
Research article

Comprehensive Comparative Study of Non-Bayesian and Bayesian Estimators for the DUS Topp-Leone Burr-Hatke-Exponential Model

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ABSTRACT

This study presents a comprehensive comparison of non-Bayesian and Bayesian estimation methods for the parameters of the DUS Topp-Leone Burr-Hatke Exponential distribution. Nineteen non-Bayesian estimation procedures are considered alongside Bayesian estimators obtained through Markov chain Monte Carlo sampling under squared error, Linear exponential, and generalized entropy loss functions. The finite-sample performance of the competing estimators is investigated through an extensive Monte Carlo simulation study under four parameter configurations and varying sample sizes. Estimation accuracy is evaluated using average bias and root mean squared error. The results reveal that all estimators improve as sample size increases, confirming their asymptotic consistency. Among the classical procedures, the maximum product spacing and Anderson–Darling estimators generally exhibit best performance relative to the other non-Bayesian methods. However, the Bayesian estimators consistently produce smaller measure values especially in small and moderate samples. The Markov chain Monte Carlo convergence diagnostics indicate excellent chain mixing and convergence across all parameter settings. The practical usefulness of the estimators is further illustrated using two COVID-19 mortality datasets. Goodness-of-fit assessment based on the Kolmogorov–Smirnov statistic and corresponding p-values indicates that the Anderson–Darling estimator provides the best fit for the first dataset, whereas the percentile estimator performs best for the second dataset. Overall, the findings demonstrate that Bayesian estimation offers a robust and efficient alternative for inference, while Anderson–Darling and percentile estimators remain competitive choices in real-data applications.

1. Introduction

Statistical modeling serves as a cornerstone for data-driven decision-making across diverse scientific disciplines. The constant evolution of various datasets has necessitated the development of more sophisticated probability distributions that can capture a wider range of behaviours, including skewness, kurtosis, and various forms of hazard functions. In this context, the method of generating new distributions by compounding existing families has proven immensely fruitful. The DUS Topp-Leone Burr-Hatke exponential (DUSTL_{BHE}) model introduced by [1], represents a significant contribution to this field, with properties that offer remarkable flexibility in modeling various datasets often observed in biomedical research and reliability engineering.

The introduction of new models is considered a theoretical advancement; however, the practical implementation hinges on the accurate and efficient estimation of the parameters. Therefore, non-Bayesian methods, like maximum likelihood estimation (MLE), maximum product spacing estimation (MPSE), Least squares estimation (LSE), Weighted least squares estimation (WLSE), and others, are known for their well-established properties. Nonetheless, these estimators can be biased for small samples and may suffer from convergence issues during parameter estimation in complex models with multiple parameters [10, 12]. In contrast, the Bayesian framework incorporates prior knowledge or expert opinion through a prior distribution, offering a coherent mechanism for uncertainty quantification. This framework is particularly advantageous for small sample sizes, models with many parameters, and the availability of historical data [9, 11]. The comparison of these two approaches is a research area of great interest and practical importance, as evidenced by numerous comparative studies for other complex distributions, see [2, 15].

Given that the theoretical properties of the DUS Topp-Leone Burr-Hatke exponential (DUSTL_{BHE}) model, as well as data modeling in comparison to other existing distributions, have been explored using the MLE (see, [1]), a rigorous comparison of various estimation approaches for its parameters remains an open and critical research question. This gap in the literature leaves practitioners without guidance on the most reliable method for implementing the DUSTL_{BHE} model. The aim of this study is to conduct a comprehensive comparative analysis of non-Bayesian and Bayesian estimation methods for the DUSTL_{BHE} model. Hence, this study intends to

- derive the functional forms of nineteen non-Bayesian estimators and establish the Bayesian estimation framework using the Markov-Chain-Monte-Carlo (MCMC) technique with informative gamma priors and three distinct loss functions.
- conduct an extensive simulation study to compare the efficiency and reliability of the estimators across different sample sizes and parameter combinations.
- demonstrate the comparative performance of the estimators through the analysis of real datasets.

This study is sectionalized as follows: Section 2 presents the distribution, density, failure rate, and quantile functions of the DUSTL_{BHE} model. Section 3 develops the functional forms of nineteen non-Bayesian estimation methods for the DUSTL_{BHE} model, while Section 4 presents the Bayesian framework with three distinct loss functions. Section 5 presents the extensive simulation study for both non-Bayesian and Bayesian methods. Section 6 presents a comprehensive estimator comparison using real-data applications, and the study concludes in Section 7.

2. The Model

Introduced by [1], the cumulative distribution function (cdf) of the DUSTL_{BHE} model is expressed as

$$F(y; \eta, \varphi) = \frac{e^{\left(1 - \left[\frac{e^{-\varphi y}}{1 + \varphi y}\right]^2\right)^\eta} - 1}{e - 1}, \quad y > 0, \quad (2.1)$$

with the corresponding probability density function (pdf) expressed as

$$f(y; \eta, \varphi) = \frac{2\varphi\eta(2 + \varphi y)e^{-2\varphi y}}{(e - 1)(1 + \varphi y)^3} e^{\left(1 - \left[\frac{e^{-\varphi y}}{1 + \varphi y}\right]^2\right)^\eta} \left(1 - \left[\frac{e^{-\varphi y}}{1 + \varphi y}\right]^2\right)^{\eta-1}, \quad (2.2)$$

where $\eta > 0$, $\varphi > 0$ and $e = \exp \approx 2.71828$. The failure rate function for reliability studies is expressed as

$$h(y; \eta, \varphi) = \frac{2\eta\varphi(2 + \varphi y)e^{-2\varphi y} \left(1 - \left[\frac{e^{-\varphi y}}{1 + \varphi y}\right]^2\right)^{\eta-1} e^{\left(1 - \left[\frac{e^{-\varphi y}}{1 + \varphi y}\right]^2\right)^\eta}}{(1 + \varphi y)^3 \left[e - e^{\left(1 - \left[\frac{e^{-\varphi y}}{1 + \varphi y}\right]^2\right)^\eta}\right]}. \quad (2.3)$$

The quantile function $Q(u)$ for generating pseudo-random samples from the DUSTL_{BHE} distribution is expressed as

$$Q(u) = \frac{1}{\varphi} \left\{ W \left(\frac{e}{\sqrt{1 - [\log(1 + u(e - 1))]^{1/\eta}}} \right) - 1 \right\}, \quad 0 < u < 1, \quad (2.4)$$

where W is the Lambert W function (See, [7]).

3. Non-Bayesian Estimation

Here, the estimation methods for the DUSTL_{BHE} parameters using uncensored datasets are presented. These include the method of moments (MOM), method of maximum likelihood (MLE), method of maximum product of spacing (MPS), methods of least squares (LS) and weighted least squares (WLS), method of Cramer-Von Mises (CVM), method of Anderson Darling (ANDA), method of right-tailed Anderson Darling (RTANDA), method of left-tailed Anderson Darling (LTANDA), method of second order Anderson Darling (SOANDA), method of second order right-tailed Anderson Darling (OSRTANDA), method of second order left-tailed Anderson Darling (OSLTANDA), method of minimum spacing absolute distance (MSAD), method of minimum spacing absolute-log distance (MSALD), method of minimum spacing squared distance (MSSD), method of minimum spacing squared log distance (MSSLD), method of minimum spacing Linex distance (MSLD), method of percentile (PE) and method of Kolmogorov-Smirnov (KS) estimation methods.

3.1. The MOM and MLE

Let $y_1, y_2, \dots, y_n$ be the random observed values of size (n) from $\text{DUSTL}_{BHE}(\eta, \varphi)$. The moment estimates (MEs) are found by equating the first two sample moments with the first two theoretical moments, see [1]. The $\hat{\eta}_{ME}$ and $\hat{\varphi}_{ME}$ are found by solving

$$\frac{1}{n} \sum_{i=1}^n y_i - \sum_{i=0}^{\infty} A_{jkl}^* \left\{ \frac{2\Gamma(l+2)}{[2\varphi(i+k+1)]^{l+2}} + \frac{\varphi\Gamma(l+3)}{[2\varphi(i+k+1)]^{l+3}} \right\} = 0, \quad (3.1)$$

and

$$\frac{1}{n} \sum_{i=1}^n y_i^2 - \sum_{i=0}^{\infty} A_{jkl}^* \left\{ \frac{2\Gamma(l+3)}{[2\varphi(i+k+1)]^{l+3}} + \frac{\varphi\Gamma(l+4)}{[2\varphi(i+k+1)]^{l+4}} \right\} = 0. \quad (3.2)$$

Likewise, the ML estimates (MLE) are obtained from the log-likelihood function of Eq. (2.2), specified as

$$\begin{aligned} \ell(\eta, \varphi) = & n \ln(2\varphi\eta - (e - 1)) + \sum_{i=1}^n \ln(2 + \varphi y_i) - 3 \sum_{i=1}^n \ln(1 + \varphi y_i) \\ & - 2\varphi \sum_{i=1}^n y_i + (\eta - 1) \sum_{i=1}^n \ln(1 - u_i^2) + \sum_{i=1}^n [1 - u_i^2]^\eta, \end{aligned} \quad (3.3)$$

where $u_i = \frac{\exp(-\varphi y_i)}{1 + \varphi y_i}$. The partial derivative of Eq. (3.3) with respect to η and φ are given by

$$\frac{\partial \ell}{\partial \eta} = \frac{n}{\eta} + \sum_{i=1}^n \ln(1 - u_i^2) [1 + (1 - u_i^2)^\eta] = 0, \quad (3.4)$$

and

$$\begin{aligned} \frac{\partial \ell}{\partial \varphi} = & \frac{n}{\varphi} + \sum_{i=1}^n \frac{y_i}{2 + \varphi y_i} - 3 \sum_{i=1}^n \frac{y_i}{1 + \varphi y_i} - 2 \sum_{i=1}^n y_i + 2(\eta - 1) \\ & \times \sum_{i=1}^n \frac{u_i y_i \exp(-\varphi y_i)(2 + \varphi y_i)}{(1 - u_i^2)(1 + \varphi y_i)^2} + 2\eta \sum_{i=1}^n \frac{(1 - u_i^2)^{\eta-1} u_i y_i \exp(-\varphi y_i)(2 + \varphi y_i)}{(1 + \varphi y_i)^2} = 0. \end{aligned} \quad (3.5)$$

The ML estimates $\hat{\eta}_{ML}$ and $\hat{\varphi}_{ML}$ can be found by simultaneously solving the non-linear functions.

3.2. The LS and WLS

Let $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ be the ordered sample of size (n) from the $DUSTL_{BHE}$ model. Then, the LS estimates $\hat{\eta}_{LS}$ and $\hat{\varphi}_{LS}$ are found by minimizing, with respect to η and φ , the function

$$LS(\eta, \varphi) = \sum_{i=1}^n \left[\frac{e^{(1-u_i^2)^\eta} - 1}{e - 1} - \frac{i}{n+1} \right]^2. \quad (3.6)$$

Equivalently, the LS estimates can be found by solving the following non-linear equation

$$\sum_{i=1}^n \left[\frac{e^{(1-u_i^2)^\eta} - 1}{e - 1} - \frac{i}{n+1} \right] \xi_j(y_i | \eta, \varphi) = 0; \quad j = 1, 2, \quad (3.7)$$

where $\xi_j(y_i | \eta, \varphi)$ is the partial derivative of Eq. (2.1) with respect to η and φ , given as

$$\xi_1(y_i | \eta, \varphi) = \frac{(1 - u^2)^\eta \ln(1 - u^2) e^{(1-u^2)^\eta}}{e - 1}, \quad (3.8)$$

and

$$\xi_2(y_i | \eta, \varphi) = \frac{2\eta(1 - u_i^2)^{\eta-1} u_i y_i \exp(-\varphi y_i)(2 + \varphi y_i) e^{(1-u_i^2)^\eta}}{(1 + \varphi y_i)^2 (e - 1)}. \quad (3.9)$$

The WLS estimates $\hat{\eta}_{\text{WLS}}$ and $\hat{\varphi}_{\text{WLS}}$ can be found by minimizing with respect to η and φ , the function

$$WLE(\eta, \varphi) = \sum_{j=1}^n \zeta(i, n) \left[\frac{e^{(1-u_i^2)^\eta} - 1}{e - 1} - \frac{i}{n+1} \right]^2, \quad (3.10)$$

where $\zeta(i, n) = (n+1)^2(n+2)/i(n-i+1)$. Equivalently, the WLS estimates can also be found by solving the following non-linear equation

$$\sum_{i=1}^n \zeta(i, n) \left[\frac{e^{(1-u_i^2)^\eta} - 1}{e - 1} - \frac{i}{n+1} \right] \xi_j(y_i | \eta, \varphi) = 0; \quad j = 1, 2, \quad (3.11)$$

where $\xi_1(\cdot | \eta, \varphi)$ and $\xi_2(\cdot | \eta, \varphi)$ is specified in Eqs. (3.8) and (3.9).

