
Research article

Probability density functions of quotient-product of two generalized order statistics from the generalized exponential distribution

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ABSTRACT

A major area of statistical research, especially in fields like dependability analysis and stochastic modeling, is the investigation of the Quotient-Product (Q&P) of generalized order statistics. These functions are crucial for determining the properties of different statistical distributions and for comprehending the dynamics of complex systems. The generalized exponential distribution (GED) is one of them that is particularly adaptable and can handle a variety of data kinds. The purpose of this work is to obtain explicit equations for the Quotient-Product of two generalized order statistics and its probability density function. We provide a thorough framework for these derivations by using the Mellin transform and its inverse. The findings advance theoretical understanding of generalized order statistics and their relationships by offering insightful examples and useful insights, like Quotient-Product of extreme generalized order statistics and consecutive generalized order statistics, which are pertinent to applications in engineering reliability, risk assessment, and survival analysis. The consistency of the derived results with existing literature is confirmed through several corollaries that recover well-known special cases. The paper concludes with a discussion of the theoretical and practical limitations of the current work and outlines promising directions for future research, including extensions to dependent GOS models, other lifetime distributions, and Bayesian estimation frameworks.

1. Introduction

The central research question motivating this work is: Can explicit, closed-form probability density functions be derived for the Quotient-Product of two Generalized Order Statistics from the Generalized Exponential Distribution, and can the Fox H-function provide a unified representation of such distributions? This question is significant for several reasons. First, despite the growing body of literature on Q&P distributions for various families (e.g., Weibull by [1], Kumaraswamy by [2]), no results exist for GOS from the GED, which is a widely used alternative to the gamma and Weibull distributions for modeling lifetime data [3]. Second, the GOS framework unifies many practically important models, including ordinary order statistics, record values, and progressively censored data, making any result derived within it broadly applicable. Third, in stress-strength reliability models, the probability that a component survives (i.e., that its strength exceeds the applied stress) is directly linked to the distribution of the quotient of order statistics; deriving such PDFs explicitly thus has direct engineering relevance. The present work fills this gap by establishing the theoretical machinery for these derivations using the Mellin transform approach pioneered by [4] and extended by [5].

The PDF and CDF of the two-parameter generalized exp. distribution (GED) are expressed as follows:

$$f(x) = \lambda \theta e^{-\lambda x} (1 - e^{-\lambda x})^{\theta-1}, \quad (1.1)$$

and

$$F(x) = (1 - e^{-\lambda x})^\theta, \quad (1.2)$$

where, $\theta, \lambda > 0$ are shape & scale parameters respectively.

When $\theta = 1$, GED simplifies to the standard exp. distribution. This is highly effective in scenarios requiring a skewed distribution. Its failure rates can either increase or decrease, contingent upon the shape parameter. [3] demonstrated that the GED is particularly useful for analyzing various lifetime data, serving as a viable alternative to the 2-parameters gamma and WDs. Consequently, this distribution is applicable in contexts where the progression of a disease results in mortality peaking after a certain period, followed by a gradual decline, as observed in cases of curable breast cancer.

The Quotient-Product (Q&P) distributions of two random variables play a critical role in statistical analysis, with broad applications in various fields. For instance, products of random variables are integral to modeling investments across international markets, while quotients of order statistics are essential in reliability analysis, particularly in stress-strength. This model evaluates the lifespan of component exposed to random stress Y and random strength X , with failure occurring when the stress exceeds the strength.

Numerous studies on Q&P distributions have been carried out over time. [6], and [7] have all made further progress. These studies find applications in ranking and selection rules, as discussed by [8] and [9]. Such distributions also naturally arise in scenarios involving ratios, such as cost per unit weight or fuel efficiency per mile.

A number of distributions have been thoroughly investigated. [10] examined the Q&P of order statistics (OS) that were obtained from exponential and uniform distributions. This analysis was expanded by [11] to include Pareto, power, and Weibull distributions. Weibull distribution generalized order statistics (GOS) were studied by [1], and Kumaraswamy distribution GOS were studied by [2]. PDF for the Q&P of two GOS are derived from the GED in this study. In addition to providing useful applications in dependability analysis and statistical modeling, these discoveries add to the corpus of current knowledge.

