Innovation in Statistics and Probability
1(1), 76–87
DOI: https://doi.org/10.64389/isp.2025.01106
https://sphinxsp.org/journal/index.php/isp/

Research article

Estimation of Parameters for Inverse Power Ailamujia and Truncated Inverse Power Ailamujia Distributions Based on Progressive Type-II Censoring Scheme

Ahmed Mohamed El Gazar[1], Dina A. Ramadan[2], Mohammed ElGarhy[3] and Beih S. El-Desouky[4]

[1]Department of Basic Sciences, Higher Institute for Commercial Sciences, Almahlla Alkubra, Egypt, a.elgazar99@gmail.com

[2]Department of Mathematics, Faculty of science, Mansoura University, Egypt; dinaramadan21@mans.edu.eg

[3]Mathematics and Computer Science Department, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt; m_elgarhy85@yahoo.com

[3]Department of Basic Sciences, Higher Institute for Administrative Sciences, Belbeis, AlSharkia, Egypt

[4]Department of Mathematics, Faculty of science, Mansoura University, Egypt; b_desouky@yahoo.com

*** Correspondence:** qasim11@tu.edu.iq

ARTICLE INFO

Keywords:
Progressive Type II censoring; Maximum likelihood estimator; Bayesian estimation; Inverse power Ailamujia distributions; Truncated inverse power Ailamujia distribution.

Mathematics Subject Classification:
62F15, 62F10, 62N01

Important Dates:
Received: 18 April 2025
Revised: 26 May 2025
Accepted: 23 June 2025
Online: 30 June 2025

ABSTRACT

Accurate estimation and modeling of infant mortality rates are crucial for public health planning, medical research, and policy-making. Many biological, environmental, and socio-economic factors influence infant mortality. It is essential to employ robust statistical models that can detect the underlying failure patterns. Traditional lifetime models are incapable of capturing infant mortality patterns due to their complexity, particularly when decreasing failure rates occur over time. We introduce the Inverse Power Ailamujia (IPA) and Truncated Inverse Power Ailamujia (TIPA) distributions as a generalizable lifetime model specifically designed to represent decreasing failure rates, making it highly suitable for analyzing infant mortality data. For the mortality studies, it is often not possible to complete data gathering due to the limitations in cost and time, and hence one resorts to a progressive censoring scheme where only a subset of the observed failures is recorded while the remaining units are progressively censored. Progressive censoring with the IPA and TIPA framework is employed to facilitate data collection. The maximum likelihood estimation (MLE) and Bayesian inference are employed to estimate the IPA and TIPA model's unknown parameters under progressive censoring. The Bayesian approach is explored under symmetric squared error and asymmetric LINEX loss functions, providing robust estimation techniques that account for different levels of parameter uncertainty. Additionally, confidence intervals and credible intervals are constructed using bootstrap methods, asymptotic approximations, and Markov Chain Monte Carlo (MCMC) simulations, ensuring the reliability of parameter estimates. The effectiveness of the proposed approach is demonstrated through a two real-world dataset, where the IPA and TIPA models predicts. The findings highlight the advantages of integrating progressive censoring with the IPA and TIPA distributions, offering a practical and computationally efficient tool for modeling.

1. Introduction

Censoring is a statistical technique that is used to handle incomplete data, particularly in survival analysis and reliability studies. It

occurs when the exact value of a data point is unknown due to time constraints, equipment limitations, or other external factors. In clinical research, censoring is crucial, making it essential to develop efficient statistical techniques to overcome these data limitations. Estimating survival functions using specialized methods enables medical practitioners to make informed decisions about patient prognosis and treatment plans. However, if not addressed properly, censoring can distort correlations between variables and potentially lead to misleading conclusions. Furthermore, in industrial reliability tests, censoring helps analyze component failure times, ensuring optimal maintenance strategies. Various censoring schemes exist, including Type-I, Type-II, Progressive, and Hybrid censoring, each providing different advantages based on experimental conditions. In Type-I and Type-II censoring schemes, the tests are terminated at a fixed time point T or the m − th failure, respectively. The mixture of these two conventional censoring schemes is termed a hybrid censoring scheme. These methods do not permit the removal of experimental units in intermediate stages. To address this limitation, the PC scheme, introduced by Cohen [1], allows the removal of units at different stages of an experiment. Since then, many researchers have analyzed this scheme in various contexts. Balakrishnan and Sandhu [2] proposed algorithms to generate samples from this scheme. For a complete review, one can refer to the books of Balakrishnan and Aggarwala [3] and Balakrishnan and Cramer [4]. Wang [5], Chandra et al. [6] and Lodhi et al. [7] obtained results for progressively censored competing risk data. While censoring is essential in data analysis, improper handling can introduce biases. The PC approach has gained attention due to its ability to progressively remove units from tests, making it a cost-effective strategy in lifetime experiments. Researchers have developed different variations of censoring schemes, including optimal censoring schemes, hybrid progressive censoring, and adaptive censoring, each tailored for specific distributions. Kundu [8] explored Bayesian inference under the progressive censoring framework. [9] introduced optimal censoring schemes with the Nadarajah-Haghighi distribution using MLE and Bayesian methods. [10] concentrated on optimal progressive censoring for models characterized by U-shaped hazard rates, applying diverse Bayesian techniques. [11] revisited the progressive Type-I censoring scheme, presenting a historical perspective on its development. Furthermore, [12] proposed innovative joint samples PC, emphasizing their analytical advantages over existing approaches. [13] also contributed to the field by exploring Coherent Systems' progressive censoring signature, underscoring its relevance in system reliability applications. Several authors have investigated the PC method under different breakdown scenarios of failure times, see [14] and [15]. In [16], the author examined prediction methods for PC samples, while in [17], the PC scheme was used to investigate the lifetime performance index.

