loss function corresponding to the parameters α,β and λ. Additionally, the hazard and reliability functions are shown at time $t = 0.4$. The 95% credibility intervals ($ACIs$ and $CRIs$) for α,β and λ are calculated. Moreover, using the delta approach, the 95% approximate confidence intervals ($ACIs$) and credible intervals ($CRIs$) for the reliability and hazard functions were computed and are displayed in Table 3. The 95% bootstrap confidence intervals for $Boot-p$ and $Boot-t$ are also provided in Table 3. It is widely recognized as shape of parameter c approaches zero, the $LINEX$ loss function becomes symmetric. In Table 3 confirm this by demonstrating that at $c = 0.0001$, the equality of the SEL and $LINEX$ loss functions validates the accuracy of the proposed approaches.

Table 1. Progressive type-II failure data

0.08	2.26	2.54	4.23	3.88
12.63	13.29	4.51	5.49	11.79

Table 2. Estimates for the point values of parameters and the reliability and hazard functions.

Parameters	MLE	Boot-p	Boot-t	Bayesian			
				SEL	LINEX		
					$c = -2$	$c = 2$	$c = 0.0001$
α	14.519	25.072	19.064	24.722	33.130	18.081	24.0875
β	1.34312	1.5478	1.32139	1.45139	1.45298	1.44983	1.45139
λ	0.04861	0.07169	0.0524	0.06169	0.06173	0.06165	0.06169
S	0.992	0.989	0.978	0.994	0.995	0.991	0.994
h	0.00279	0.00538	0.0032	0.00237	0.00235	0.00239	0.00238

Table 3. For parameters α, β, λ and the reliability and hazard functions, the 95% CIs.

Parameter	MLE	Boot-p	Boot-t	MCMC
α	$(0, 29.090)$	$(0, 35.087)$	$(0, 30.887)$	$(19.368, 31.5832)$
β	$(0.4147, 2.2752)$	$(1.947, 5.664)$	$(0.9987, 3.752)$	$(1.38724, 1.5219)$
λ	$(0, 0.1072)$	$(0, 0.7821)$	$(0, 0.0953)$	$(0.00518, 0.0689)$
$S(t)$	$(0.9286, 1.0055)$	$(0.999286, 1.00055)$	$(0.999286, 1.558)$	$(0.9939, 0.9961)$
$h(t)$	$(0, 0.02497)$	$(0, 0.02497)$	$(0, 0.02497)$	$(0.001968, 0.003154)$

6. Simulated Data

To compare the estimators of the $APB - XII$ distribution, simulation tests were conducted using one hundred $Prog - II - C$ samples in each test. The initial values of α, β and λ were set to $\alpha_0 = 10$, $\beta_0 = 0.5$ and $\lambda_0 = 0.2$, to generate $Prog - II - C$ samples from the $APB - XII$ distribution. The estimators for α, β and λ were compared and obtained the mean square error (MSE) in the calculations. For each parameter, denoted as Ψ_i where $i = 1, 2, 3(\Psi_1 = \alpha, \Psi_2 = \beta, \Psi_3 = \lambda)$, the MSE($\Psi_i$) was calculated as $MSE(\Psi_i) = \frac{1}{M(\hat{\Psi}_i - \hat{\Psi}_i)^2}$, M denotes the quantity of simulated samples ($M = 1000$ in this case). To obtain 95% confidence intervals (CIs), the asymptotic distribution of ($MLEs$) was employed and the resulting CIs were compared against a separate standard. The study employed average confidence interval lengths ($ACLs$) and coverage probability (CP) as measures to evaluate and compare the performance of the confidence intervals (CIs). Three progressive schemes were investigated in this section: Schema $I : R_i = n - m$ for $i \neq 1, R_i = 0$. Schema $II : R_{m/2} = n - m/2$ for $i \neq m/2$ and $i \neq m/2 + 1, R_i = 0$. Schema $III : R_m = n - m$ for $i \neq m, R_i = 0$. By evaluating the estimators under different schemes and analyzing the resulting $ACLs$ and CP. The objective of the study was conduct to a comparison of the performance of parameters estimators in progressive type-II censoring

Tables (4)-(8) display the estimate parameters findings together with their MSE, while Tables 9 and 10 display the results of the ACL and CP of 95% CIs.

The results from the tables can be summarized and described as follows:

(1) Tables 4-8 show that $MSEs$ drop with increasing sample size, with Bayes estimates having the lowest $MSEs$ for parameters. For all censoring examined, Bayes estimates thus outperform MLE techniques.

(2) Better estimates are produced by Bayes estimates under $LINEX$ with $c = 2$ since their $MSEs$ are less.