3.3. The MPS, MSAD, MSALD, MSSD, MSSLD and MSLD

Here, the MPS, MSAD, MSALD, MSSD, MSSLD and MSLD estimations of unknown parameters with ordered sample $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ from the DUSTL_{BHE} model are considered. The MPS estimates $\hat{\eta}_{\text{MPS}}$ and $\hat{\varphi}_{\text{MPS}}$ can be found by maximizing relative to η and φ , the logarithm of the geometric mean of the spacing specified as

$$MPS(\eta, \varphi) = \frac{1}{n+1} \sum_{i=1}^{n+1} \ln D_i(\eta, \varphi), \quad (3.12)$$

where

$$\begin{aligned} D_i(\eta, \varphi) &= F_n(y_i | \eta, \varphi) - F_n(y_{i-1} | \eta, \varphi) \\ &= \frac{e^{(1-u_i^2)^\eta} - 1}{e - 1} - \frac{e^{(1-u_{i-1}^2)^\eta} - 1}{e - 1}, \end{aligned} \quad (3.13)$$

with $F(y_{(0)} | \eta, \varphi) = 0$, $F(y_{(n+1)} | \eta, \varphi) = 1$ and $\sum_{i=1}^{n+1} D_i(\eta, \varphi) = 1$.

The MSAD estimates $\hat{\eta}_{\text{MSAD}}$ and $\hat{\varphi}_{\text{MSAD}}$ can be found by minimizing with respect to η and φ , the function

$$MSAD(\eta, \varphi) = \sum_{i=1}^{n+1} \left| D_i(\eta, \varphi) - \frac{1}{n+1} \right|. \quad (3.14)$$

The MSALD estimates $\hat{\eta}_{\text{MSALD}}$ and $\hat{\varphi}_{\text{MSALD}}$ can be found by minimizing with respect to η and φ , the function

$$MSALD(\eta, \varphi) = \sum_{i=1}^{n+1} \left| \ln D_i(\eta, \varphi) - \ln \left(\frac{1}{n+1} \right) \right|. \quad (3.15)$$

Likewise, the MSSD estimates $\hat{\eta}_{\text{MSSD}}$ and $\hat{\varphi}_{\text{MSSD}}$ can be found by minimizing with respect to η and φ , the function

$$MSSD(\eta, \varphi) = \sum_{i=1}^{n+1} \left(D_i(\eta, \varphi) - \frac{1}{n+1} \right)^2. \quad (3.16)$$

The MSSLD estimates $\hat{\eta}_{\text{MSSLD}}$ and $\hat{\varphi}_{\text{MSSLD}}$ can be found by minimizing with respect to η and φ , the function

$$MSSLD(\eta, \varphi) = \sum_{i=1}^{n+1} \left[\ln D_i(\eta, \varphi) - \ln \left(\frac{1}{n+1} \right) \right]^2. \quad (3.17)$$

The MSLD estimates $\hat{\eta}_{\text{MSLD}}$ and $\hat{\varphi}_{\text{MSLD}}$ can be found by minimizing with respect to η and φ , the function

$$MSLD(\eta, \varphi) = \sum_{i=1}^{n+1} \left[e^{(D_i(\eta, \varphi) - \frac{1}{n+1})} - \left(D_i(\eta, \varphi) - \frac{1}{n+1} \right) - 1 \right]. \quad (3.18)$$

Equivalently, the parameter estimates can be found by finding the first derivative of Eqs. (3.12), (3.14), (3.15), (3.16), (3.17) and (3.18) with respect to η and φ , then equating to zero.

3.4. The ANDA, RTANDA, LTANDA, OSANDA, OSRTANDA and OSLTANDA

The ANDA estimates $\hat{\eta}_{\text{ANDA}}$ and $\hat{\varphi}_{\text{ANDA}}$ can be found for $\text{DUSTL}_{\text{BHE}}$ by minimizing relative to η and φ , the function

$$\begin{aligned} AD(\eta, \varphi) &= -n - \frac{1}{n} \sum_{i=1}^n (2i-1) \left[\ln F(y_i | \eta, \varphi) \right. \\ &\quad \left. + \ln(1 - F(y_{n+1-i} | \eta, \varphi)) \right] \\ &= -n - \frac{1}{n} \sum_{i=1}^n (2i-1) \left[\ln \left(\frac{e^{(1-u_i^2)^\eta} - 1}{e - 1} \right) \right. \\ &\quad \left. + \ln \left(1 - \frac{e^{(1-u_{n+1-i}^2)^\eta} - 1}{e - 1} \right) \right]. \end{aligned} \quad (3.19)$$

Equivalently, the ANDA estimates can also be found by solving the following non-linear equation

$$\sum_{i=1}^n (2i-1) \left[\frac{\xi_j(y_i | \eta, \varphi)}{\frac{e^{(1-u_i^2)^\eta} - 1}{e - 1}} - \frac{\xi_j(y_{n+1-i} | \eta, \varphi)}{1 - \frac{e^{(1-u_{n+1-i}^2)^\eta} - 1}{e - 1}} \right] = 0; \quad j = 1, 2, \quad (3.20)$$

where $\xi_1(\cdot | \eta, \varphi)$ and $\xi_2(\cdot | \eta, \varphi)$ is specified in Eqs. (3.8) and (3.9). The RTANDA estimates $\hat{\eta}_{\text{RTANDA}}$ and $\hat{\varphi}_{\text{RTANDA}}$ can be found for $\text{DUSTL}_{\text{BHE}}$ by minimizing relative to η and φ , the function

$$\begin{aligned} RAD(\eta, \varphi) &= \frac{n}{2} - 2 \sum_{i=1}^n F(y_i | \eta, \varphi) \\ &\quad - \frac{1}{n} \sum_{i=1}^n (2i-1) \ln(1 - F(y_{n+1-i} | \eta, \varphi)) \\ &= \frac{n}{2} - 2 \sum_{i=1}^n \left(\frac{e^{(1-u_i^2)^\eta} - 1}{e - 1} \right) \\ &\quad - \frac{1}{n} \sum_{i=1}^n (2i-1) \ln \left(1 - \frac{e^{(1-u_{n+1-i}^2)^\eta} - 1}{e - 1} \right). \end{aligned} \quad (3.21)$$

Equivalently, the RTANDA estimates can also be found by solving the following non-linear equation

$$-2 \sum_{i=1}^n \xi_j(y_i | \eta, \varphi) - \frac{1}{n} \sum_{i=1}^n (2i-1) \left[\frac{\xi_j(y_{n+1-i} | \eta, \varphi)}{1 - \frac{e^{(1-u_{n+1-i}^2)^\eta} - 1}{e - 1}} \right] = 0; \quad j = 1, 2, \quad (3.22)$$

where $\xi_1(\cdot|\eta, \varphi)$ and $\xi_2(\cdot|\eta, \varphi)$ is specified in Eqs. (3.8) and (3.9). The LTANDA estimates $\hat{\eta}_{\text{LTANDA}}$ and $\hat{\varphi}_{\text{LTANDA}}$ can be found for $\text{DUSTL}_{\text{BHE}}$ by minimizing relative to α and η , the function

$$\begin{aligned} LAD(\eta, \varphi) &= -\frac{3}{2}n + 2 \sum_{i=1}^n F(y_i | \eta, \varphi) \\ &\quad - \frac{1}{n} \sum_{i=1}^n (2i-1) \ln F(y_i | \eta, \varphi) \\ &= -\frac{3}{2}n + 2 \sum_{i=1}^n \left(\frac{e^{(1-u_i^2)^\eta} - 1}{e-1} \right) \\ &\quad - \frac{1}{n} \sum_{i=1}^n (2i-1) \ln \left(\frac{e^{(1-u_i^2)^\eta} - 1}{e-1} \right). \end{aligned} \quad (3.23)$$

Equivalently, the LTANDA estimates can also be found by solving the following non-linear equation

$$2 \sum_{i=1}^n \xi_j(y_i | \eta, \varphi) - \frac{1}{n} \sum_{i=1}^n (2i-1) \left[\frac{\xi_j(y_i | \eta, \varphi)}{\frac{e^{(1-u_i^2)^\eta} - 1}{e-1}} \right] = 0; \quad j = 1, 2, \quad (3.24)$$

where $\xi_1(\cdot|\eta, \varphi)$ and $\xi_2(\cdot|\eta, \varphi)$ are specified in Eqs. (3.8) and (3.9). The OSANDA estimates $\hat{\eta}_{\text{OSANDA}}$ and $\hat{\varphi}_{\text{OSANDA}}$ can be found for $\text{DUSTL}_{\text{BHE}}$ by minimizing relative to η and φ , the function

$$\begin{aligned} OSAD(\eta, \varphi) &= \frac{1}{n} \sum_{i=1}^n (2i-1) \left[\frac{1}{F(y_i | \eta, \varphi)} + \frac{1}{1 - F(y_{n+1-i} | \eta, \varphi)} \right], \\ &= \frac{1}{n} \sum_{i=1}^n (2i-1) \left[\frac{1}{\left(\frac{e^{(1-u_i^2)^\eta} - 1}{e-1} \right)} + \frac{1}{1 - \frac{e^{(1-u_{n+1-i}^2)^\eta} - 1}{e-1}} \right]. \end{aligned} \quad (3.25)$$

Equivalently, the OSANDA estimates can also be found by solving the following non-linear equation

$$\sum_{i=1}^n (2i-1) \left[-\frac{\xi_j(y_i | \eta, \varphi)}{\left(\frac{e^{(1-u_i^2)^\eta} - 1}{e-1} \right)^2} + \frac{\xi_j(y_{n+1-i} | \eta, \varphi)}{\left(1 - \frac{e^{(1-u_{n+1-i}^2)^\eta} - 1}{e-1} \right)^2} \right] = 0; \quad j = 1, 2, \quad (3.26)$$

where $\xi_1(\cdot|\eta, \varphi)$ and $\xi_2(\cdot|\eta, \varphi)$ are specified in Eqs. (3.8) and (3.9). The OSRTANDA estimates $\hat{\eta}_{\text{OSRTANDA}}$ and $\hat{\varphi}_{\text{OSRTANDA}}$ can be found for $\text{DUSTL}_{\text{BHE}}$ by minimizing relative to η and φ , the function

$$\begin{aligned} OSRAD(\eta, \varphi) &= -2 \sum_{i=1}^n \log F(y_i | \eta, \varphi) + \frac{1}{n} \sum_{i=1}^n \frac{(2i-1)}{1 - F(y_{n+1-i} | \eta, \varphi)}, \\ &= -2 \sum_{i=1}^n \log \left(\frac{e^{(1-u_i^2)^\eta} - 1}{e-1} \right) + \frac{1}{n} \sum_{i=1}^n \frac{(2i-1)}{\left(1 - \frac{e^{(1-u_{n+1-i}^2)^\eta} - 1}{e-1} \right)}. \end{aligned} \quad (3.27)$$