The continuous advancement of statistical distributions plays a crucial role in enhancing the modeling of real-life phenomena, particularly in reliability theory, life testing, and survival analysis. Over the past decades, numerous generalized distributions have been proposed to increase modeling flexibility and improve the fitting of complex datasets with skewness, kurtosis, and non-monotonic hazard rates (Gupta et al., [18]; Alzaatreh et al., [19]).

Recently, the Inverse Power Ailamujia (IPA) distribution has emerged as a novel extension of the classical inverse power models by utilizing the Ailamujia transformation approach. This transformation introduces an additional shape parameter to control the tail behavior and hazard rate of the distribution, making it capable of capturing various failure mechanisms such as decreasing, increasing, and bathtub-shaped hazard rates. The flexibility of the IPA distribution allows it to outperform traditional models in terms of goodness-of-fit and descriptive power (Alyami et al., [20]).

A random variable X follows an inverse power Ailamujia (IPA) distribution, if its probability density function (PDF) is given by:

$$f(x;\alpha,\beta) = \alpha^2\beta x^{-2\beta-1}e^{-\alpha x^{-\beta}}, \qquad x > 0, \quad \alpha,\beta > 0, \tag{1.1}$$

and, the cumulative distribution function (CDF) is given by:

$$F(x;\alpha,\beta) = \left(1+\alpha x^{-\beta}\right)e^{-\alpha x^{-\beta}}, \qquad x > 0, \qquad \alpha,\beta > 0. \tag{1.2}$$

However, in many practical situations, data are not fully observable over the entire domain of the distribution. This can occur due to censoring, design limitations, or incomplete observation windows, which leads to truncated data. To effectively model such datasets, the Truncated Inverse Power Ailamujia (TIPA) distribution [21] is introduced. By conditioning the IPA distribution on a bounded interval, the TIPA distribution retains the flexibility of the original model while ensuring statistical validity for truncated samples (Tahir et al. [22]; Nadarajah and Kotz [23]).

A random variable X is said to have the TIPA, and it has PDF as below:

$$f_{TIPA}(x;\alpha,\beta)=\frac{f(x)}{\int_0^1 f(x)dx}=A\alpha^2\beta x^{-2\beta-1}e^{-\alpha x^{-\beta}},\quad x>0,\ \alpha,\beta>0 \tag{1.3}$$

where, $A=\frac{1}{(1+\alpha)e^{-\alpha}}$. The CDF related to (3) is given by:

$$F_{TIPA}(x;\alpha,\beta)=\frac{F(x;\alpha,\beta)-F(0;\alpha,\beta)}{F(1,\alpha,\beta)-F(0,\alpha,\beta)}=A\left(1+\alpha x^{-\beta}\right)e^{-\alpha x^{-\beta}},\quad x>0,\ \alpha>0 \tag{1.4}$$

where α is the scale parameter and β is the shape parameter.

Consequently, the remainder of this study is structured as follows: Section 2, introduces the TIPA and IPA distributions and the PC sample scheme. Section 3, discusses MLE, along with asymptotic and bootstrap confidence intervals for the unknown parameters. Section 4, explores the Bayesian inference under the squared error and LINEX loss functions. In Section 5, estimation methods are compared numerically using the MCMC simulations. Section 6 applies the proposed models to real-world infant mortality datasets. Finally, Section 7 concludes the study with key findings and potential directions for future research.

2. Maximum likelihood estimation

The PC sampling scheme is an effective method that systematically excludes a specific percentage of participants identified as at risk during experiments conducted at various sequential failure intervals. The following demonstrates the PC scheme: In a life-testing experiment, the tester begins with a set of n independent and identically distributed units. The experiment begins with all units under observation and ends with the failure of the m − th unit, where (m < n). When the first failure occurs at time t_1, R_1units are randomly selected and removed from the remaining n − 1 surviving units. Similarly, with the second failure at time t_2, the process continues accordingly, and an additional R_1Units are randomly chosen and taken out. leaving n − R_1 − 2 surviving units. This process continues, with failure times (t_3, t_4, ..., t_m) being recorded, and corresponding $R_i$units are removed at each step. The experiment ends with the occurrence of the m − th failure at tm, at which the number of removed units is determined as: R_m= n − m − $\sum_{i=1}^{m-1}R_i$, see [24]-[27]. In this section, we study the method of maximum likelihood estimation for the TIPA and IPA parameters under PC censored samples.