(3) Scheme I outperforms schemes II and III in terms of smaller $MSEs$ for fixed values of the sample n and effective sizes m.

(4) From Tables 9 and 10 for various sample sizes, observed failures and schemes, it can be seen that the $CRIs$ provide more accurate results than the $ACIs$.

Table 4. MSE of ML and Bayesian parameter estimates of α with $\alpha_0 = 10$.

				LINEX	
(n, m)	CS	MLE	SEL	$c = -2$	$c = 2$
(20, 10)	I	10.2205	10.0305	10.0834	9.987
		(0.1841)	(0.3196)	(0.3795)	(0.2853)
	II	10.0034	9.9807	10.006	9.9568
		(0.0152)	(0.1822)	(0.1808)	(0.1813)
	III	10	9.9831	9.9881	9.9783
		(0.0012)	(0.013)	(0.0132)	(0.0128)
(30, 20)	I	8.1188	8.1165	8.211	8.0242
		(5.1274)	(5.4896)	(5.1741)	(5.8944)
	II	9.7419	9.7045	9.8166	9.5903
		(0.2905)	(0.743)	(0.7535)	(0.7608)
	III	17.6756	17.7473	17.787	17.7071
		(0.8379)	(5.1626)	(5.8293)	(4.4914)
(40, 20)	I	10.0216	10.0096	10.0302	9.9886
		(0.312)	(0.0939)	(0.0937)	(0.0958)
	II	9.9463	9.9331	9.9413	9.9249
		(0.0321)	(0.0546)	(0.053)	(0.0565)
	III	8.1188	8.1165	8.211	8.0242
		(5.1274)	(5.4896)	(5.1741)	(5.8944)
(40, 30)	I	6.5891	6.6105	6.7824	6.3945
		(1.655)	(12.1436)	(12.0459)	(12.4272)
	II	7.6893	7.7105	7.824	7.5945
		(1.655)	(12.1436)	(12.0459)	(12.4272)
	III	8.8443	8.8754	9.0183	8.7043
		(3.039)	(3.6688)	(4.0903)	(3.7882)
(50, 20)	I	10	9.9748	9.9797	9.9698
		(0.0125)	(0.0324)	(0.0317)	(0.0333)
	II	10.022	10.102	10.314	10.063
		(0.5974)	(0.478)	(0.5798)	(0.6763)
	III	10.5022	10.5102	10.514	10.5063
		(0.6974)	(0.778)	(0.7798)	(0.7763)

Table 5. MSE of ML and Bayesian parameter estimates of β with $\beta_0 = 0.5$.

				LINEX	
(n, m)	CS	MLE	SEL	$c = -2$	$c = 2$
(20, 10)	I	0.3395	0.3373	0.3378	0.3378
		(0.037)	(0.039)	(0.0389)	(0.039)
	II	0.4692	0.4844	0.4861	0.4861
		(0.0047)	(0.008)	(0.0082)	(0.0078)
	III	0.5	0.4938	0.4964	0.4964
		(0.004)	(0.0038)	(0.0038)	(0.0039)
(30, 20)	I	0.3519	0.3432	0.3436	0.3436
		(0.0329)	(0.0356)	(0.0355)	(0.0357)
	II	0.4478	0.4572	0.4589	0.4589
		(0.0103)	(0.0147)	(0.0144)	(0.0151)
	III	0.7355	0.7337	0.7341	0.7341
		(0.063)	(0.0643)	(0.0645)	(0.0641)
(40, 20)	I	0.4232	0.428	0.4285	0.4285
		(0.0135)	(0.0155)	(0.0155)	(0.0155)
	II	0.4547	0.4572	0.4573	0.4573
		(0.0093)	(0.0092)	(0.0092)	(0.0092)
	III	0.3519	0.3432	0.3436	0.3436
		(0.0329)	(0.0356)	(0.0355)	(0.0357)
(40, 30)	I	0.4539	0.4525	0.4526	0.4526
		(0.0079)	(0.0084)	(0.0084)	(0.0084)
	II	0.4266	0.43	0.4304	0.4304
		(0.0127)	(0.0143)	(0.0142)	(0.0143)
	III	0.4626	0.453	0.4504	0.4404
		(0.2127)	(0.3143)	(0.2142)	(0.1143)
(50, 20)	I	0.5	0.4959	0.4971	0.4971
		(0.001)	(0.0015)	(0.0016)	(0.0015)
	II	0.4184	0.4123	0.4125	0.4125
		(0.0171)	(0.0204)	(0.0204)	(0.0204)
	III	0.4284	0.4223	0.4215	0.4425
		(0.1171)	(0.2104)	(0.0224)	(0.0244)

Table 6. MSE of ML and Bayesian estimates for the parameter λ with $\lambda_0 = 0.6$.