Equivalently, the OSRTANDA estimates can also be found by solving the following non-linear equation

$$-2 \sum_{i=1}^n \frac{1}{\left(\frac{e^{(1-u_i^2)^\eta}-1}{e-1}\right)} \xi_j(y_i|\eta, \varphi) + \frac{1}{n} \sum_{i=1}^n \frac{(2i-1)}{\left(1 - \frac{e^{(1-u_{n+1-i}^2)^\eta}-1}{e-1}\right)} \xi_j(y_i|\eta, \varphi) = 0; \quad j = 1, 2, \quad (3.28)$$

where $\xi_1(\cdot|\eta, \varphi)$ and $\xi_2(\cdot|\eta, \varphi)$ are specified in Eqs. (3.8) and (3.9). The OSLTANDA estimates $\hat{\eta}_{\text{OSLTANDA}}$ and $\hat{\varphi}_{\text{OSLTANDA}}$ can be found for DUSTL_{BHE} by minimizing relative to η and φ , the function

$$\begin{aligned} \text{OSLAD}(\eta, \varphi) &= 2 \sum_{i=1}^n \log F(y_i | \eta, \varphi) + \frac{1}{n} \sum_{i=1}^n \frac{(2i-1)}{F(y_i | \eta, \varphi)} \\ &= 2 \sum_{i=1}^n \log \left(\frac{e^{(1-u_i^2)^\eta}-1}{e-1} \right) + \frac{1}{n} \sum_{i=1}^n \frac{(2i-1)}{\left(\frac{e^{(1-u_i^2)^\eta}-1}{e-1}\right)}. \end{aligned} \quad (3.29)$$

Equivalently, the OSLTANDA estimates can also be found by solving the following non-linear equation

$$2 \sum_{i=1}^n \frac{1}{\left(\frac{e^{(1-u_i^2)^\eta}-1}{e-1}\right)} \xi_j(y_i|\eta, \varphi) - \frac{1}{n} \sum_{i=1}^n \frac{(2i-1)}{\left(\frac{e^{(1-u_i^2)^\eta}-1}{e-1}\right)^2} \xi_j(y_i|\eta, \varphi) = 0; \quad j = 1, 2, \quad (3.30)$$

where $\xi_1(\cdot|\eta, \varphi)$ and $\xi_2(\cdot|\eta, \varphi)$ are specified in Eqs. (3.8) and (3.9).

3.5. The CVM

The CVM estimates $\hat{\eta}_{\text{CVM}}$ and $\hat{\varphi}_{\text{CVM}}$ are found by minimizing relative to η and φ the function

$$\begin{aligned} CV(\eta, \varphi) &= \frac{1}{12n} + \sum_{i=1}^n \left[F(y_i | \eta, \varphi) - \frac{2(i-1)+1}{2n} \right]^2 \\ &= \frac{1}{12n} n + \sum_{i=1}^n \left[\left(\frac{e^{(1-u_i^2)^\eta}-1}{e-1} \right) - \frac{2(i-1)+1}{2n} \right]^2. \end{aligned} \quad (3.31)$$

Equivalently, the CVM estimates can also be found by solving the following non-linear equation

$$\sum_{i=1}^n \left[\frac{e^{(1-u_i^2)^\eta}-1}{e-1} - \frac{2(i-1)+1}{2n} \right] \xi_j(y_i|\eta, \varphi) = 0; \quad j = 1, 2, \quad (3.32)$$

where $\xi_1(\cdot|\eta, \varphi)$ and $\xi_2(\cdot|\eta, \varphi)$ are specified in Eqs. (3.8) and (3.9).

3.6. The PE

The DUSTL_{BHE} model parameters can be estimated by equating the sample percentile points with the population percentile points. An estimate of $F(y_i|\eta, \varphi)$ is given by u_i , then the percentile estimates $\hat{\eta}_{\text{PCB}}$ and $\hat{\varphi}_{\text{PCB}}$ can be found by minimizing with respect to η and φ , the function

$$P(\eta, \varphi) = \sum_{i=1}^n [y_i - Q(u_i|\eta, \varphi)]^2, \quad (3.33)$$

where $u_i = \frac{i}{n+1}$ is the unbiased estimator of $F(y_i|\eta, \varphi)$ and $Q(u_i|\eta, \varphi)$ is specified in Eq. (2.4). Equivalently, the parameter estimates can be found by deriving the first derivative of Eq. (3.33) with respect to η and φ , then equating to zero.

3.7. The KS

The DUSTL_{BHE} model parameters can be estimated by minimizing the maximum absolute distance. The Kolmogorov estimates $\hat{\eta}_{KS}$ and $\hat{\varphi}_{KS}$ can be found by minimizing with respect to η and φ , the function

$$K(\eta, \varphi) = \max_{1 \leq i \leq n} \left[\max \left(\left| \frac{i}{n} - F(y_i | \eta, \varphi) \right|, \left| \frac{i-1}{n} - F(y_i | \eta, \varphi) \right| \right) \right], \quad (3.34)$$

where $F(y_i | \eta, \varphi)$ is the cdf of the DUSTL_{BHE} model in Eq. (2.1). Additionally, the parameter estimates in all the estimation methods are numerically solved with iterative method utilizing the R-program (*Optim function*).

4. Bayesian Estimation

This section presents the Bayesian framework for estimating the unknown parameters η and φ of the DUSTL_{BHE} distribution. The Bayesian approach incorporates prior information about the parameters and utilizes various loss functions to obtain optimal estimates. Three distinct loss functions are considered: squared error loss (SEL), Linear-Exponential (LINEX) loss, and generalized entropy loss (GEL). We assume independent gamma prior distributions for the parameters η and φ . The prior probability density functions are specified as:

$$\pi_1(\eta) \propto \eta^{a_1-1} e^{(-b_1\eta)}, \quad \eta > 0, a_1 > 0, b_1 > 0, \quad (4.1)$$

$$\pi_2(\varphi) \propto \varphi^{a_2-1} e^{(-b_2\varphi)}, \quad \varphi > 0, a_2 > 0, b_2 > 0, \quad (4.2)$$

where a_j and b_j ($j = 1, 2$) represent hyper-parameters chosen based on available prior knowledge about the parameters. The joint prior distribution for $\theta = (\eta, \varphi)$ takes the form

$$\pi(\theta) = \pi_1(\eta)\pi_2(\varphi) \propto \eta^{a_1-1} \varphi^{a_2-1} e^{(-b_1\eta-b_2\varphi)}, \quad (4.3)$$

Given an independent random sample $\mathbf{y} = (y_1, y_2, \dots, y_n)$ from the DUSTL_{BHE} distribution, the likelihood function is specified as:

$$L(\theta) = \prod_{i=1}^n f(y_i; \eta, \varphi), \quad (4.4)$$

where $f(y_i; \eta, \varphi)$ represents the pdf given in Eq. (2.2). Following Bayes' theorem, the posterior distribution of θ given the observed data $\mathbf{y}$ is proportional to:

$$\pi(\theta|\mathbf{y}) \propto \pi(\theta) \cdot L(\theta). \quad (4.5)$$

This yields the posterior density specified as

$$\pi(\theta|\mathbf{y}) \propto \frac{2^n \eta^{a_1+n-1} \varphi^{a_2+n-1} e^{-b_1\eta-b_2\varphi-2\varphi \sum_{i=1}^n y_i + \sum_{i=1}^n \left\{ 1 - \left[\frac{e^{-\varphi y_i}}{1+\varphi y_i} \right]^2 \right\}}{(e-1)^n}$$

$$\times \prod_{i=1}^n \left[\frac{(2 + \varphi y_i) \left[1 - \left(\frac{e^{-\varphi y_i}}{1 + \varphi y_i} \right)^2 \right]^{\eta-1}}{(1 + \varphi y_i)^3} \right]. \quad (4.6)$$

Under the SEL function, which treats underestimation and overestimation symmetrically, the Bayes estimator is the posterior mean:

$$\hat{\theta}_{\text{SEL}} = \mathbb{E}[\theta|\mathbf{y}] = \int_{\theta} \theta \pi(\theta|\mathbf{y}) d\theta. \quad (4.7)$$

However, the LINEX loss function provides asymmetric treatment of estimation errors and is defined as

$$L(\hat{\theta}, \theta) = e^{\{c(\hat{\theta}-\theta)\}} - c(\hat{\theta} - \theta) - 1, \quad (4.8)$$

where $c \neq 0$ is a shape parameter controlling the asymmetry. Positive values of c penalize overestimation more severely than underestimation, while negative values have the opposite effect, see [14, 8]. The corresponding Bayes estimator is specified as

$$\hat{\theta}_{\text{LINEX}} = -\frac{1}{c} \ln \left[\int_{\theta} e^{-c\theta} \pi(\theta|\mathbf{y}) d\theta \right]. \quad (4.9)$$

More so, the GEL function is characterized by

$$L(\hat{\theta}, \theta) = \left(\frac{\hat{\theta}}{\theta} \right)^{-q} - q \ln \left(\frac{\hat{\theta}}{\theta} \right) - 1, \quad (4.10)$$

where $q \neq 0$ is a shape parameter. $q > 0$, overestimation is penalized more than underestimation, and vice versa for $q < 0$, see [6]. The Bayes estimator under GEL is specified as

$$\hat{\theta}_{\text{GEL}} = [\mathbb{E}[\theta^{-q}|\mathbf{y}]]^{-1/q} = \left[\int_{\theta} \theta^{-q} \pi(\theta|\mathbf{y}) d\theta \right]^{-1/q}. \quad (4.11)$$

Due to the mathematical complexity of the posterior distribution, analytical computation of the Bayes estimators is intractable. Therefore, we employ Markov Chain Monte Carlo (MCMC) method to generate samples from the posterior distribution and compute the estimators numerically, see [5, 13]. The MCMC algorithm generates a sequence of dependent samples $\{\theta^{(j)}\}_{j=1}^N$ from the posterior distribution. After discarding the initial λ_A samples as burn-in to ensure convergence, the remaining samples are used to approximate the Bayes estimators:

$$\hat{\theta}_{\text{SEL}} = \frac{1}{N - \lambda_A} \sum_{j=\lambda_A+1}^N \theta^{(j)}, \quad (4.12)$$

$$\hat{\theta}_{\text{LINEX}} = -\frac{1}{c} \ln \left[\frac{1}{N - \lambda_A} \sum_{j=\lambda_A+1}^N e^{-c\theta^{(j)}} \right], \quad (4.13)$$

$$\hat{\theta}_{\text{GEL}} = \left[\frac{1}{N - \lambda_A} \sum_{j=\lambda_A+1}^N (\theta^{(j)})^{-q} \right]^{-1/q}, \quad (4.14)$$

where N denotes the total number of MCMC iterations and λ_A represents the burn-in period length.

5. Comparison of estimators via simulation

This study utilized multiple non-Bayesian estimators for comparative analysis. Parameter estimates were calculated for 1,500 samples generated from the $DUSTL_{BHE}$ across four distinct sample sizes using each methodology. Beyond the non-Bayesian simulation framework, the Markov chain Monte Carlo (MCMC) approach was implemented to generate posterior samples for Bayesian parameter estimation, necessitated by the intractability of obtaining closed-form solutions from Eqs. (4.12), (4.13), and (4.14). To enhance validation of the proposed estimation procedures, a bootstrap methodology was integrated with MCMC techniques. Additionally, Independent Gamma priors with shape 2 and rate 1 were assigned to the model parameters to reflect weak prior information while preserving positivity. The selected priors possess finite means and variances and provide stable posterior computation. Specifically, 30,000 iterations were generated for sample sizes $n = 25, 75, 150$, and 250. The Metropolis–Hastings algorithm was applied to each sample to estimate the parameter vector $\theta^{(j)} = (\eta^{(j)}, \varphi^{(j)})$, with 10,000 burn-in iterations and proposal standard deviations of 0.2 for both parameters to ensure Markov chain convergence. Subsequently, Bayes estimates were derived under SEL, LINEX, and GEL frameworks using the retained samples. Furthermore, the MCMC convergence was assessed through Gelman–Rubin ($\hat{R}$) diagnostics and effective sample sizes (ESS).

Table 1. Average bias and RMSE for non-Bayesian and Bayesian estimations at various n , with $DUSTL_{BHE}$ parameters ($\eta = 2.0, \varphi = 1.5$).