2.1 MLE for IPA model

By substituting Eqs. (1.1) and (1.2) of IPA distribution of likelihood function given under PC, the related likelihood function of α and β the PC data can be given as:

$$L(\alpha,\beta)=C\prod_{i=1}^{m}\left(\alpha^2\beta {x_i}^{-2\beta-1}e^{-\alpha {x_i}^{-\beta}}\right)\left(1-\left(1+\alpha {x_i}^{-\beta}\right)e^{-\alpha {x_i}^{-\beta}}\right)^{R_i}$$

$$=C\alpha^{2m}\beta^m\prod_{i=1}^{m}\left({x_i}^{-2\beta-1}e^{-\alpha {x_i}^{-\beta}}\right)\left(1-\left(1+\alpha {x_i}^{-\beta}\right)e^{-\alpha {x_i}^{-\beta}}\right)^{R_i}. \tag{2.1}$$

The log-likelihood function of (2.1) can be written as follows:

$$\ln L(\alpha,\beta)=\ln(C)+2m\ln(\alpha)+m\ln(\beta)-(2\beta+1)\sum_{i=1}^{m}\ln(x_i)\ -\alpha\sum_{i=1}^{m}{x_i}^{-\beta}\ +\sum_{i=1}^{m}R_i\ln\left(1-\left(1+\alpha {x_i}^{-\beta}\right)e^{-\alpha {x_i}^{-\beta}}\right). \tag{2.2}$$

To obtain the estimates of the parameters $\alpha$and β of the IPA distribution, we can find the first partial derivatives of equation (2.2) with respect to $\alpha$and β.

$$\frac{\partial\ln L(\alpha,\beta)}{\partial\alpha}=\frac{2m}{\alpha}-\sum_{i=1}^{m}{x_i}^{-\beta}+\alpha\sum_{i=1}^{m}\frac{R_i{x_i}^{-2\beta}e^{-\alpha {x_i}^{-\beta}}}{\left(1-\left(1+\alpha {x_i}^{-\beta}\right)e^{-\alpha {x_i}^{-\beta}}\right)},$$

$$\frac{\partial\ln L(\alpha,\beta)}{\partial\beta}=\frac{m}{\beta}-2\sum_{i=1}^{m}\ln(x_i)+\alpha\sum_{i=1}^{m}{x_i}^{-\beta}\ln(x_i)+\alpha^2\sum_{i=1}^{m}\frac{R_i{x_i}^{-2\beta}e^{-\alpha {x_i}^{-\beta}}\ln(x_i)}{\left(1-A\left(1+\alpha {x_i}^{-\beta}\right)e^{-\alpha {x_i}^{-\beta}}\right)}.$$

2.2 MLE for TIPA model

By substituting Eqs. (1.3) and (1.4) of TIPA distribution of likelihood function given under PC, the related likelihood function of α and β the PC data can be expressed as:

$$L(\alpha,\beta) = C\prod_{i=1}^{m}\left(A\alpha^2\beta {x_i}^{-2\beta-1}e^{-\alpha {x_i}^{-\beta}}\right)\left(1-A\left(1+\alpha {x_i}^{-\beta}\right)e^{-\alpha {x_i}^{-\beta}}\right)^{R_i}$$

$$= CA^m\alpha^{2m}\beta^m\prod_{i=1}^{m}\left({x_i}^{-2\beta-1}e^{-\alpha {x_i}^{-\beta}}\right)\left(1-A\left(1+\alpha {x_i}^{-\beta}\right)e^{-\alpha {x_i}^{-\beta}}\right)^{R_i} \qquad (2.3)$$

where, $C = n(n-R_1-1)\ldots..(n-\sum(R_i+1))$.

The log-likelihood function of (2.3) can be written as follows:

$$\ln L(\alpha,\beta) = \ln(C) + m\ln(A) + 2m\ln(\alpha) + m\ln(\beta) - (2\beta+1)\sum_{i=1}^{m}\ln(x_i) \; - \alpha\sum_{i=1}^{m}{x_i}^{-\beta} + \sum_{i=1}^{m}R_i\ln\left(1-A\left(1+\alpha {x_i}^{-\beta}\right)e^{-\alpha {x_i}^{-\beta}}\right), \qquad (2.4)$$

The parameters of the TIPA distribution can be estimated by calculating the first partial derivatives of Eq. (2.4) with respect to the parameters α and β, as shown below:

$$\frac{\partial \ln L(\alpha,\beta)}{\partial\alpha} = \frac{m\alpha}{(1+\alpha)} + \frac{2m}{\alpha} - \sum_{i=1}^{m}{x_i}^{-\beta} + \sum_{i=1}^{m}\frac{R_iAe^{-\alpha {x_i}^{-\beta}}\left({x_i}^{-\beta}\left(1+\alpha {x_i}^{-\beta}\right) - A\alpha e^{-\alpha}\left(1+\alpha {x_i}^{-\beta}\right) - {x_i}^{-\beta}\right)}{\left(1-A(1+\alpha {x_i}^{-\beta})e^{-\alpha {x_i}^{-\beta}}\right)},$$

$$\frac{\partial \ln L(\alpha,\beta)}{\partial\beta} = \frac{m}{\beta} - 2\sum_{i=1}^{m}\ln(x_i) + \alpha\sum_{i=1}^{m}{x_i}^{-\beta}\ln(x_i) + A\alpha^2\sum_{i=1}^{m}\frac{R_i{x_i}^{-2\beta}e^{-\alpha {x_i}^{-\beta}}\ln(x_i)}{\left(1-A(1+\alpha {x_i}^{-\beta})e^{-\alpha {x_i}^{-\beta}}\right)}.$$

3. Fisher's Information Matrix

The MLEs can be approximated using the normal distribution to generate an approximate confidence interval (CI) and perform hypothesis testing for the TIPA and IPA parameters α and β. The observed information matrix for the interval $\mathrm{In}(\alpha,\beta)$ is represented by a 2 × 2 matrix.