Record values and ordinary order statistics (OOS) are all included in the framework provided by GOS. [12] was the first to propose this idea in 1995. The following is a summary of the framework:

The random variables (r.v.) $X(1, n, m, r), \dots, X(n, n, m, r)$ be GOS from an absolutely continuous CDF $F(x)$ and (pdf) $f(x)$, with noting that $(X(0, n, m, r) = 0, r \geq 1, m \in R^{n-1})$, then their joint PDF can be written in the form:

$$f(x_1, x_2, \dots, x_n) = r \prod_{i=1}^{n-1} \gamma_i f(x_i) \cdot [1 - F(x_i)]^m [1 - F(x_n)]^{r-1} f(x_n), \quad (1.3)$$

on the cone, $F^{-1}(0) < x_1 < \dots < x_n < F^{-1}(1)$ of R^n , where $\gamma_i = r + (n - i)(m + 1)$ and $\gamma_n = r > 0$. If $m = 0$ and $k = 1$, then (1.3) is the joint PDF of the GOS $X(i, n, m, r)$ and $X(j, n, m, r)$, $i < j$ is given by

$$f_{i,j}(x, y) = \frac{c_{j-1}}{(i-1)!(j-i-1)!} (1 - F(x))^m f(x) g_m^{i-1}(F(x)) \cdot [h_m(F(y)) - h_m(F(x))]^{j-i-1} [1 - F(y)]^{\gamma_j-1} f(y), \quad (1.4)$$

for $x < y$. Here $C_{j-1} = \prod_{i=1}^j \gamma_i$ $j = 1, 2, \dots, n$, and for $x \in [0, 1]$,

$$g_m(x) = h_m(x) - h_m(0),$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ -\log(1-x), & m = -1. \end{cases}$$

2. The Mellin Transform (MT)

Ref. [4] introduced a systematic methodology for analyzing the quotients-products of independent r.v. using the MT technique. This pioneering work established a foundational framework that has since influenced extensive research in statistical theory and applications involving multiplicative and ratio-based distributions. The MT of $f(x, y)$ is

$$M_{s_1, s_2}(f(x, y)) = \int_0^\infty \int_0^\infty x^{s_1-1} y^{s_2-1} f(x, y) dx dy, \quad (2.1)$$

where s_1 and s_2 are complex variables. As outlined by [5] under appropriate conditions, it holds the inverse property.

$$f(x, y) = \frac{1}{(2\pi i)^2} \int_{h-i\infty}^{h+i\infty} \int_{k-i\infty}^{k+i\infty} x^{-s_1} y^{-s_2} M_{s_1, s_2}(f(x, y)) ds_1 ds_2. \quad (2.2)$$

The integration paths are composed of two vertical lines in the complex plane, positioned to the right of the origin. We will study in this work two cases:

1. when $s_1 = s_2 = t$, we obtain

$M_{t,t}(g(u)) = M(t|u)$ This represents the MT for PDF of the product $U = XY$. Hence,

$$g(u) = \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} u^{-t} M(t|u) dt \quad (2.3)$$

1. When $s_1 = t$ and $s_2 = -t + 2$, get the following:

$M_{t,-t+2}(h(z)) = M(t|z)$, which is equivalent to the MT for the quotient's pdf $Z = X/Y$. Thus

$$h(z) = \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} z^{-t} M(t|z) dt. \quad (2.4)$$

3. The Fox H-Function

Ref. [13] pointed out that this is an extension of the G-function, which is frequently called a Mellin-Barnes integral in literature. The G-function is defined as

$$G_{p,q}^{m,n} \left[z \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} z^{-t} g(t) dt,$$

$$g(t) = \frac{\prod_{j=1}^m \Gamma(b_j + t) \prod_{j=1}^n \Gamma(1 - a_j - t)}{\prod_{j=m+1}^q \Gamma(1 - b_j - t) \prod_{j=n+1}^p \Gamma(a_j + t)}.$$

It is defined that the H-function is

$$H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} z^{-t} h(t) dt,$$

$$h(t) = \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j t) \prod_{j=1}^n \Gamma(1 - a_j - \alpha_j t)}{\prod_{j=m+1}^q \Gamma(1 - b_j - \beta_j t) \prod_{j=n+1}^p \Gamma(a_j + \alpha_j t)}.$$

The non-negative numbers m, n, p , and q satisfy, $0 \leq n \leq p$, $1 \leq m \leq q$; $\alpha_j (j = 1, \dots, p)$ and $\beta_j (j = 1, \dots, q)$ are taken to be positive numbers in order to maintain uniformity. Theorem 4.1 will make use of the Fox function's product relation.