Method	Pa.	n = 25		n = 75		n = 150		n = 250	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	η	0.461	1.229	0.119	0.487	0.062	0.323	0.046	0.242
	φ	0.148	0.447	0.043	0.221	0.022	0.152	0.015	0.117
MPS	η	-0.160	0.790	-0.126	0.423	-0.079	0.300	-0.047	0.229
	φ	-0.117	0.383	-0.071	0.215	-0.044	0.152	-0.028	0.116
LS	η	0.253	1.561	0.053	0.585	0.032	0.393	0.024	0.295
	φ	0.026	0.482	0.006	0.256	0.005	0.181	0.003	0.138
WLS	η	0.266	1.399	0.068	0.523	0.044	0.349	0.035	0.264
	φ	0.045	0.455	0.017	0.235	0.013	0.163	0.010	0.125
CVM	η	0.682	2.196	0.159	0.647	0.083	0.414	0.054	0.305
	φ	0.174	0.557	0.052	0.269	0.027	0.185	0.017	0.140
ANDA	η	0.259	1.084	0.069	0.501	0.040	0.339	0.031	0.259
	φ	0.064	0.420	0.019	0.228	0.011	0.160	0.008	0.124
RTANDA	η	0.556	2.087	0.122	0.663	0.065	0.430	0.044	0.319
	φ	0.107	0.495	0.027	0.256	0.015	0.179	0.010	0.137
LTANDA	η	0.396	1.284	0.102	0.547	0.057	0.354	0.043	0.272
	φ	0.128	0.500	0.036	0.256	0.021	0.179	0.015	0.138
SOANDA	η	-1.091	1.136	-1.126	1.138	-1.131	1.137	-1.133	1.136
	φ	-0.686	0.728	-0.715	0.715	-0.704	0.711	-0.706	0.709
RTOSANDA	η	1.413	1.414	1.432	1.434	1.433	1.435	1.424	1.425
	φ	1.475	1.475	-1.477	1.477	-1.477	1.477	-1.476	1.476

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Table 1: (contd...)

Method	Pa.	n = 25		n = 75		n = 150		n = 250	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
LTOSANDA	η	-1.998	1.998	-1.997	1.997	-1.997	1.997	-1.997	1.997
	φ	1.830	1.830	1.822	1.822	1.813	1.814	1.806	1.806
PE	η	-0.050	1.234	-0.136	0.703	-0.111	0.511	-0.089	0.402
	φ	-0.199	0.498	-0.096	0.308	-0.068	0.224	-0.050	0.175
MOM	η	0.742	1.858	0.224	0.781	0.121	0.515	0.079	0.388
	φ	0.200	0.566	0.063	0.301	0.034	0.211	0.022	0.164
MSAD	η	0.057	1.194	-0.048	0.548	-0.039	0.401	-0.012	0.302
	φ	-0.049	0.464	-0.041	0.264	-0.027	0.190	-0.012	0.150
MSALD	η	-0.160	0.790	-0.125	0.423	-0.079	0.300	-0.048	0.229
	φ	-0.117	0.383	-0.071	0.215	-0.044	0.152	-0.028	0.116
MSSD	η	-0.044	1.986	-0.135	0.646	-0.105	0.420	-0.070	0.327
	φ	-0.133	0.580	-0.085	0.305	-0.062	0.209	-0.042	0.164
MSSLD	η	0.034	0.918	-0.054	0.461	-0.041	0.331	-0.027	0.253
	φ	-0.032	0.412	-0.041	0.230	-0.029	0.166	-0.020	0.127
MSLD	η	-0.041	1.973	-0.134	0.648	-0.105	0.421	-0.070	0.327
	φ	-0.133	0.582	-0.085	0.306	-0.062	0.209	-0.042	0.164
KS	η	0.392	1.231	0.137	0.599	0.070	0.393	0.055	0.312
	φ	0.187	0.493	0.070	0.269	0.035	0.184	0.025	0.144
BE_{SEL}	η	0.227	0.590	0.133	0.409	0.062	0.269	0.031	0.170
	φ	0.085	0.223	0.022	0.179	0.042	0.116	0.012	0.092
$BE_{LINEX} (c = -1.5)$	η	-0.001	0.385	0.037	0.282	0.007	0.273	0.038	0.167
	φ	0.051	0.221	0.036	0.127	0.026	0.123	0.029	0.099
$BE_{LINEX} (c = 1.5)$	η	0.995	1.097	0.331	0.444	0.144	0.312	0.121	0.213
	φ	0.227	0.296	0.098	0.152	0.055	0.127	0.048	0.109
$BE_{GEL} (q = -0.5)$	η	0.206	0.473	0.120	0.314	0.050	0.281	0.064	0.179
	φ	0.105	0.239	0.056	0.134	0.034	0.124	0.035	0.102
$BE_{GEL} (q = 0.5)$	η	0.517	0.685	0.217	0.369	0.095	0.294	0.092	0.195
	φ	0.164	0.263	0.077	0.142	0.044	0.125	0.041	0.105

Table 2. Average bias and RMSE for non-Bayesian and Bayesian estimations at various n, with $DUSTL_{BHE}$ parameters ($\eta = 2.5, \varphi = 1.75$).

Method	Pa.	n = 25		n = 75		n = 150		n = 250	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	η	0.637	1.683	0.162	0.647	0.084	0.426	0.062	0.319
	φ	0.167	0.504	0.048	0.250	0.025	0.172	0.017	0.132
MPS	η	-0.199	1.051	-0.163	0.556	-0.104	0.390	-0.063	0.300
	φ	-0.133	0.434	-0.081	0.244	-0.051	0.171	-0.032	0.132

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Table 2: (contd. . .)

Method	Pa.	n = 25		n = 75		n = 150		n = 250	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
LS	η	0.372	2.213	0.077	0.784	0.046	0.522	0.033	0.391
	φ	0.028	0.545	0.006	0.290	0.005	0.205	0.004	0.157
WLS	η	0.383	1.958	0.096	0.698	0.061	0.462	0.047	0.349
	φ	0.050	0.514	0.019	0.266	0.014	0.185	0.011	0.142
CVM	η	0.967	3.190	0.220	0.871	0.113	0.551	0.073	0.405
	φ	0.195	0.629	0.059	0.305	0.031	0.210	0.019	0.159
ANDA	η	0.366	1.525	0.097	0.667	0.055	0.448	0.042	0.341
	φ	0.071	0.477	0.021	0.258	0.012	0.181	0.009	0.140
RTANDA	η	0.792	3.102	0.169	0.892	0.090	0.572	0.060	0.423
	φ	0.121	0.562	0.030	0.292	0.017	0.204	0.011	0.156
LTANDA	η	0.493	1.583	0.140	0.726	0.076	0.467	0.060	0.359
	φ	0.131	0.542	0.039	0.288	0.022	0.201	0.018	0.155
SOANDA	η	-1.412	1.468	-1.461	1.475	-1.465	1.472	-1.467	1.471
	φ	-0.783	0.831	-0.802	0.817	-0.804	0.811	-0.804	0.809
RTOSANDA	η	1.532	1.533	1.513	1.514	1.499	1.501	1.484	1.486
	φ	-1.688	1.688	-1.689	1.689	-1.690	1.690	-1.690	1.690
LTOSANDA	η	-2.496	2.496	-2.496	2.496	-2.496	2.496	2.496	0.272
	φ	2.231	2.231	2.225	0.225	2.224	0.179	2.224	2.224
PE	η	-0.052	1.612	-0.169	0.894	-0.138	0.646	-0.110	0.508
	φ	-0.135	0.558	-0.108	0.343	-0.075	0.249	-0.056	0.196
MOM	η	0.991	2.516	0.293	1.019	0.156	0.665	0.103	0.500
	φ	0.219	0.633	0.068	0.337	0.037	0.237	0.024	0.184
MSAD	η	0.110	1.703	-0.059	0.720	-0.048	0.525	-0.016	0.397
	φ	-0.054	0.527	-0.046	0.297	-0.031	0.214	-0.015	0.170
MSALD	η	-0.199	1.050	-0.163	0.555	-0.104	0.394	-0.063	0.300
	φ	-0.134	0.434	-0.081	0.244	-0.051	0.172	-0.032	0.132
MSSD	η	-0.006	2.872	-0.171	0.859	-0.137	0.553	-0.092	0.430
	φ	-0.149	0.658	-0.097	0.346	-0.070	0.236	-0.048	0.186
MSSLD	η	0.059	1.233	-0.067	0.607	-0.053	0.434	-0.035	0.331
	φ	-0.039	0.465	-0.048	0.261	-0.033	0.188	-0.023	0.144
MSLD	η	-0.001	2.909	-0.171	0.861	-0.137	0.554	-0.092	0.430
	φ	-0.149	0.660	-0.097	0.347	-0.070	0.237	-0.048	0.186
KS	η	0.493	1.528	0.186	0.791	0.098	0.525	0.075	0.413
	φ	0.199	0.535	0.077	0.302	0.039	0.207	0.028	0.163
BE_{SEL}	η	0.092	0.658	0.099	0.494	0.048	0.339	0.023	0.218
	φ	0.022	0.216	-0.003	0.192	0.033	0.126	0.006	0.103
$BE_{LINEX} (c = -1.5)$	η	-0.300	0.527	-0.022	0.354	-0.023	0.309	-0.007	0.206
	φ	-0.056	0.245	0.013	0.178	0.022	0.125	0.009	0.111
$BE_{LINEX} (c = 1.5)$	η	0.923	1.177	0.501	0.658	0.208	0.392	0.130	0.266

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Table 2: (contd. . .)

Method	Pa.	n = 25		n = 75		n = 150		n = 250	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
$BE_{GEL} (q = -0.5)$	φ	0.139	0.270	0.090	0.658	0.062	0.130	0.033	0.116
	η	-0.044	0.564	0.111	0.392	0.045	0.321	0.036	0.217
$BE_{GEL} (q = 0.5)$	φ	0.005	0.244	0.038	0.181	0.035	0.125	0.017	0.112
	η	0.348	0.759	0.278	0.497	0.121	0.351	0.081	0.237
	φ	0.070	0.253	0.063	0.185	0.049	0.127	0.025	0.114

Table 3. Average bias and RMSE for non-Bayesian and Bayesian estimations at various n, with $DUSTL_{BHE}$ parameters ($\eta = 1.7, \varphi = 2.5$) .

Method	Pa.	n = 25		n = 75		n = 150		n = 250	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	η	0.365	0.980	0.094	0.396	0.050	0.264	0.037	0.198
	φ	0.254	0.766	0.073	0.378	0.037	0.260	0.026	0.200
MPS	η	-0.135	0.643	-0.104	0.347	-0.065	0.247	-0.039	0.188
	φ	-0.197	0.653	-0.120	0.367	-0.074	0.259	-0.048	0.199
LS	η	0.192	1.216	0.040	0.473	0.025	0.319	0.019	0.241
	φ	0.046	0.824	0.010	0.437	0.008	0.308	0.006	0.236
WLS	η	0.204	1.099	0.053	0.425	0.035	0.284	0.028	0.216
	φ	0.079	0.779	0.030	0.400	0.022	0.279	0.017	0.214
CVM	η	0.531	1.687	0.126	0.521	0.066	0.335	0.043	0.249
	φ	0.299	0.955	0.089	0.460	0.047	0.316	0.029	0.239
ANDA	η	0.206	0.890	0.055	0.408	0.032	0.277	0.025	0.212
	φ	0.112	0.723	0.033	0.390	0.019	0.273	0.014	0.211
RTANDA	η	0.432	1.645	0.095	0.534	0.051	0.348	0.035	0.259
	φ	0.183	0.842	0.046	0.435	0.025	0.304	0.017	0.232
LTANDA	η	0.317	1.045	0.081	0.444	0.045	0.289	0.035	0.222
	φ	0.224	0.863	0.062	0.440	0.036	0.307	0.026	0.237
SOANDA	η	-0.904	0.942	-0.933	0.943	-0.936	0.941	-0.937	0.940
	φ	-1.163	1.234	-1.193	1.215	-1.197	1.208	-1.198	1.204
RTOSANDA	η	2.148	2.148	2.140	2.140	2.140	2.140	2.140	2.140
	φ	-2.460	2.460	-2.465	2.465	-2.465	2.465	-2.465	2.465
LTOSANDA	η	0.396	1.284	0.102	0.547	0.057	0.354	0.043	0.272
	φ	0.128	0.500	0.036	0.256	0.021	0.179	0.015	0.138
PE	η	-0.047	1.022	-0.117	0.592	-0.095	0.433	-0.077	0.340
	φ	-0.201	0.858	-0.166	0.531	-0.117	0.386	-0.088	0.303
MOM	η	0.605	1.498	0.186	0.646	0.101	0.429	0.066	0.323
	φ	0.352	0.977	0.111	0.517	0.060	0.363	0.039	0.281
MSAD	η	0.050	1.039	-0.041	0.451	-0.033	0.328	-0.012	0.247

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Table 3: (contd. . .)