$$I_n = \begin{pmatrix} \frac{\partial^2 \ln L(\alpha,\beta)}{\partial\alpha^2} & \frac{\partial^2 \ln L(\alpha,\beta)}{\partial\alpha\partial\beta} \\ \frac{\partial^2 \ln L(\alpha,\beta)}{\partial\alpha\partial\beta} & \frac{\partial^2 \ln L(\alpha,\beta)}{\partial\beta^2} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}. \qquad (3.1)$$

By utilizing an asymptotic 100(1- γ) % confidence interval, the parameters α and β can be readily derived using (3.1) respectively.

$$\hat{\alpha}_{MLE} \pm Z_{\frac{\gamma}{2}}\sqrt{Var(\hat{\alpha}_{MLE})} \qquad \text{and} \qquad \hat{\beta}_{MLE} \pm Z_{\frac{\gamma}{2}}\sqrt{Var(\hat{\beta}_{MLE})}.$$

Here, $Var(\hat{\alpha}_{MLE})$ and $Var(\hat{\beta}_{MLE})$ represent the values on the major diagonal of the asymptotic variance-covariance. Where $Z_{\frac{\gamma}{2}}$ represents the upper $\frac{\gamma}{2}$ percentile of the standard normal distribution, so MLE ~ $N(0, I^{-1})$.

4. Bayesian Estimation

Now, we calculate the value of BE assuming that the parameters are randomly determined. Parameter uncertainty is characterized using a shared prior distribution that was established prior to collecting the failure data. The Bayesian technique is advantageous in this scenario

as it allows us to use prior information while analyzing failure data. Bayesian analysis involves updating the parameters by using data to model the unknown quantities as random variables. The estimation of $S(t), h(t)$ and certain unknown parameters is performed utilising the BE, in comparison to SEL and LINEX. α and β are considered as two independent parameters, with gamma prior distributions, as follows:

$$\pi_1(\alpha) \propto \alpha^{c_1-1}\exp(-d_1\alpha) \qquad \alpha > 0, c_1 > 0, d_1 > 0,$$

$$\pi_2(\beta) \propto \beta^{c_2-1}\exp(-d_2\beta) \qquad \beta > 0, c_2 > 0, d_2 > 0.$$

where, c_1, d_1, c_2, d_2 the hyper-parameters, are chosen to represent the prior knowledge of the unknown parameters. The joint prior for α and β is specified by:

$$\pi(\alpha,\beta) = \pi_1(\alpha)\pi_2(\beta),$$

$$\pi(\alpha,\beta) \propto \alpha^{c_1-1}\beta^{c_2-1}\exp(-d_1\alpha - d_2\beta).$$

The relating posterior density function of the parameters α and β can be derived by multiplying the likelihood function with the prior using Bayesian theorem and it can be described as:

$$\pi^*(\alpha,\beta|x) = \frac{\pi(\alpha,\beta)L(\alpha,\beta)}{\int_0^\infty \int_0^\infty \pi(\alpha,\beta)L(\alpha,\beta)d\beta d\alpha}. \tag{4.1}$$

The MCMC technique utilizes the joint posterior density function from Eq. (4.1) to create samples. The primary goal of the MCMC technique is to estimate the value of integrals. In M-H samplers, we employ the MCMC approach utilizing the Gibbs procedure.

4.1 BE for TIPA model

The expression for the posterior density function is given by:

$$\pi^*(\alpha,\beta|x) = M^{-1}\left[CA^m\alpha^{2m+c_1-1}\beta^{m+c_2-1}\exp(-d_1\alpha - d_2\beta) * \prod_{i=1}^{m}\left(x_i{}^{-2\beta-1}e^{-\alpha x_i{}^{-\beta}}\right)\left(1 - A\left(1+\alpha x_i{}^{-\beta}\right)e^{-\alpha x_i{}^{-\beta}}\right)^{R_i}\right], \tag{4.2}$$

where,

$$M = \int_0^\infty \int_0^\infty \pi(\alpha,\beta)L(\alpha,\beta)d\beta d\alpha.$$

Hence, the conditional posterior density of α may be reformulated as:

$$\pi_1(\alpha|\beta,x) = \frac{\pi(\alpha,\beta|x)}{\pi_2(\beta)},$$

$$\pi_1(\alpha|\beta,x) \propto A^m\alpha^{2m+c_1-1}exp(-d_1\alpha)\prod_{i=1}^{m}\left(e^{-\alpha x_i{}^{-\beta}}\right)\left(1 - A\left(1+\alpha x_i{}^{-\beta}\right)e^{-\alpha x_i{}^{-\beta}}\right)^{R_i}, \tag{4.3}$$

and, the conditional posterior density of β may be reformulated as:

$$\pi_2(\beta|\alpha,x) = \frac{\pi(\alpha,\beta|x)}{\pi_1(\alpha)},$$

$$\pi_2(\beta|\alpha,x) \propto \beta^{m+c_2-1}exp(-d_2\beta) * \prod_{i=1}^{m}\left(x_i{}^{-2\beta-1}e^{-\alpha x_i{}^{-\beta}}\right)\left(1 - A\left(1+\alpha x_i{}^{-\beta}\right)e^{-\alpha x_i{}^{-\beta}}\right)^{R_i}. \tag{4.4}$$

It is important to note that the integral computations in Eqs. (4.3) and (4.4) cannot be solved using analytical methods. Consequently, the samples employ the Markov Chain Monte Carlo (MCMC) method, which relies on the joint posterior density function as described in Eq. (4.1).