$$H_{p,q}^{m,n} \left[w z \left| \begin{matrix} (a_i, \alpha_i), i = 1, p \\ (b_j, \beta_j), j = 1, q \end{matrix} \right. \right] = w^{b_1} \sum_{n=0}^{\infty} \frac{(1-w)^n}{n!} H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_i, \alpha_i), i = 1, p \\ (b_j, \beta_j), j = 1, q \end{matrix} \right. \right] \quad (3.1)$$

4. The Distribution of product of two-GOS

Theorem 4.1. Consider $X(i, n, m, r)$ and $X(j, n, m, r)$ be the i th and j th GOS with ($i < j$), derived from a r.s. of size n taken from the GED, then pdf of the product $U = X(i, n, m, r) \cdot X(j, n, m, r)$ is:

$$g(u) = \frac{\theta^2 c_{j-1}}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{r,s}^* \sum_{l_1, l_2}^* \sum_{l_3, l_4}^* H_{1,2}^{2,0} \left[\lambda^2 (l_4 + 1)(l_4 + l_3 + 2)u \left| \begin{matrix} (1+n, 1) \\ (n, 1), (n, 2) \end{matrix} \right. \right]. \quad (4.1)$$

Proof.

By substituting equations (1.1) and (1.2) into (1.4), the joint PDF of these GOS is expressed as:

$$f(x, y) = \frac{c_{j-1}(\lambda\theta)^2}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{k=0}^{i-1} \sum_{s=0}^{j-i-1} (-1)^{k+s} \binom{i-1}{k} \binom{j-i-1}{s} \\ \cdot \sum_{l_1=0}^{(m+1)(j+k-i-s)-1} \sum_{l_2=0}^{(m+1)s+\gamma_j-1} (-1)^{l_1+l_2} \binom{(m+1)(j+k-i-s)-1}{l_1} \binom{(m+1)s+\gamma_j-1}{l_2} \\ \cdot \sum_{l_3=0}^{\theta(l_1+1)-1} \sum_{l_4=0}^{\theta(l_2+1)-1} (-1)^{l_3+l_4} \binom{\theta(l_1+1)-1}{l_3} \binom{\theta(l_2+1)-1}{l_4} e^{-\lambda(l_3+1)x} e^{-\lambda(l_4+1)y}. \quad (4.2)$$

The MT of (4.2) is given by

$$M_{s_1, s_2}(f(x, y)) = K \sum_{r,s}^* \sum_{l_1, l_2}^* \sum_{l_3, l_4}^* \int_0^\infty y^{s_2-1} e^{-\lambda(l_4+1)y} I(y) dy,$$

where $K = \frac{c_{j-1}(\lambda\theta)^2}{(i-1)!(j-i-1)!(m+1)^{j-2}}$, $\sum_{r,s}^* = \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} (-1)^{r+s} \binom{i-1}{r} \binom{j-i-1}{s}$,
 $\sum_{l_1, l_2}^* = \sum_{l_1=0}^{(m+1)(j+r-i-s)-1} \sum_{l_2=0}^{(m+1)s+\gamma_j-1} (-1)^{l_1+l_2} \binom{(m+1)(j+r-i-s)-1}{l_1} \binom{(m+1)s+\gamma_j-1}{l_2}$, $\sum_{l_3, l_4}^* = \sum_{l_3=0}^{\theta(l_1+1)-1} \sum_{l_4=0}^{\theta(l_2+1)-1} (-1)^{l_3+l_4} \binom{\theta(l_1+1)-1}{l_3} \binom{\theta(l_2+1)-1}{l_4}$, and

$$I(y) = \int_0^y x^{s_1-1} e^{-\lambda(l_3+1)x} dx.$$

Let $h = \lambda x(l_3 + 1)$ and $n = \lambda y(l_3 + 1)$. Thus

$$I(y) = (\lambda(l_3 + 1))^{-s_1} \int_0^w h^{s_1-1} e^{-v} dh = (\lambda(l_3 + 1))^{-s_1} \gamma(s_1, n),$$

where $\gamma(s_1, n)$ is the incomplete gamma function. Thus, we can rewrite MT as

$$M_{s_1, s_2}(f(x, y)) = K \sum_{r,s}^* \sum_{l_1, l_2}^* \sum_{l_3, l_4}^* (\lambda(l_3 + 1))^{-s_1-s_2} \int_0^\infty n^{s_2-1} e^{-\left(\frac{l_4+1}{l_3+1}\right)n} \gamma(s_1, n) dn. \quad (4.3)$$