Method	Pa.	n = 25		n = 75		n = 150		n = 250	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MSALD	φ	-0.079	0.797	-0.069	0.453	-0.046	0.323	-0.021	0.258
	η	-0.135	0.644	-0.104	0.347	-0.065	0.247	-0.039	0.188
MSSD	φ	-0.197	0.653	-0.120	0.367	-0.075	0.259	-0.048	0.199
	η	-0.059	1.443	-0.113	0.526	-0.087	0.344	-0.058	0.268
MSSLD	φ	-0.225	0.986	-0.144	0.521	-0.104	0.356	-0.071	0.280
	η	0.021	0.741	-0.046	0.378	-0.034	0.271	-0.022	0.208
MSLD	φ	-0.052	0.703	-0.070	0.393	-0.048	0.283	-0.034	0.217
	η	-0.058	1.447	-0.113	0.527	-0.087	0.345	-0.058	0.268
KS	φ	-0.224	0.990	-0.144	0.523	-0.105	0.356	-0.071	0.280
	η	0.315	0.989	0.109	0.494	0.057	0.323	0.043	0.254
BE_{SEL}	φ	0.328	0.844	0.121	0.462	0.062	0.314	0.043	0.246
	η	0.161	0.493	0.097	0.334	0.043	0.222	0.021	0.140
$BE_{LINEX} (c = -1.5)$	φ	0.072	0.364	0.011	0.300	0.059	0.197	0.011	0.157
	η	0.035	0.351	-0.028	0.224	-0.011	0.163	0.026	0.153
$BE_{LINEX} (c = 1.5)$	φ	-0.039	0.343	-0.095	0.261	-0.034	0.189	0.028	0.169
	η	0.782	0.896	0.156	0.282	0.077	0.176	0.081	0.178
$BE_{GEL} (q = -0.5)$	φ	0.487	0.569	0.074	0.259	0.054	0.197	0.084	0.188
	η	0.191	0.430	0.025	0.232	0.017	0.163	0.043	0.160
$BE_{GEL} (q = 0.5)$	φ	0.113	0.363	-0.041	0.249	-0.005	0.190	0.046	0.174
	η	0.423	0.593	0.086	0.250	0.046	0.167	0.062	0.168
	φ	0.287	0.442	0.015	0.246	0.024	0.191	0.065	0.181

Table 4. Average bias and RMSE for non-Bayesian and Bayesian estimations at various n, with $DUSTL_{BHE}$ parameters ($\eta = 1.5, \varphi = 2.0$) .

Method	Pa.	n = 25		n = 75		n = 150		n = 250	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	η	0.304	0.824	0.079	0.338	0.042	0.226	0.031	0.170
	φ	0.208	0.627	0.059	0.309	0.030	0.213	0.022	0.163
MPS	η	-0.119	0.550	-0.090	0.298	-0.056	0.212	-0.033	0.162
	φ	-0.159	0.533	-0.098	0.300	-0.060	0.211	-0.039	0.162
LS	η	0.154	1.006	0.032	0.402	0.020	0.273	0.016	0.206
	φ	0.039	0.674	0.009	0.357	0.007	0.251	0.005	0.192
WLS	η	0.167	0.914	0.044	0.362	0.029	0.243	0.023	0.185
	φ	0.066	0.637	0.025	0.327	0.018	0.228	0.014	0.174
CVM	η	0.437	1.379	0.105	0.442	0.055	0.286	0.036	0.212
	φ	0.245	0.782	0.073	0.375	0.039	0.258	0.024	0.195
ANDA	η	0.170	0.749	0.045	0.348	0.027	0.237	0.021	0.182

Continued on the next page. . .

Table 4: (contd. . .)

Method	Pa.	n = 25		n = 75		n = 150		n = 250	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
RANDA	φ	0.093	0.592	0.027	0.318	0.016	0.223	0.012	0.172
	η	0.359	1.382	0.078	0.451	0.042	0.297	0.029	0.222
LTANDA	φ	0.150	0.686	0.038	0.352	0.021	0.247	0.014	0.189
	η	0.278	0.943	0.068	0.379	0.038	0.248	0.029	0.191
SOANDA	φ	0.193	0.727	0.052	0.362	0.030	0.252	0.022	0.194
	η	-0.782	0.816	-0.806	0.815	-0.808	0.813	-0.808	0.811
RTOSANDA	φ	-0.946	1.002	-0.969	0.986	-0.972	0.980	-0.972	0.977
	η	1.712	1.712	1.712	1.712	1.712	1.712	1.712	1.712
LTOSANDA	φ	-1.972	1.972	-1.972	1.972	-1.972	1.972	-1.972	1.972
	η	-1.498	1.498	-1.498	1.498	-1.498	1.498	-1.498	1.498
PE	φ	1.516	1.516	1.515	1.515	1.513	1.513	1.512	1.512
	η	-0.044	0.886	-0.103	0.520	-0.085	0.382	-0.069	0.300
MOM	φ	-0.163	0.705	-0.137	0.437	-0.097	0.318	-0.072	0.249
	η	0.518	1.273	0.161	0.559	0.088	0.373	0.058	0.282
MSAD	φ	0.294	0.804	0.093	0.424	0.051	0.298	0.033	0.231
	η	0.042	0.886	-0.036	0.388	-0.029	0.282	-0.010	0.213
MSALD	φ	-0.061	0.654	-0.055	0.370	-0.037	0.265	-0.017	0.211
	η	-0.119	0.550	-0.090	0.298	-0.056	0.212	-0.033	0.162
MSSD	φ	-0.159	0.533	-0.098	0.300	-0.061	0.211	-0.039	0.162
	η	-0.061	1.188	-0.099	0.449	-0.075	0.295	-0.050	0.280
MSSLD	φ	-0.1810	0.810	-0.117	0.426	-0.085	0.290	-0.058	0.228
	η	0.014	0.629	-0.040	0.325	-0.030	0.233	-0.019	0.179
MSLD	φ	-0.041	0.575	-0.056	0.320	-0.039	0.231	-0.027	0.177
	η	-0.059	1.191	-0.099	0.450	-0.075	0.296	-0.049	0.230
KS	φ	-0.180	0.813	-0.117	0.427	-0.085	0.290	-0.057	0.228
	η	0.268	0.874	0.090	0.419	0.049	0.282	0.035	0.214
BE_{SEL}	φ	0.273	0.699	0.010	0.377	0.052	0.260	0.034	0.199
	η	0.216	0.467	0.109	0.299	0.048	0.194	0.025	0.122
$BE_{LINEX} (c = -1.5)$	φ	0.134	0.329	0.035	0.253	0.062	0.165	0.017	0.129
	η	0.103	0.299	-0.006	0.227	0.015	0.149	0.032	0.138
$BE_{LINEX} (c = 1.5)$	φ	0.075	0.304	-0.080	0.205	-0.008	0.166	0.036	0.140
	η	0.640	0.691	0.132	0.281	0.084	0.156	0.073	0.154
$BE_{GEL} (q = -0.5)$	φ	0.444	0.523	0.032	0.213	0.053	0.173	0.074	0.153
	η	0.229	0.365	0.035	0.239	0.037	0.151	0.045	0.142
$BE_{GEL} (q = 0.5)$	φ	0.185	0.355	-0.044	0.205	0.012	0.168	0.048	0.143
	η	0.399	0.487	0.081	0.258	0.060	0.153	0.059	0.147
	φ	0.308	0.428	-0.006	0.207	0.032	0.170	0.061	0.148

Tables 1–4 present the average bias and root mean squared error (RMSE) of the non-Bayesian and

Bayesian estimators for the parameters of the DUSTL_{BHE} distribution under four distinct parameter settings, namely $(\eta, \phi) = (2.0, 1.5), (2.5, 1.75), (1.7, 2.5), \text{ and } (1.5, 2.0)$. The simulation study was conducted across sample sizes $n = 25, 75, 150, \text{ and } 250$ to evaluate the finite-sample performance and asymptotic behaviour of the competing estimation methods. A common pattern observed throughout is the progressive reduction in both bias and RMSE as the sample size increases. This behaviour is evident for nearly all estimation procedures and confirms the consistency of the estimators. As larger samples provide more information about the underlying distribution, parameter estimates become increasingly concentrated around their true values, leading to improved estimation accuracy and precision. The decline in RMSE values with increasing sample size further demonstrates the asymptotic efficiency of the estimators and highlights the benefit of larger sample sizes in statistical inference.

Among the non-Bayesian estimators, the MPS estimator consistently exhibited superior performance. Across all parameter configurations, MPS produced relatively small biases and among the lowest RMSE values for both parameters. The performance of MPS was particularly impressive in small samples where likelihood-based procedures often encounter numerical instability. The close agreement between the results of MPS and the minimum spacing-based estimators (MSALD and related variants) is expected since these procedures are founded on similar spacing principles. The ANDA also demonstrated competitive performance, frequently yielding lower RMSE values than the classical MLE, particularly for moderate and large sample sizes. The MLE generally performed satisfactorily and exhibited the expected improvement as sample size increased. However, for small samples, the MLE often produced larger biases and RMSE values than MPS and several Bayesian estimators. This behaviour reflects the sensitivity of likelihood-based estimation to sample variability when limited information is available. Similar observations were made for the LS, WLS, CVM, PE, and MOM estimators, which generally displayed higher RMSE values, especially when $n = 25$. The poorest performance among the non-Bayesian methods was observed for SOANDA, RTOSANDA, and LTOSANDA. These estimators produced extremely large and persistent biases that remained nearly unchanged as sample size increased. Such behaviour indicates substantial systematic bias and suggests that these estimation procedures are unsuitable for estimating the parameters of the DUSTL_{BHE} distribution. The lack of improvement with increasing sample size further implies that these estimators may not possess desirable consistency properties in this setting.

The Bayesian estimators consistently outperformed the non-Bayesian competitors across all parameter configurations. Under the SELF loss function, the Bayes estimator (BES) generally produced the smallest RMSE values while maintaining very low bias. This performance was particularly evident in small samples where Bayesian shrinkage effectively reduces estimation variability. For example, when $n = 25$, the Bayesian estimators typically achieved RMSE values substantially lower than those of MLE, LS, WLS, CVM, and MOM, indicating greater estimation efficiency. The performance of the Bayesian estimators under LINEX and GEL loss functions depended on the choice of asymmetry parameter. The BELINEX estimator with $c = -1.5$ consistently produced some of the smallest RMSE values across all sample sizes and parameter configurations. In many cases, it yielded the best overall performance among all competing estimators, suggesting that the asymmetric loss structure effectively penalizes estimation errors in a manner that aligns well with the characteristics of the DUSTL_{BHE} distribution. Conversely, BELINEX with $c = 1.5$ generally produced larger biases and RMSE values, particularly for small sample sizes, indicating that the direction of asymmetry in the loss function has a substantial influence on estimator performance. A similar pattern was observed for the GEL-based estimators. The BEGEL estimator with $q = -0.5$ frequently outperformed its counterpart with $q = 0.5$, producing smaller biases and lower RMSE values across most

scenarios. This finding suggests that negative values of the asymmetry parameter provide a more appropriate balance between overestimation and underestimation errors for the DUSTL_{BHE} distribution. Nevertheless, both GEL estimators demonstrated clear improvements over most classical estimation procedures, particularly when sample sizes were small or moderate. Comparing the four parameter settings reveals that the relative ranking of the estimation methods remained largely unchanged. Whether the true parameter values were (2.0, 1.5), (2.5, 1.75), (1.7, 2.5), or (1.5, 2.0), the Bayesian estimators under SELF, BELINEX ($c = -1.5$), and BEGEL ($q = -0.5$) consistently achieved the lowest RMSE values. This stability across diverse parameter combinations indicates the robustness of the Bayesian framework and demonstrates that the good performance of these estimators is not dependent on a particular parameter configuration. Overall, the simulation results clearly demonstrate that Bayesian estimation methods provide the most accurate and efficient inference for the DUSTL_{BHE} distribution. Although several non-Bayesian procedures, particularly MPS and ANDA, yielded competitive results, the Bayesian estimators especially those based on SELF, BELINEX ($c = -1.5$), and BEGEL ($q = -0.5$) consistently produced smaller biases and lower RMSE values. The results therefore support the adoption of Bayesian estimation as the preferred inferential approach for the DUSTL_{BHE} distribution, particularly in situations involving small or moderate sample sizes where estimation efficiency is most critical.