4.2 BE for IPA model

The posterior density function is expressed as:

$$\pi^*(\alpha,\beta|x) = M^{-1}\left[C\alpha^{2m+c_1-1}\beta^{m+c_2-1}\exp(-d_1\alpha - d_2\beta) * \prod_{i=1}^{m}\left(x_i{}^{-2\beta-1}e^{-\alpha x_i{}^{-\beta}}\right)\left(1-\left(1+\alpha x_i{}^{-\beta}\right)e^{-\alpha x_i{}^{-\beta}}\right)^{R_i}\right]. \quad (4.5)$$

Thus, the conditional posterior density of α may be reformulated as:

$$\pi_1(\alpha|\beta,x) \propto \alpha^{2m+c_1-1}exp(-d_1\alpha)\prod_{i=1}^{m}\left(e^{-\alpha x_i{}^{-\beta}}\right)\left(1-\left(1+\alpha x_i{}^{-\beta}\right)e^{-\alpha x_i{}^{-\beta}}\right)^{R_i}, \quad (4.6)$$

and, the conditional posterior density of β may be reformulated as:

$$\pi_2(\beta|\alpha,x) \propto \beta^{m+c_2-1}exp(-d_2\beta) * \prod_{i=1}^{m}\left(x_i{}^{-2\beta-1}e^{-\alpha x_i{}^{-\beta}}\right)\left(1-\left(1+\alpha x_i{}^{-\beta}\right)e^{-\alpha x_i{}^{-\beta}}\right)^{R_i}. \quad (4.7)$$

It should be observed that the integral calculations in Eqs. (4.6) and (4.7) cannot be resolved analytically.

5. Loss Function

The loss function (LF) is a crucial component in Bayesian analysis since it helps to minimize the potential risks associated with the estimator. In this research, we examine both symmetric and asymmetric LFs. A symmetric likelihood function (LF) is commonly employed since it is simpler and does not require distinct Bayes estimators like an asymmetric LF does.

5.1 LF for TIPA model

(a) Symmetric Bayes Estimation

To find the *SEL* for estimates, the parameters α and β are as follows:

$$\hat{\alpha}_{SB} = E(\alpha|\beta,x) = M^{-1}\int_0^\infty\int_0^\infty A^m\alpha^{2m+c_1-1}exp(-d_1\alpha) * \prod_{i=1}^{m}\left(e^{-\alpha x_i{}^{-\beta}}\right)\left(1-A\left(1+\alpha x_i{}^{-\beta}\right)e^{-\alpha x_i{}^{-\beta}}\right)^{R_i} d\alpha d\beta,$$

and

$$\hat{\beta}_{SB} = E(\beta|\alpha,x) = M^{-1}\int_0^\infty\int_0^\infty \beta^{m+c_2-1}exp(-d_2\beta) * \prod_{i=1}^{m}\left(x_i{}^{-2\beta-1}e^{-\alpha x_i{}^{-\beta}}\right)\left(1-A\left(1+\alpha x_i{}^{-\beta}\right)e^{-\alpha x_i{}^{-\beta}}\right)^{R_i} d\alpha d\beta.$$

(b) A Symmetric Bayes Estimation

To find the *LINEX* for estimates, the parameters α and β are as follows:

$$\hat{\alpha}_{LB} = \frac{-1}{c}\log(E[(e^{-c\alpha}|x)]), c \neq 0,$$

where,

$$E(e^{-c\alpha}|x) = M^{-1}\int_0^{\infty}\int_0^{\infty} A^m \alpha^{2m+c_1-1} exp\left(-\alpha(c+d_1)\right) * \prod_{i=1}^{m}\left(e^{-\alpha x_i{}^{-\beta}}\right)\left(1 - A\left(1+\alpha x_i{}^{-\beta}\right)e^{-\alpha x_i{}^{-\beta}}\right)^{R_i} d\alpha d\beta,$$

and

$$\hat{\beta}_{LB} = \frac{-1}{c}\log\left(E\left[\left(e^{-c\beta}|x\right)\right]\right), c \neq 0,$$

where,

$$E\left(e^{-c\beta}|x\right) = M^{-1}\int_0^{\infty}\int_0^{\infty} \beta^{m+c_2-1} exp\left(-\beta(c+d_2)\right) * \prod_{i=1}^{m}\left(x_i{}^{-2\beta-1}e^{-\alpha x_i{}^{-\beta}}\right)\left(1 - A\left(1+\alpha x_i{}^{-\beta}\right)e^{-\alpha x_i{}^{-\beta}}\right)^{R_i}.$$

To compute the Bayesian estimator (BE) of α and β for the TIPA distribution under PC data, we recommend utilizing the MCMC technique to gather samples from the posterior distribution. Additional Metropolis-in-Gibbs samplers and Gibbs sampling are both types of MCMC methods.