				LINEX	
(n, m)	CS	MLE	SEL	$c = -2$	$c = 2$
(20, 10)	I	0.0688	0.264	0.2642	0.2642
		(0.0071)	(0.0076)	(0.0076)	(0.0075)
	II	0.2119	0.2222	0.2224	0.2224
		(0.007)	(0.0016)	(0.0016)	(0.0015)
	III	0.2	0.2079	0.2085	0.2085
		(0.0076)	(0.0022)	(0.0022)	(0.0022)
(30, 20)	I	0.2972	0.2943	0.2946	0.2946
		(0.0159)	(0.0156)	(0.0157)	(0.0155)
	II	0.2257	0.2243	0.2246	0.2246
		(0.0029)	(0.0047)	(0.0047)	(0.0047)
	III	0.0146	0.0145	0.0145	0.0145
		(0.001)	(0.0012)	(0.0021)	(0.0031)
(40, 20)	I	0.2339	0.2358	0.236	0.236
		(0.0028)	(0.0041)	(0.0041)	(0.004)
	II	0.2231	0.2252	0.2252	0.2252
		(0.0026)	(0.0032)	(0.0032)	(0.0032)
	III	0.2972	0.2943	0.2946	0.2946
		(0.0159)	(0.0156)	(0.0157)	(0.0155)
(40, 30)	I	0.2404	0.2406	0.2406	0.2406
		(0.0053)	(0.0057)	(0.0057)	(0.0057)
	II	0.2417	0.2451	0.2454	0.2454
		(0.0047)	(0.0064)	(0.0064)	(0.0064)
	III	0.2427	0.2452	0.2455	0.2456
		(0.0048)	(0.0069)	(0.0068)	(0.0067)
(50, 20)	I	0.2	0.1829	0.1832	0.1832
		(0.0021)	(0.0009)	(0.0008)	(0.0009)
	II	0.2227	0.21	0.2101	0.2101
		(0.0013)	(0.0004)	(0.0005)	(0.0006)
	III	0.2371	0.21	0.2101	0.2101
		(0.0031)	(0.004)	(0.0004)	(0.0005)

Table 7. MSE of ML and Bayesian estimates for the $S(t)$ with $t = 0.4$.

				LINEX	
(n, m)	CS	MLE	SEL	$c = -2$	$c = 2$
(20, 10)	I	0.8498	0.8474	0.8478	0.8474
		(0.002)	(0.003)	(0.003)	(0.003)
	II	0.8597	0.8579	0.8579	0.8579
		(0.003)	(0.004)	(0.005)	(0.004)
	III	0.8622	0.8607	0.8607	0.8607
		(0.004)	(0.005)	(0.015)	(0.14)
(30, 20)	I	0.8001	0.7988	0.799	0.799
		(0.0062)	(0.0066)	(0.0065)	(0.0066)
	II	0.8522	0.851	0.851	0.851
		(0.0005)	(0.0006)	(0.0006)	(0.0006)
	III	0.9376	0.9378	0.9378	0.9378
		(0.0027)	(0.0028)	(0.0028)	(0.0028)
(40, 20)	I	0.8552	0.8546	0.8546	0.8546
		(0.0001)	(0.0003)	(0.0004)	(0.0003)
	II	0.8564	0.8559	0.8559	0.8559
		(0.0002)	(0.0004)	(0.0006)	(0.0004)
	III	0.8001	0.7988	0.799	0.799
		(0.0062)	(0.0066)	(0.0065)	(0.0066)
(40, 30)	I	0.7644	0.7631	0.7633	0.7633
		(0.0357)	(0.0357)	(0.0357)	(0.0357)
	II	0.8292	0.8273	0.8275	0.8275
		(0.0027)	(0.0031)	(0.0031)	(0.0032)
	III	0.8292	0.8273	0.8275	0.8275
		(0.0037)	(0.0032)	(0.0033)	(0.0032)
(50, 20)	I	0.8622	0.8641	0.8641	0.8641
		(0.001)	(0.002)	(0.0022)	(0.0023)
	II	0.8631	0.865	0.865	0.865
		(0.002)	(0.0024)	(0.0025)	(0.0024)
	III	0.8731	0.885	0.867	0.885
		(0.0021)	(0.0029)	(0.0028)	(0.0031)

Table 8. MSE of ML and Bayesian estimations for the $h(t)$ with $t = 0.4$.