The results presented in Tables 5–8 demonstrate the convergence behavior and sampling efficiency of the MCMC algorithm used to obtain Bayesian estimates of the parameters η and φ under SELF, LINEX, and GEL loss functions. The assessment was based on the Gelman–Rubin ($\hat{R}$) and ESS, two widely accepted diagnostics for evaluating convergence and mixing performance of MCMC chains. Across all parameter combinations and sample sizes, the univariate ($\hat{R}$) values for both η and φ , as well as the multivariate ($\hat{R}$) values, were consistently close to unity, ranging between 1.00 and 1.02. Since ($\hat{R}$) values below 1.10 are generally regarded as evidence of convergence, these findings indicate that the multiple MCMC chains mixed adequately and converged successfully to the target posterior distributions.

The absence of large ($\hat{R}$) values suggests that the between-chain variability and within-chain variability were nearly identical, confirming the stability and reliability of the posterior samples obtained. A clear trend is observed in the effective sample sizes as the sample size increases. For all parameter configurations, ESS values increased substantially from $n = 25$ to $n = 250$. Under the SELF loss function, for example, the ESS for η increased from approximately 244 at $n = 25$ to more than 3000 at $n = 250$ for the parameter combination $(\eta, \varphi) = (2.0, 1.5)$. Similar improvements were observed for φ , where ESS increased from about 577 to 3823 over the same sample-size range. This pattern indicates that larger sample sizes provide more information about the model parameters, reducing posterior uncertainty and autocorrelation among successive MCMC draws. Consequently, a larger proportion of the generated samples behave like independent observations, leading to higher ESS values. The parameter combination (1.5, 2.0) exhibited the largest effective sample sizes among all scenarios. Under SELF, ESS values reached approximately 4549 for η and 4570 for φ when $n = 250$. Such large ESS values indicate highly efficient posterior sampling and minimal autocorrelation in the MCMC chains. Similarly, the parameter combination (1.7, 2.5) also produced remarkably high ESS values, particularly for larger sample sizes, suggesting that the posterior distribution associated with these parameter settings is easier for the MCMC algorithm to explore. For the parameter combination (2.5, 1.75), ESS values were generally lower than those obtained for the other configurations, especially at smaller sample sizes. Nevertheless, all reported ESS values remained sufficiently large to ensure reliable posterior inference. The relatively smaller ESS values may be attributed to increased posterior dependence or a more concentrated posterior surface, which can reduce the efficiency of the sampling process. Despite this,

Table 5. MCMC Convergence Diagnostics under SELF Loss Function

(η, ϕ)	n	$\hat{R}_\eta$	$\hat{R}_\phi$	Multivariate $\hat{R}$	ESS(η)	ESS(ϕ)
(2.0, 1.5)	25	1.01	1.01	1.01	244.2	576.9
	75	1.00	1.00	1.00	1535	2339
	150	1.00	1.00	1.00	2087	2914
	250	1.00	1.00	1.00	3021	3823
(2.5, 1.75)	25	1.01	1.01	1.01	200.8	506.1
	75	1.00	1.00	1.00	1068	1679
	150	1.00	1.00	1.00	1446	2163
	250	1.00	1.00	1.00	2420	3383
(1.7, 2.5)	25	1.02	1.01	1.01	384.3	642.4
	75	1.01	1.01	1.01	1699	1660
	150	1.00	1.00	1.00	2711	2775
	250	1.00	1.00	1.00	4130	3962
(1.5, 2.0)	25	1.01	1.01	1.01	414.0	694.6
	75	1.00	1.00	1.00	2079	2152
	150	1.00	1.00	1.00	3294	3301
	250	1.00	1.00	1.00	4549	4570

Table 6. MCMC Convergence Diagnostics under LINEX Loss Function ($c = -1.5$)

(η, ϕ)	n	$\hat{R}_\eta$	$\hat{R}_\phi$	Multivariate $\hat{R}$	ESS(η)	ESS(ϕ)
(2.0, 1.5)	25	1.01	1.01	1.01	244.5	608.3
	75	1.00	1.00	1.00	1006	1414
	150	1.00	1.00	1.00	2411	3314
	250	1.00	1.00	1.00	2919	3804
(2.5, 1.75)	25	1.01	1.01	1.01	200.7	530.1
	75	1.01	1.01	1.01	971	1474
	150	1.00	1.00	1.00	1447	2116
	250	1.00	1.00	1.00	2210	3031
(1.7, 2.5)	25	1.01	1.01	1.01	377.8	642.0
	75	1.00	1.00	1.00	519.3	742.3
	150	1.00	1.00	1.00	2998	3024
	250	1.00	1.00	1.00	4218	3838
(1.5, 2.0)	25	1.00	1.00	1.00	428.5	703.9
	75	1.00	1.00	1.00	1569	1907
	150	1.00	1.00	1.00	3019	3085
	250	1.00	1.00	1.00	4295	4353

Table 7. MCMC Convergence Diagnostics under LINEX Loss Function ($c = 1.5$)

(η, ϕ)	n	$\hat{R}_\eta$	$\hat{R}_\phi$	Multivariate $\hat{R}$	ESS(η)	ESS(ϕ)
(2.0, 1.5)	25	1.01	1.01	1.01	244.5	608.3
	75	1.00	1.00	1.00	1006	1414
	150	1.00	1.00	1.00	2411	3314
	250	1.00	1.00	1.00	2919	3804
(2.5, 1.75)	25	1.01	1.01	1.01	200.7	530.1
	75	1.01	1.01	1.01	971	1474
	150	1.00	1.00	1.00	1447	2116
	250	1.00	1.00	1.00	2210	3031
(1.7, 2.5)	25	1.01	1.01	1.01	377.8	642.0
	75	1.00	1.00	1.00	519.3	742.3
	150	1.00	1.00	1.00	2998	3024
	250	1.00	1.00	1.00	4218	3838
(1.5, 2.0)	25	1.00	1.00	1.00	428.5	703.9
	75	1.00	1.00	1.00	1569	1907
	150	1.00	1.00	1.00	3019	3085
	250	1.00	1.00	1.00	4295	4353

Table 8. MCMC Convergence Diagnostics under GEL Loss Function ($q = -0.5$)

(η, ϕ)	n	$\hat{R}_\eta$	$\hat{R}_\phi$	Multivariate $\hat{R}$	ESS(η)	ESS(ϕ)
(2.0, 1.5)	25	1.00	1.00	1.00	244.5	608.3
	75	1.00	1.00	1.00	1006	1414
	150	1.00	1.00	1.00	2411	3314
	250	1.00	1.00	1.00	2919	3804
(2.5, 1.75)	25	1.00	1.00	1.00	200.7	530.1
	75	1.00	1.00	1.00	971	1474
	150	1.00	1.00	1.00	1447	2116
	250	1.00	1.00	1.00	2210	3031
(1.7, 2.5)	25	1.01	1.01	1.01	377.8	642.0
	75	1.00	1.00	1.00	519.3	742.3
	150	1.00	1.00	1.00	2998	3024
	250	1.00	1.00	1.00	4218	3838
(1.5, 2.0)	25	1.00	1.00	1.00	428.5	703.9
	75	1.00	1.00	1.00	1569	1907
	150	1.00	1.00	1.00	3019	3085
	250	1.00	1.00	1.00	4295	4353

Table 9. MCMC Convergence Diagnostics under GEL Loss Function ($q = 0.5$)

(η, ϕ)	n	$\hat{R}_\eta$	$\hat{R}_\phi$	Multivariate $\hat{R}$	ESS(η)	ESS(ϕ)
(2.0, 1.5)	25	1.00	1.00	1.00	244.5	608.3
	75	1.00	1.00	1.00	1006	1414
	150	1.00	1.00	1.00	2411	3314
	250	1.00	1.00	1.00	2919	3804
(2.5, 1.75)	25	1.00	1.00	1.00	200.7	530.1
	75	1.00	1.00	1.00	971	1474
	150	1.00	1.00	1.00	1447	2116
	250	1.00	1.00	1.00	2210	3031
(1.7, 2.5)	25	1.01	1.01	1.01	377.8	642.0
	75	1.00	1.00	1.00	519.3	742.3
	150	1.00	1.00	1.00	2998	3024
	250	1.00	1.00	1.00	4218	3838
(1.5, 2.0)	25	1.00	1.00	1.00	428.5	703.9
	75	1.00	1.00	1.00	1569	1907
	150	1.00	1.00	1.00	3019	3085
	250	1.00	1.00	1.00	4295	4353

convergence diagnostics remained satisfactory, indicating that the generated posterior samples were still adequate for Bayesian estimation. An important observation is that the convergence diagnostics under the LINEX and GEL loss functions are virtually identical. This result is expected because the $(\hat{R})$ and ESS are properties of the posterior samples generated by the MCMC algorithm and do not depend on the choice of the Bayesian loss function. The loss function only affects the final Bayes estimator computed from the posterior draws. Consequently, the posterior chains used to obtain SELF, LINEX, and GEL estimators possess the same convergence characteristics, leading to nearly identical $(\hat{R})$ and ESS values across the three estimation frameworks. Overall, the results provide strong evidence that the implemented MCMC algorithm converged satisfactorily and generated highly efficient posterior samples. The consistently low $(\hat{R})$ values and the large ESS values observed across all parameter settings confirm the robustness of the Bayesian estimation procedure. Furthermore, the increasing ESS values with larger sample sizes indicate that the efficiency of the posterior sampling process improves as more information becomes available, thereby enhancing the precision and reliability of the Bayesian estimators under SELF, LINEX, and GEL loss functions. These findings support the validity of the subsequent simulation results and inferential conclusions derived from the Bayesian framework.

6. Comparison of estimators via real dataset

To assess the practical applicability of the DUSTL_{BHE} distribution and the competing estimation procedures, two COVID-19 mortality datasets were analysed. The estimation methods were compared using the Kolmogorov–Smirnov statistic (KSV), the corresponding p-value (KSPV), and probability–probability (P–P) plots. These measures evaluate how closely the fitted DUSTL_{BHE} distribution agrees with the empirical distribution of the observed data.

Dataset-1. The daily number of confirmed deaths attributed to COVID-19, analysed by [4]. The observa-

tions are as follows: 1.00, 1.00, 2.00, 4.00, 5.00, 1.00, 1.00, 3.00, 6.00, 6.00, 4.00, 1.00, 5.00, 6.00, 6.00, 8.00, 5.00, 7.00, 7.00, 9.00, 9.00, 15.00, 17.00, 11.00, 13.00, 5.00, 14.00, 5.00, 13.00, 9.00, 19.00, 15.00, 11.00, 14.00, 12.00, 11.00, 7.00, 13.00, 10.00, 20.00, 22.00, 21.00, 12.00, 14.00, 9.00, 14.00, 7.00, 16.00, 17.00, 13.00, 21.00, 11.00, 11.00, 8.00, 11.00, 12.00, 15.00, 21.00, 20.00, 18.00, 15.00, 14.00, 21.00, 16.00, 11.00, 28.00, 29.00, 19.00, 14.00, 19.00, 29.00, 34.00, 34.00, 46.00, 46.00, 47.00, 36.00, 38.00, 40.00, 32.00, 39.00, 34.00, 35.00, 36.00, 35.00, 45.00, 62.00.