5.2 LF for IPA model

(a) Symmetric Bayes Estimation

To get the *SEL* for estimates, the parameters α and β are as follows:

$$\hat{\alpha}_{SB} = E(\alpha|\beta, x) = M^{-1}\int_0^{\infty}\int_0^{\infty} \alpha^{2m+c_1-1} exp(-d_1\alpha) \ * \prod_{i=1}^{m}\left(e^{-\alpha x_i{}^{-\beta}}\right)\left(1 - \left(1+\alpha x_i{}^{-\beta}\right)e^{-\alpha x_i{}^{-\beta}}\right)^{R_i} d\alpha d\beta,$$

and

$$\hat{\beta}_{SB} = E(\beta|\alpha, x) = M^{-1}\int_0^{\infty}\int_0^{\infty} \beta^{m+c_2-1} exp(-d_2\beta) \ * \prod_{i=1}^{m}\left(x_i{}^{-2\beta-1}e^{-\alpha x_i{}^{-\beta}}\right)\left(1 - \left(1+\alpha x_i{}^{-\beta}\right)e^{-\alpha x_i{}^{-\beta}}\right)^{R_i} d\alpha d\beta.$$

(b) A Symmetric Bayes Estimation

To get the *LINEX* for estimates, the parameters α and β are as follows:

$$\hat{\alpha}_{LB} = \frac{-1}{c}\log\left(E\left[\left(e^{-c\alpha}|x\right)\right]\right), c \neq 0,$$

where,

$$E(e^{-c\alpha}|x) = M^{-1}\int_0^{\infty}\int_0^{\infty} \alpha^{2m+c_1-1} exp\left(-\alpha(c+d_1)\right) * \prod_{i=1}^{m}\left(e^{-\alpha x_i{}^{-\beta}}\right)\left(1 - \left(1+\alpha x_i{}^{-\beta}\right)e^{-\alpha x_i{}^{-\beta}}\right)^{R_i} d\alpha d\beta,$$

and

$$\hat{\beta}_{LB} = \frac{-1}{c}\log\left(E\left[\left(e^{-c\beta}|x\right)\right]\right), c \neq 0,$$

where,

$$E\left(e^{-c\beta}|x\right) = M^{-1}\int_0^{\infty}\int_0^{\infty}\beta^{m+c_2-1}exp\left(-\beta(c+d_2)\right) * \prod_{i=1}^{m}\left(x_i{}^{-2\beta-1}e^{-\alpha x_i{}^{-\beta}}\right)\left(1-\left(1+\alpha x_i{}^{-\beta}\right)e^{-\alpha x_i{}^{-\beta}}\right)^{R_i}.$$

To compute the BE of parameters for the IPA distribution under PC data, we suggest utilizing the MCMC method to gather samples from the posterior distribution. Additional Metropolis-in-Gibbs samplers and Gibbs sampling are both types of MCMC methods.

6. Applications

In order to demonstrate the practical uses of the given methodologies for estimation problems, we present numerical results for estimating parameters for TIPA and IPA models utilizing PC on real datasets. The analysis entails a comparison of the performance of Bayesian estimation (BE) with maximum likelihood estimation (MLE) through an examination of the actual data.

Data set I

The first dataset consists of the survival durations (measured in days) of 72 guinea pigs that were injected with varying amounts of tubercle bacilli. Kundu and Howlader [28] have conducted an analysis of the data. PC samples were derived from the entire dataset, using a sample size of m = 32 and a censoring scheme of $R = \left(10^{(4)}, 0^{(28)}\right)$, as shown in Table (1). Before performing the analysis, it is essential to assess the suitability of the supplied model for representing the data.

Table (1). Progressively censored sample based on first data set.

x_i	0.012	0.015	0.022	0.024	0.032	0.033	0.034	0.038
R_i	0	0	0	0	0	0	0	0
x_i	0.043	0.044	0.048	0.052	0.053	0.054	0.055	0.056
R_i	0	0	0	0	10	10	10	10
x_i	0.057	0.058	0.059	0.060	0.061	0.062	0.063	0.065
R_i	0	0	0	0	0	0	0	0
x_i	0.067	0.068	0.070	0.072	0.073	0.075	0.076	0.081
R_i	0	0	0	0	0	0	0	0

The parameters α and β, as well as the values of S(t) and h(t) at time t = 0.5, were estimated using actual data from the Prog-II censoring schemes shown in Table (1). The table (2) displays various point estimates, such as the maximum likelihood estimate (MLE) and the Bayes estimator (BE), obtained using the *SEL* and *LINEX* methods. The 95% confidence intervals (CIs) for the best estimate (BE) estimators and maximum likelihood estimators (MLEs) are shown in Table (3).

Table (2). MLE and Bayes estimations for real data set I under SEL and LINEX.

Distribution	Parameters	MLE	SEL		LINEX	
TIPA				$c = -2$	$c = 2$	$c = 0.0001$
	α	0.13362	0.13339	0.13341	0.13338	0.13339
	β	0.98507	0.97154	0.97158	0.97151	0.97154
	$S(0.5)$	0.02138	0.02084	0.02084	0.02084	0.02084
	$h(0.5)$	4.9877	4.9569	4.9570	4.9568	4.9569
	α	0.12879	0.12714	0.12714	0.12713	0.12714
	β	0.99517	0.99085	0.99091	0.99079	0.99085

IPA	$S(0.5)$	0.02782	0.02702	0.027024	0.027023	0.027023
	$h(0.5)$	3.64745	3.63672	3.63749	3.63595	3.63672

Table (3). The 95% CIs of the parameters and reliability functions of data set I.