				LINEX	
(n, m)	CS	MLE	SEL	$c = -2$	$c = 2$
(20, 10)	I	0.0282	0.0274	0.0274	0.0274
		(0.0157)	(0.0153)	(0.0153)	(0.0153)
	II	0.0288	0.0309	0.0309	0.0309
		(0.0113)	(0.0114)	(0.0114)	(0.0117)
	III	0.0286	0.0298	0.0299	0.0299
		(0.0212)	(0.0201)	(0.0241)	(0.0241)
(30, 20)	I	0.037	0.0364	0.0364	0.0364
		(0.013)	(0.023)	(0.023)	(0.023)
	II	0.0297	0.0302	0.0302	0.0302
		(0.0112)	(0.0124)	(0.0124)	(0.0124)
	III	0.0016	0.0016	0.0016	0.0016
		(0.0001)	(0.0001)	(0.0001)	(0.0001)
(40, 20)	I	0.0294	0.0299	0.0299	0.0299
		(0.0221)	(0.0002)	(0.0002)	(0.0002)
	II	0.0294	0.0299	0.0299	0.0299
		(0.0197)	(0.0475)	(0.0475)	(0.0475)
	III	0.037	0.0364	0.0364	0.0364
		(0.0112)	(0.0001)	(0.0001)	(0.0001)
(40, 30)	I	0.0399	0.0399	0.0399	0.0399
		(0.0254)	(0.0254)	(0.0254)	(0.0254)
	II	0.0332	0.034	0.034	0.034
		(0.0131)	(0.0001)	(0.0001)	(0.0001)
	III	0.0332	0.034	0.034	0.034
		(0.0457)	(0.0461)	(0.0461)	(0.0461)
(50, 20)	I	0.0289	0.0257	0.0257	0.0257
		(0.001)	(0.002)	(0.002)	(0.002)
	II	0.0268	0.0256	0.0255	0.0256
		(0.002)	(0.0025)	(0.0024)	(0.0025)
	III	0.0269	0.0257	0.0256	0.0257
		(0.0321)	(0.0301)	(0.0332)	(0.0322)

Table 9. The ACL and CP of 95% CIs for the parameters α, β and λ.

		α		β		λ	
(n, m)	CS	MLE	MCMC	MLE	MCMC	MLE	MCMC
(20, 10)	I	31.8725	0.6638	2.5009	0.0574	1.6277	0.9479
		(0.9506)	(0.9669)	(0.9748)	(0.954)	(0.9604)	(0.9669)
	II	24.5472	0.5582	5.5674	0.1241	2.2316	0.0477
		(0.9362)	(0.9728)	(0.9581)	(0.9682)	(0.975)	(0.9733)
	III	18.1127	0.2569	7.0282	0.1569	3.2625	0.0763
		(0.9578)	(0.9376)	(0.9741)	(0.9541)	(0.9669)	(0.955)
(30, 20)	I	38.8192	0.8388	2.4395	0.0489	3.0555	0.0546
		(0.9403)	(0.9484)	(0.969)	(0.9609)	(0.9395)	(0.9312)
	II	41.21	0.9345	3.6231	0.0825	1.8559	0.0476
		(0.9494)	(0.9278)	(0.9346)	(0.9503)	(0.9453)	(0.9336)
	III	29.7257	0.6474	2.8529	0.0624	0.0012	0.0013
		(0.9468)	(0.9521)	(0.9491)	(0.9325)	(0.9446)	(0.9609)
(40, 20)	I	22.9705	0.4696	2.8379	0.0628	1.136	0.0441
		(0.9442)	(0.9631)	(0.9324)	(0.9507)	(0.9656)	(0.9716)
	II	14.4258	0.2932	1.4542	0.0303	0.3337	0.0186
		(0.9672)	(0.9621)	(0.9669)	(0.974)	(0.9637)	(0.965)
	III	38.8192	0.8388	2.4395	0.0489	3.0555	0.0546
		(0.9678)	(0.9288)	(0.9519)	(0.9296)	(0.9711)	(0.9675)
(40, 30)	I	0.8102	40.1146	0.268	1.2395	0.0218	0.4057
		(0.9278)	(0.9464)	(0.959)	(0.9739)	(0.9741)	(0.934)
	II	5.8381	1.0489	2.5684	0.0543	1.4848	0.0427
		(0.9714)	(0.9699)	(0.9444)	(0.9407)	(0.9552)	(0.9705)
	III	5.7381	1.0289	2.2684	0.0143	1.4748	0.0417
		(0.9479)	(0.9667)	(0.9254)	(0.9436)	(0.9462)	(0.9743)
(50, 20)	I	14.2722	0.2595	3.8443	0.1034	0.9822	0.0575
		(0.9731)	(0.9495)	(0.9405)	(0.9308)	(0.9333)	(0.9696)
	II	12.6208	0.2013	2.0892	0.0441	0.3368	0.0264
		(0.934)	(0.9358)	(0.9437)	(0.9535)	(0.9631)	(0.93)
	III	11.6808	0.2004	2.0792	0.0432	0.3367	0.0244
		(0.9585)	(0.964)	(0.9632)	(0.938)	(0.9536)	(0.9354)

Table 10. The ACL and CP of 95% CIs for the $S(t)$ and $h(t)$.