Table 10. Estimation methods and KSV test for the COVID-19 dataset-1.

Method	Pa.	Est.	LL	KSV	KSPV
MLE	η	1.608	-328.2	0.070	0.80
	φ	0.027			
MPS	η	1.479	-322.6	0.073	0.74
	φ	0.025			
LS	η	1.786	-328.4	0.076	0.70
	φ	0.029			
WLS	η	1.738	-328.3	0.072	0.76
	φ	0.028			
CVM	η	1.788	-328.4	0.076	0.70
	φ	0.029			
ANDA	η	1.583	-328.2	0.067	0.83
	φ	0.026			
RTANDA	η	1.537	-328.2	0.071	0.78
	φ	0.026			
LTANDA	η	10.00	-431.1	0.356	0.00
	φ	0.100			
SOANDA	η	1.514	-328.3	0.073	0.74
	φ	0.026			
SORTANDA	η	1.577	-328.2	0.069	0.79
	φ	0.026			
SOLTANDA	η	1.507	-328.3	0.072	0.75
	φ	0.026			
PE	η	1.706	-328.3	0.072	0.76
	φ	0.028			
MOM	η	3.332	-338.1	0.134	0.09
	φ	0.036			
MSAD	η	1.225	-359.5	0.402	1.1E-12
	φ	0.010			
MSALD	η	1.061	-394.0	0.534	5.7E-22
	φ	0.005			
MSSD	η	1.054	-331.7	0.115	0.20
	φ	0.019			
MSSLD	η	0.923	-392.3	0.519	9.1E-21

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Table 10: (contd. . .)

Method	Pa.	Est.	LL	KSV	KSPV
MSLD	φ	0.005			
	η	1.049	-331.7	0.114	0.21
KS	φ	0.019			
	η	1.659	-328.2	0.071	0.77
BE_{SEL}	φ	0.028			
	η	1.604	-328.2	0.074	0.73
$BE_{LINEX} (c = -1.5)$	φ	0.027			
	η	1.662	-328.2	0.069	0.80
$BE_{LINEX} (c = 1.5)$	φ	0.027			
	η	1.544	-328.2	0.086	0.54
$BE_{GEL} (q = -0.5)$	φ	0.027			
	η	1.601	-328.2	0.074	0.72
$BE_{GEL} (q = 0.5)$	φ	0.027			
	η	1.604	-328.2	0.074	0.73
	φ	0.027			

Dataset-2. The mortality rates of COVID-19 in Italy, recorded over a 59-day period between 27th February and 27th April 2020, due to [3]. The observations are as follows: 4.571, 7.201, 3.606, 8.479, 11.410, 8.961, 10.919, 10.908, 6.503, 18.474, 11.010, 17.337, 16.561, 13.226, 15.137, 8.697, 15.787, 13.333, 11.822, 14.242, 11.273, 14.330, 16.046, 11.950, 10.282, 11.775, 10.138, 9.037, 12.396, 10.644, 8.646, 8.905, 8.906, 7.407, 7.445, 7.214, 6.194, 4.640, 5.452, 5.073, 4.416, 4.859, 4.408, 4.639, 3.148, 4.040, 4.253, 4.011, 3.564, 3.827, 3.134, 2.780, 2.881, 3.341, 2.686, 2.814, 2.508, 2.450, 1.518.

Table 11. Estimation methods and KSV test for the COVID-19 dataset-2.

Method	Pa.	Est.	LL	KSV	KSPV
MLE	η	3.824	-168.3	0.112	0.41
	φ	0.083			
MPS	η	3.311	-168.5	0.101	0.55
	φ	0.077			
LS	η	2.469	-169.7	0.082	0.79
	φ	0.065			
WLS	η	3.455	-168.4	0.104	0.51
	φ	0.079			
CVM	η	2.468	-169.7	0.082	0.79
	φ	0.065			
ANDA	η	2.860	-168.9	0.091	0.67
	φ	0.071			
RTANDA	η	3.104	-168.7	0.102	0.53

Continued on the next page...

Table 11: (contd. . .)

Method	Pa.	Est.	LL	KSV	KSPV
LTANDA	φ	0.074			
	η	4.727	-169.4	0.167	0.07
SOANDA	φ	0.100			
	η	2.753	-169.1	0.090	0.68
SORTANDA	φ	0.071			
	η	2.885	-168.9	0.091	0.67
SOLTANDA	φ	0.072			
	η	2.723	-169.1	0.090	0.69
PE	φ	0.070			
	η	2.340	-170.1	0.081	0.80
MOM	φ	0.063			
	η	5.039	-168.9	0.135	0.21
MSAD	φ	0.094			
	η	3.386	-168.9	0.136	0.20
MSALD	φ	0.072			
	η	3.443	-168.7	0.132	0.23
MSSD	φ	0.074			
	η	2.572	-169.4	0.103	0.53
MSSLD	φ	0.070			
	η	4.178	-168.4	0.113	0.41
MSLD	φ	0.088			
	η	2.566	-169.4	0.103	0.52
KS	φ	0.070			
	η	2.303	-170.0	0.086	0.74
BE_{SEL}	φ	0.064			
	η	3.330	-168.4	0.103	0.53
$BE_{LINEX} (c = -1.5)$	φ	0.079			
	η	3.812	-168.5	0.136	0.20
$BE_{LINEX} (c = 1.5)$	φ	0.079			
	η	2.846	-168.5	0.145	0.15
$BE_{GEL} (q = -0.5)$	φ	0.079			
	η	3.320	-168.4	0.102	0.53
$BE_{GEL} (q = 0.5)$	φ	0.078			
	η	3.332	-168.4	0.103	0.53
	φ	0.079			

The empirical results presented in Tables 10 and 11 reveal that the $DUSTL_{BHE}$ distribution provides an adequate representation of both datasets regardless of the estimation technique employed, demonstrating the flexibility of the model for analyzing lifetime and mortality data. However, substantial differences were

observed among the competing estimators, indicating that estimator selection plays an important role in determining the quality of the fitted model.

For Dataset 1, the ANDA estimator produced the most satisfactory fit. The estimator attained the smallest Kolmogorov–Smirnov statistic and the largest Kolmogorov–Smirnov p-value among all competing methods, indicating the closest agreement between the theoretical $DUSTL_{BHE}$ distribution and the empirical observations. This result suggests that the distributional characteristics of Dataset 1, particularly its tail behaviour, are effectively captured by the Anderson–Darling estimation criterion. Since the ANDA places greater emphasis on discrepancies in the tails of the distribution, its superior performance indicates that accurate modeling of the extreme observations is crucial for obtaining an adequate fit for this dataset. Notably, the ANDA estimator outperformed not only conventional methods such as MLE and MPS, but also the Bayesian estimators obtained under SELF, LINEX, and GEL loss functions. In contrast, the PE estimator emerged as the best-performing method for Dataset 2. The PE estimator achieved the minimum KSV value together with a relatively large KSPV, indicating that the fitted distribution closely reproduces the empirical quantile structure of the observed data. The superior performance of the percentile estimator suggests that the second dataset is characterized more effectively through quantile matching than through likelihood-based optimization. Consequently, the PE estimator was able to capture the underlying distributional shape more accurately than the competing estimators.

An important implication of these findings is that the optimal estimation method is dataset-dependent. While several estimators produced reasonably good fits, no single method dominated across both datasets. The ANDA was preferable for Dataset 1, whereas the PE was best for Dataset 2. This variability reflects differences in the distributional characteristics of the datasets and highlights the need to evaluate multiple estimation procedures when fitting complex lifetime models. The adequacy of these findings is further reinforced by the P–P plots in Figures 1 and 2, which provide a visual assessment of the model fit. In both figures, the plotted points corresponding to the best-performing estimators lie closely along the 45-degree reference line, indicating strong agreement between the empirical and theoretical cumulative probabilities. Deviations from this line are minimal for the ANDA estimator in Dataset-1 and the PE estimator in Dataset-2, confirming they performed better as observed in the tabulated KSV statistics. Conversely, estimators associated with larger KSV values exhibit noticeable departures from the diagonal line in the P–P plots, reflecting poorer fit and potential model misspecification in the tails or central regions of the distribution. Thus, the graphical evidence aligns well with the numerical results, collectively demonstrating that the $DUSTL_{BHE}$ model provides an excellent fit to both datasets when appropriate estimation techniques, particularly ANDA and PE are utilized.

A noteworthy observation is that the real-data findings differ from the simulation results. The simulation study identified the Bayesian estimators, particularly the Bayes estimator under SELF, BELINEX ($c = -1.5$), and BEGEL ($q = -0.5$), as the most efficient procedures in terms of bias and RMSE. In contrast, the real-data applications favored the Anderson–Darling and Percentile estimators. This difference is expected because the simulation study evaluates parameter recovery when the data are generated exactly from the $DUSTL_{BHE}$ model, whereas the real-data analysis evaluates empirical goodness-of-fit when the true data-generating mechanism is unknown. Consequently, estimators that minimize estimation error under idealized conditions do not necessarily provide the best fit to observed datasets. Overall, the real-data analysis confirms the practical usefulness of the $DUSTL_{BHE}$ distribution for modeling COVID-19 mortality data and demonstrates that estimator performance depends on the characteristics of the dataset under consideration. The results suggest that Anderson–Darling estimation is particularly effective when tail be-

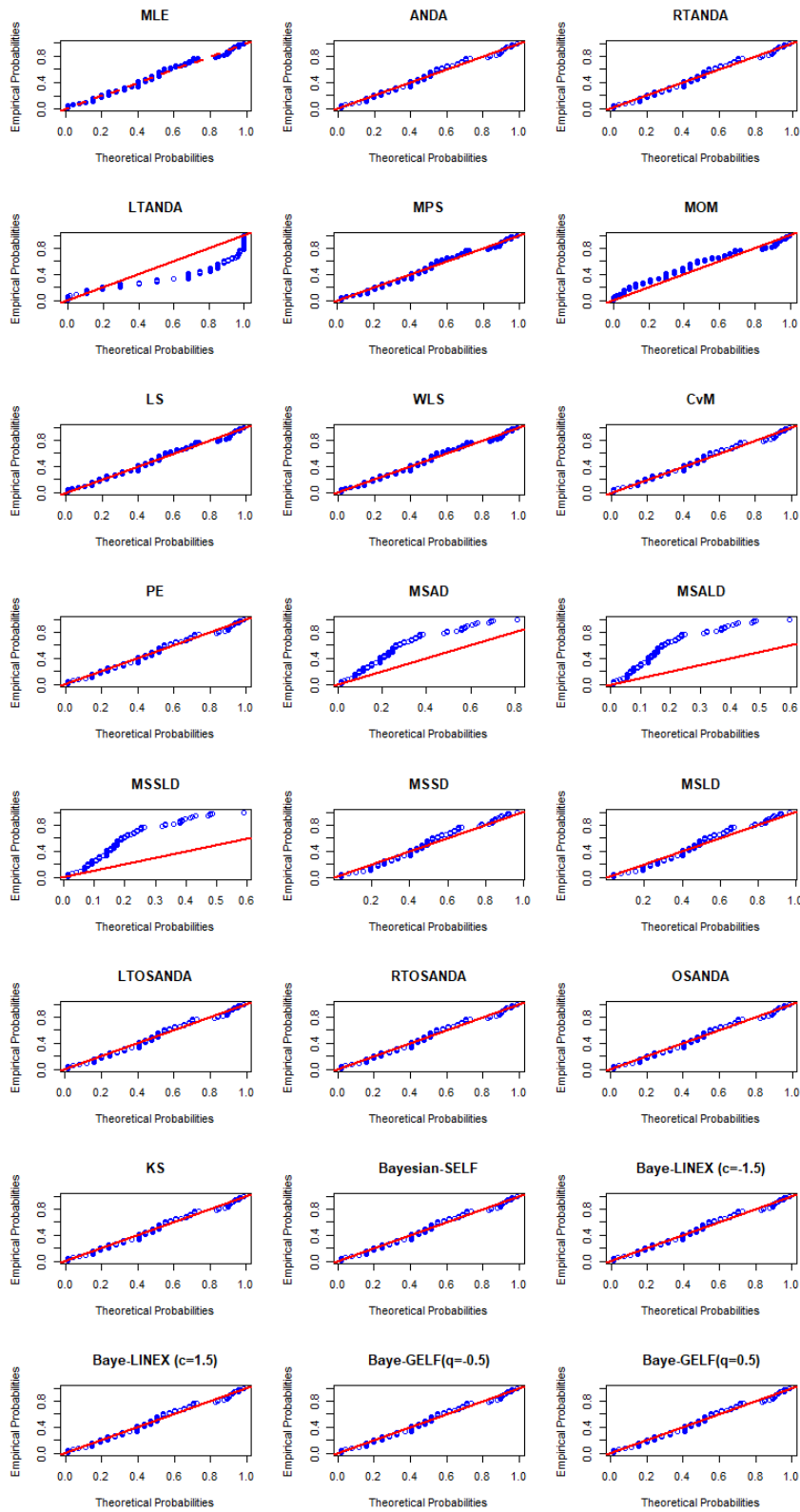


Figure 1. Probability-Probability plots of the various estimators for the dataset-1.