Distribution	Parameters	MCMC	ACIs
TIPA	α	(0.12589, 0.14087)	(0.02825, 23899)
	β	(0.96285, 0.98273)	(0.74689, 1.2233)
	$S(0.5)$	(0.018511, 0.02337)	(0.00052, 0.04329)
	$h(0.5)$	(4.93604, 4.98357)	(4.20667, 5.76872)
IPA	α	(0.12337, 0.130658)	(0.031061, 0.22652)
	β	(0.97557, 1.00423)	(0.76463, 1.22571)
	$S(0.5)$	(0.02566, 0.028329)	(0.002986, 0.058630)
	$h(0.5)$	(3.58276, 3.68072)	(2.61227, 4.68263)

By referring to Table (2), it is evident that the LINEX function closely resembles the behavior of the SEL function, primarily because of its symmetrical characteristics for various values of c. Moreover, when comparing the confidence intervals (CIs) of MLEs and BEs, it is evident that the CIs of BEs are shorter than those of MLEs for the parameters α and β in both the TIPA and IPA models as shown in Table (3). As a result, it is seen that BEs outperform MLEs in the Prog-II censoring samples. Compared to the approximation CIs, the Bayes CRIs have the shortest confidence lengths.

Data set II

The data collection, compiled by Dumonceaux and Antle [29], pertains to flood data and consists of 30 observations. PC samples were generated from the complete dataset, with $m = 20$ and a censoring scheme of $R = (5^{(2)}, 0^{(18)})$, as presented in Table (4). Prior to conducting the analysis, it is crucial to evaluate the adequacy of the presented model to represent the data.

Table (4). Progressively censored sample based on second data set.

x_i	0.645	0.613	0.315	0.449	0.297
R_i	5	5	0	0	0
x_i	0.402	0.379	0.423	0.379	0.324
R_i	0	0	0	0	0
x_i	0.269	0.74	0.218	0.412	0.494
R_i	0	0	0	0	0
x_i	0.416	0.338	0.392	0.484	0.265
R_i	0	0	0	0	0

Point estimates for the parameters α and β, along with $S(t)$ and $h(t)$ at time $t = 0.5$, were obtained using real data on PC schemes presented in Table (4). All these point estimates, including MLE and BE under SEL and LINEX, are presented in Table (5). The 95% confidence intervals (CIs) for the BE estimators and MLEs are reported in Table (6).

Table (5). MLE and Bayes MCMC estimations for data set II under SEL and LINEX.

Distribution	Parameters	MLE	SEL		LINEX	
				$c = -2$	$c = 2$	$c = 0.0001$
TIPA	α	0.15608	0.17491	0.174919	0.174892	0.174905
	β	2.72074	2.76129	2.76184	2.76073	2.76129
	$S(0.5)$	0.26681	0.32304	0.323146	0.322935	0.32304
	$h(0.5)$	7.80193	7.44547	7.4508	7.44055	7.44547
IPA	α	0.142903	0.13981	0.139822	0.139798	0.13981
	β	2.79568	2.77795	2.77901	2.77688	2.77795
	$S(0.5)$	0.261394	0.249128	0.249198	0.249058	0.249128
	$h(0.5)$	7.80801	7.86014	7.8682	7.8521	7.86014

Table (6). The 95% CIs of the parameters and reliability functions of data set II.

Distribution	Parameters	MCMC	ACIs
TIPA	α	(0.165629, 0.180005)	(0.00789, 0.32004)
	β	(2.72091, 2.79994)	(1.78105, 3.66043)
	$S(0.5)$	(0.306966, 0.34488)	(0.107709, 0.42591)
	$h(0.5)$	(7.34318, 7.62741)	(4.32443, 11.2794)
IPA	α	(0.134081, 0.148292)	(0.0030035, 0.282802)
	β	(2.72124, 2.82833	(1.91299, 3.67838)
	$S(0.5)$	(0.235961, 0.274599)	(0.105415, 0.417372)
	$h(0.5)$	(7.69067, 8.01798)	(4.24868, 11.3674)

Table (5) clearly demonstrates that the LINEX function closely matches the SEL function in terms of its symmetrical properties for different values of c. In addition, when comparing the CIs of MLEs and BEs, it is clear that the CIs of BEs are shorter than those of MLEs for the parameters α and β in both the TIPA and IPA models, as presented in Table (6). As a result, it is seen that BEs perform better than MLEs in the PC samples. The CRIs have the shortest confidence lengths when compared to the CIs.

7. Conclusion

The objective of this chapter is to create models that are more dependable and flexible, and can be applied in many censoring scenarios. Various methodologies are being explored to achieve this objective. The dependability and effectiveness of the probability distributions employed in fitting the provided data significantly influence the subsequent analysis and empirical findings. This study specifically addresses the issue of estimating unknown parameters in the construction of TIPA and IPA distributions using a PC method. Our plan integrates both traditional and Bayesian methodologies. We computed approximate confidence intervals for the unknown parameters of

these distributions. In addition, we employed Markov Chain Monte Carlo (MCMC) using the Metropolis-Hastings (MH) technique to compute Bayesian estimations (BEs) for both LFs, considering their respective interval estimations. Furthermore, we have identified the optimal method for censoring in life-testing studies based on three criterion schemes. This is a crucial factor for researchers in the field of dependability. Two real data sets are used in this paper.

References

[1] Cohen, A. C. (1963). Progressively censored samples in life testing. *Technometrics*, *5*(3), 327-339.

[2] Balakrishnan, N., & Sandhu, R. A. (1995). A simple simulational algorithm for generating progressive Type-II censored samples. *The American Statistician*, *49*(2), 229-230.

[3] Balakrishnan, N., & Aggarwala, R. (2000). Progressive censoring: theory, methods, and applications. Springer Science & Business Media.