		$S(t)$		$h(t)$	
(n, m)	CS	MLE	MCMC	MLE	MCMC
(20, 10)	I	0.8663	0.0119	0.2052	0.0069
		(0.9292)	(0.9656)	(0.9342)	(0.9483)
	II	0.8212	0.0103	0.2556	0.0112
		(0.9682)	(0.9311)	(0.9327)	(0.9485)
	III	0.8135	0.0127	0.3138	0.0141
		(0.9479)	(0.9595)	(0.9587)	(0.9671)
(30, 20)	I	1.4406	0.0242	0.3182	0.0088
		(0.9336)	(0.9259)	(0.941)	(0.9599)
	II	1.0095	0.0166	0.2577	0.0088
		(0.9489)	(0.933)	(0.9574)	(0.9595)
	III	0.1095	0.0024	0.0039	0.0002
		(0.9533)	(0.9645)	(0.9674)	(0.9367)
(40, 20)	I	0.6542	0.0098	0.1932	0.0073
		(0.9703)	(0.9448)	(0.955)	(0.9278)
	II	0.3125	0.005	0.0833	0.0033
		(0.9255)	(0.9275)	(0.9283)	(0.9739)
	III	1.4406	0.0242	0.3182	0.0088
		(0.973)	(0.9732)	(0.9581)	(0.9582)
(40, 30)	I	0.0205	0.0048	0.9122	0.1965
		(0.9624)	(0.9408)	(0.9748)	(0.9485)
	II	1.1637	0.0214	0.2603	0.0075
		(0.9436)	(0.9621)	(0.9586)	(0.9426)
	III	1.1337	0.0244	0.2601	0.0065
		(0.9283)	(0.9547)	(0.9644)	(0.9473)
(50, 20)	I	0.4851	0.0086	0.1839	0.0094
		(0.9587)	(0.9701)	(0.949)	(0.9539)
	II	0.3188	0.0047	0.0921	0.0045
		(0.9322)	(0.9588)	(0.9608)	(0.947)
	III	0.3078	0.0045	0.0891	0.0039
		(0.9298)	(0.9555)	(0.9303)	(0.9255)

7. Conclusion

This paper established a rigorous inference framework for the three-parameter Alpha Power Burr-XII (APB-XII) distribution under a Progressive Type-II censoring scheme. Both classical and Bayesian frameworks were developed, evaluated through extensive Monte Carlo simulations, and validated using a clinical dataset of bladder cancer patients. Based on the comprehensive empirical and numerical analyses, the definitive findings and scientific contributions of this study are structured as follows: the real data application on 30 bladder cancer patients proved that the APB-XII distribution is an exceptionally robust model for capturing complex survival data. The $K-S$ test yielded a remarkably low distance of 0.0713385 with a high $p-value$ of 0.509777, confirming an excellent, precise match between the proposed theoretical model and empirical failure rates. Across all simulated sample sizes (n, m) and censoring configurations, Bayesian estimators utilizing the $MCMC$ algorithm consistently outperformed classical $MLEs$ by showcasing systematically lower $MSEs$. Under the Bayesian framework, the asymmetric $LINEX$ loss function with a positive shape parameter ($c = 2$) provided enhanced parameter estimates with the most minimized $MSEs$. The numerical convergence where the $LINEX$ estimates at $c = 0.0001$ perfectly matched the SEL estimates validated the precision and mathematical accuracy of the implemented computational algorithms. The MCMC for $CRIs$ exhibited significant structural advantages over the $ACIs$. The $CRIs$ provided highly robust CP closest to the nominal 95% level alongside narrower $ACLs$. For instance, the $MCMC$ intervals for the parameters and reliability functions maintained tighter, more informative bounds for practical decision-making.

In summary, this research successfully bridges the gap between flexible probability modeling and practical censoring constraints, offering clinical researchers and reliability engineers a powerful predictive tool. Future research will extend this model to adaptive progressive hybrid censoring schemes and integrate it into accelerated life testing (ALT) frameworks.

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