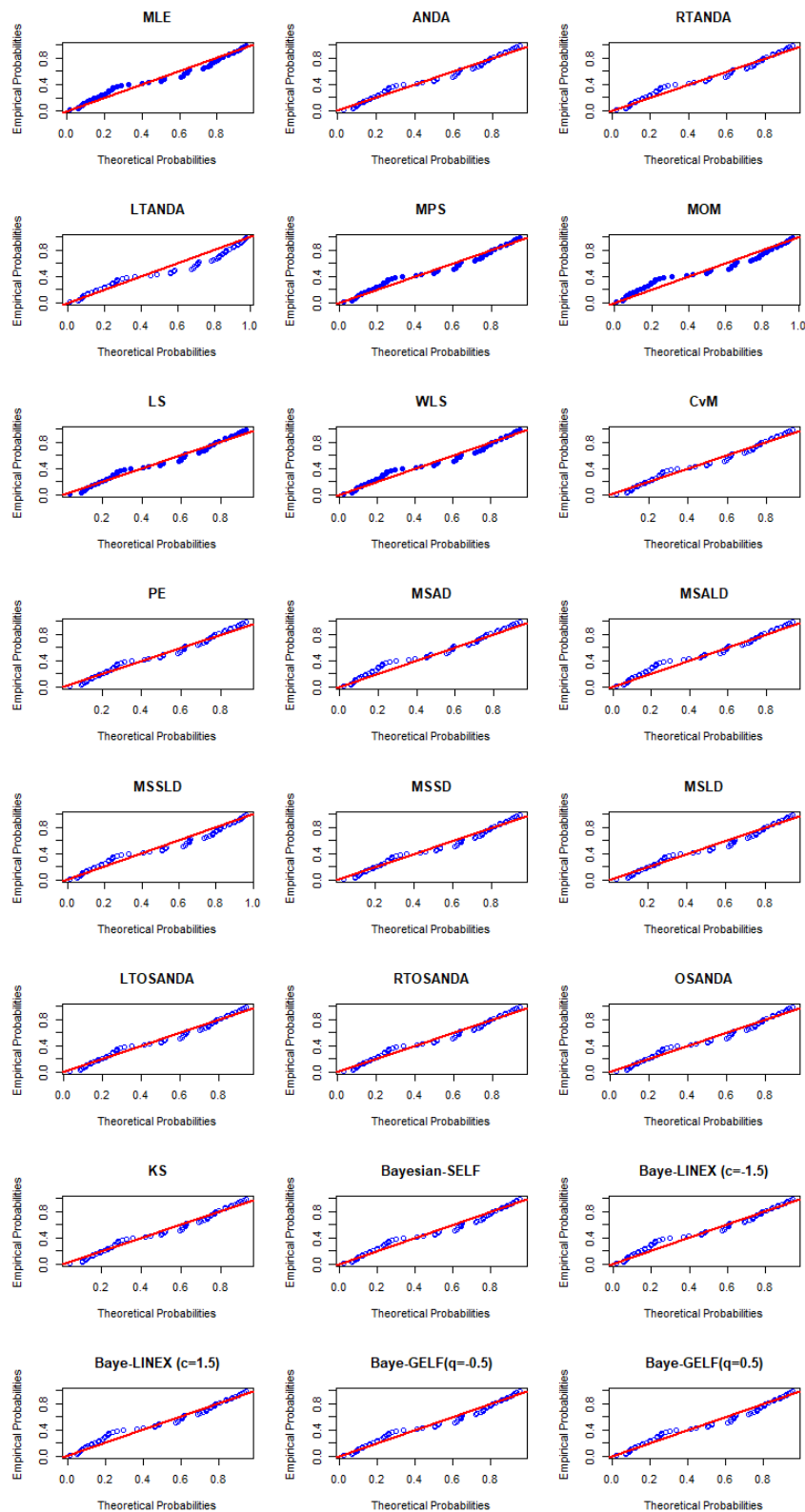


Figure 2. Probability-Probability plots of the various estimators for the dataset-2.

haviour is important, whereas percentile estimation may be preferable when accurate quantile matching is required. Therefore, although Bayesian estimators remain attractive for accurate parameter estimation, the Anderson–Darling and Percentile estimators appear to be the most effective choices when the primary objective is achieving the best empirical fit to real-life datasets.

7. Conclusion

This study presented a comprehensive investigation of parameter estimation methods for the DUSTopp–Leone Burr–Hatke Exponential (DUSTL_{BHE}) distribution. Both non-Bayesian and Bayesian inferential frameworks were considered to evaluate their effectiveness in estimating the model parameters. Nineteen non-Bayesian estimation techniques were developed and compared with Bayesian estimators obtained through Markov chain Monte Carlo (MCMC) sampling under squared error loss (SELF), LINEX, and generalized entropy loss (GEL) functions. In addition, the convergence properties of the Bayesian procedures were examined using Gelman–Rubin diagnostics and effective sample sizes. The Monte Carlo simulation study demonstrated that the performance of all estimators improves as the sample size increases, as evidenced by the general reduction in average bias and root mean squared error (RMSE). Among the non-Bayesian methods, the Maximum Product of Spacings (MPS) and Anderson–Darling (ANDA) estimators consistently exhibited competitive performance, often producing relatively small biases and RMSE values across different parameter configurations. However, the Bayesian estimators generally provided superior or comparable estimation accuracy, particularly in small and moderate sample sizes where the incorporation of prior information contributed to increased stability and efficiency. The Bayesian estimators under SELF, LINEX ($c = -1.5$), and GEL ($q = -0.5$) frequently achieved the lowest RMSE values, indicating their suitability for inference in the DUSTL_{BHE} model. The MCMC convergence diagnostics confirmed the reliability of the Bayesian estimation procedure. Across all parameter settings and sample sizes, Gelman–Rubin potential scale reduction factors were approximately equal to one, while effective sample sizes increased substantially with larger sample sizes. These findings demonstrate excellent mixing, convergence, and sampling efficiency of the posterior chains, thereby validating the resulting Bayesian estimates. To illustrate practical applicability, the proposed estimation methods were applied to two real medical datasets. The results revealed that the DUSTL_{BHE} distribution provides an adequate representation of the observed data and that estimator performance may vary depending on the underlying data characteristics. The Anderson–Darling estimator provided the best fit for one dataset, whereas the percentile estimator performed best for the second dataset according to the Kolmogorov–Smirnov goodness-of-fit criterion and graphical assessments. Overall, the findings establish the DUSTL_{BHE} distribution as a flexible and useful model for analyzing lifetime and reliability data. The study further demonstrates that Bayesian estimation constitutes a robust and efficient inferential framework for the distribution, while MPS and ANDA serve as reliable alternatives within the frequentist paradigm. Future research may extend the DUSTL_{BHE} distribution to regression settings, censored-data frameworks, accelerated life-testing models, and multivariate formulations, thereby broadening its applicability in reliability engineering, survival analysis, and other areas of applied statistics.

Authors' Contributions

Conceptualization, O.D.A; Methodology, O.D.A; Software, O.D.A.; Validation, O.D.A; Formal Analysis, O.D.A.; Investigation, O.D.A; Resources, O.D.A; Data Curation, O.D.A; Writing – Original Draft Preparation, O.D.A; Writing – Review & Editing, O.D.A; Visualization, O.D.A.; Supervision, O.D.A; Project Administration, O.D.A; Funding Acquisition, O.D.A.

Data Availability Statement

The data used can be found within the manuscript.

Conflicts of Interest

The authors declare no conflicts of interest in this study.

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References

1. Abbas, U. F. and Adubisi, O. D. (2026). Novel dustopp-leone burr-hatke exponential model with properties and application to covid-19 datasets. *Modern Journal of Statistics*, 2(2):30–56.
2. Agamy, M. A. (2022). A comparative study of bayesian and non-bayesian estimations for a new generalized inverted weibull distribution. *iKNiTO Journal of Statistics and Finance Cases*, 27(1):72–96.
3. Almongy, H. M., Almetwally, E. M., Aljohani, H. M., Alghamdi, A. S., and Hafez, E. H. (2021). A new extended rayleigh distribution with applications of covid-19 data. *Results in Physics*, 23:104012.
4. Alyami, M. H., Naser, A. Y., Orabi, M. A. A., Alwafi, H., and Alyami, H. S. (2020). Epidemiology of covid-19 in the kingdom of saudi arabia: An ecological study. *Frontiers in Public Health*, 8:506.
5. Brooks, S. P. and Gelman, A. (1998). General methods for monitoring convergence of iterative simulations. *Journal of Computational and Graphical Statistics*, 7(4):434–455.
6. Calabria, R. and Pulcini, G. (1996). Point estimation under asymmetric loss functions for left-truncated exponential samples. *Communications in Statistics – Theory and Methods*, 25(3):585–600.
7. Corless, R. M., Gonnet, G. H., Hare, D. E. G., Jeffrey, D. J., and Knuth, D. E. (1996). On the lambert w function. *Advances in Computational Mathematics*, 5(4):329–359.
8. Doostparast, M., Akbari, M. G., and Balakrishna, N. (2011). Bayesian analysis for the two-parameter pareto distribution based on record values and times. *Journal of Statistical Computation and Simulation*, 81(11):1393–1403.

9. Gelman, A., Carlin, J. B., Stern, H. S., Dunson, D. B., Vehtari, A., and Rubin, D. B. (2013). *Bayesian Data Analysis*. Chapman and Hall/CRC, 3 edition.
10. Lüdtke, O., Ulitzsch, E., and Robitzsch, A. (2021). Comparison of penalized maximum likelihood estimation and markov chain monte carlo techniques for estimating confirmatory factor analysis models with small sample sizes. *Frontiers in Psychology*, 12:615162.
11. McElreath, R. (2020). *Statistical Rethinking: A Bayesian Course with Examples in R and Stan*. Chapman and Hall/CRC, 2 edition.
12. Rangita, A. and Festus, N. (2025). Bias, calibration and stability in logistic regression models: A comparative simulation study of mle, firth and ridge methods. *International Journal of Research and Innovation in Applied Science*, 10:838–852.
13. van Ravenzwaaij, D., Cassey, P., and Brown, S. D. (2018). A simple introduction to markov chain monte-carlo sampling. *Psychonomic Bulletin & Review*, 25(1):143–154.
14. Varian, H. R. (1975). A bayesian approach to real estate assessment. In Fienberg, S. E. and Zellner, A., editors, *Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage*, pages 195–208. North-Holland.
15. Yilmaz, A., Kara, M., and Aydoğdu, H. (2020). A study on comparisons of bayesian and classical parameter estimation methods for the two-parameter weibull distribution. *Communications Faculty of Sciences University of Ankara Series A1: Mathematics and Statistics*, 69(1):576–602.



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