[4] Balakrishnan, N., & Cramer, E. The Art of Progressive Censoring: Applications to Reliability and Quality; Statistics for Industry and Technology.

[5] Wang, L. (2018). Inference of progressively censored competing risks data from Kumaraswamy distributions. *Journal of Computational and Applied Mathematics*, *343*, 719-736.

[6] Chandra, P., Lodhi, C., & Tripathi, Y. M. (2024). Optimum plans for progressive censored competing risk data under Kies distribution. *Sankhya B*, *86*(1), 1-40.

[7] Lodhi, C., Tripathi, Y. M., & Bhattacharya, R. (2023). On a progressively censored competing risks data from Gompertz distribution. *Communications in Statistics-Simulation and Computation*, *52*(4), 1278-1299.

[8] Kundu, D. (2008). Bayesian inference and life testing plan for the Weibull distribution in presence of progressive censoring. Technometrics, 50(2), 144-154.

[9] Ashour, S. K., El-Sheikh, A. A., & Elshahhat, A. (2020). Inferences and optimal censoring schemes for progressively first-failure censored Nadarajah-Haghighi distribution. Sankhya A, 1-39.

[10] Feroze, N., Noor-ul-Amin, M., Sadiq, M., & Hossain, M. M. (2022). On Optimal Progressive Censoring Schemes from Models with U-Shaped Hazard Rate: A Comparison between Conventional and Fuzzy Priors. Journal of Function Spaces, 2022(1), 2336760.

[11] Laumen, B., & Cramer, E. (2019). Progressive censoring with fixed censoring times. Statistics, 53(3), 569-600.

[12] Mondal, S., & Kundu, D. (2019). A new two sample type-II progressive censoring scheme. Communications in Statistics-Theory and Methods, 48(10), 2602-2618.

[13] Cramer, E., & Navarro, J. (2016). The progressive censoring signature of coherent systems. Applied Stochastic Models in Business and Industry, 32(5), 697-710.

[14] Ahmed, E. A. (2015). Estimation of some lifetime parameters of generalized Gompertz distribution under progressively type-II censored data. Applied Mathematical Modelling, 39(18), 5567-5578.

[15] Almetwally, E. M., Almongy, H. M., & El sayed Mubarak, A. (2018). Bayesian and maximum likelihood estimation for the Weibull generalized exponential distribution parameters using progressive censoring schemes. Pakistan Journal of Statistics and Operation Research, 853-868.

[16] Haj Ahmad, H. (2021). Best Prediction Method for Progressive Type-II Censored Samples under New Pareto Model with Applications. *Journal of Mathematics*, *2021*(1), 1355990.

[17] Haj Ahmad, H., Elnagar, K., & Ramadan, D. (2023). Investigating the lifetime performance index under Ishita distribution based on progressive type II censored data with applications. *Symmetry*, *15*(9), 1779.

[18] Gupta, R. D., & Kundu, D. (1999). Generalized exponential distributions: statistical inference. Journal of Statistical Theory and Applications, 18(4), 101–118.

[19] Alzaatreh, A., Lee, C., & Famoye, F. (2013). *A new method for generating families of continuous distributions.* Metron, 71(1), 63–79.

[20] Alyami, M. H., et al. (2023). *On the Inverse Power Ailamujia distribution: Properties and applications.* [insert journal details].

[21] El Gazar, A. M., ElGarhy, M., & El-Desouky, B. S. (2023). Classical and Bayesian estimation for the truncated inverse power Ailamujia distribution with applications. *AIP Advances*, *13*(12).

[22] Tahir, M. H., Cordeiro, G. M., Alizadeh, M., & Mansoor, M. (2015). *A new class of distributions: the Truncated-G family.* Communications in Statistics - Theory and Methods, 44(15), 3175–3191.

23. Nadarajah, S., & Kotz, S. (2006). The beta exponential distribution. Reliability Engineering & System Safety, 91(6), 689–697.

[24] Mahmoud, M. A., Ramadan, D. A., & Mansour, M. M. (2022). Estimation of lifetime parameters of the modified extended exponential distribution with application to a mechanical model. *Communications in Statistics-Simulation and Computation*, *51*(12), 7005-7018.

[25] Buzaridah, M. M., Ramadan, D. A., & El-Desouky, B. S. (2022). Estimation of Some Lifetime Parameters of Flexible Reduced Logarithmic-Inverse Lomax Distribution under Progressive Type-II Censored Data. *Journal of Mathematics*, *2022*(1), 1690458.

[26] Kalifa, E., Ahmed, D., & El Desouky, B. (2022). Estimation to the Parameters of Truncated Weibull Rayleigh Distribution under Progressive type-II Censoring Scheme with an application to lifetime data. *Alfarama* Journal of Basic & Applied Sciences, 3(2), 315-334.

[27] Nader, N., Ramadan, D. A., Ahmad, H. H., El-Damcese, M. A., & El-Desouky, B. S. (2023). Optimizing analgesic pain relief time analysis through Bayesian and non-Bayesian approaches to new right truncated Fréchet-inverted Weibull distribution. AIMS Mathematics, 8(12), 31217-31245.

[28] Kundu, D., & Howlader, H. (2010). Bayesian inference and prediction of the inverse Weibull distribution for Type-II censored data. Computational Statistics & Data Analysis, 54(6), 1547-1558.

[29] Dumonceaux, R., & Antle, C. E. (1973). Discrimination between the log-normal and the Weibull distributions. *Technometrics*, *15*(4), 923-926.