By using the integral

$$\int_0^\infty n^{s_2-1} e^{-\beta n} \gamma(s_1, \alpha n) dn = \frac{\alpha^{s_1} \Gamma(s_1 + s_2)}{s_1(\alpha + \beta)^{s_1+s_2}} {}_2F_1(1, s_1 + s_2; s_1 + 1; \frac{\alpha}{\alpha + \beta}),$$

see [13] (6.45, page 663), the MT (4.3) written as:

$$M_{s_1, s_2}(f(x, y)) = K \sum_{r,s}^* \sum_{l_1, l_2}^* \sum_{l_3, l_4}^* (\lambda(l_3 + 1))^{-s_1-s_2} \frac{\Gamma(s_1+s_2)}{s_1} \left(\frac{l_3+1}{l_4+l_3+2}\right)^{s_1+s_2} {}_2F_1(1, s_1 + s_2; s_1 + 1; \frac{l_3+1}{l_4+l_3+2}), \quad (4.4)$$

where ${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^\infty \frac{\Gamma(a+m)\Gamma(b+m)}{\Gamma(c+m)} \frac{z^n}{n!}$ is the hyper geometric function. Using the MT (4.4), with setting $s_1 = t$ and $s_2 = t$, obtain

$M_{t,t}(g(u)) = M(t|u)$, is equivalent to the MT for the product's PDF.

$$M_{t,t}(g(u)) = K \sum_{r,s}^* \sum_{l_1, l_2}^* \sum_{l_3, l_4}^* \lambda^{-2t} \frac{\Gamma(2t)}{t(l_4+l_3+2)^{2t}} {}_2F_1(1, 2t; t + 1; \frac{l_3+1}{l_4+l_3+2})$$

Using the Pfaff transform relation for the hyper geometric function,

${}_2F_1(a, b; c; z) = (1-z)^{-b} {}_2F_1(c-a, b; c; z/(z-1))$, we get

$${}_2F_1(1, 2t; t + 1; \frac{l_3 + 1}{l_4 + l_3 + 2}) = \left(\frac{l_4 + 1}{l_4 + l_3 + 2}\right)^{-2t} {}_2F_1(t, 2t; t + 1; -\frac{l_3 + 1}{l_4 + 1}),$$

thus

$$M(t|u) = K \sum_{r,s}^* \sum_{l_1, l_2}^* \sum_{l_3, l_4}^* (\lambda(l_4 + 1))^{-2t} \frac{\Gamma(2t)}{t} {}_2F_1(t, 2t; t + 1; -\frac{l_3 + 1}{l_4 + 1}).$$

Using (2.3) we get

$$\begin{aligned}
 g(u) &= K \sum_{r,s}^* \sum_{l_1,l_2}^* \sum_{l_3,l_4}^* \sum_{n=0}^{\infty} \left(-\frac{l_3+1}{l_4+1}\right)^n \frac{1}{n!} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Gamma(n+t)\Gamma(n+2t)}{\Gamma(n+1+t)} ((\lambda(l_4+1))^2 u)^{-t} dt \\
 &= K \sum_{r,s}^* \sum_{l_1,l_2}^* \sum_{l_3,l_4}^* \sum_{n=0}^{\infty} \left(-\frac{l_3+1}{l_4+1}\right)^n \frac{1}{n!} H_{1,2}^{2,0} \left[(\lambda(l_4+1))^2 u \left| \begin{matrix} (n+1, 1) \\ (n, 1), (n, 2) \end{matrix} \right. \right]. \quad (4.5)
 \end{aligned}$$

Using the product relation for the H-function (3.1) in (4.5), The product's PDF (4.1) is obtained.

5. The Distribution of quotient of two GOS

Theorem 5.1. Consider a r.s. of size n from the GED, then the quotient's pdf $Z = X(i, n, m, r)/X(j, n, m, r)$ is:

$$h(z) = \frac{\theta^2 c_{j-1}}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{r,s}^* \sum_{l_1,l_2}^* \sum_{l_3,l_4}^* [(\lambda(l_4+1) + (\lambda(l_3+1))z)]^{-2}, \quad (5.1)$$

where $0 < z \leq 1$.

Proof:

Using (4.4) by setting $s_1 = t$ and $s_2 = -t + 2$, obtain $M_{i,-t+2}(h(z)) = M(t|z)$, is equivalent to the MT for the quotient's PDF $Z = X/Y$. Thus

$$M_{i,-t+2}(h(z)) = K \sum_{r,s}^* \sum_{l_1,l_2}^* \sum_{l_3,l_4}^* \frac{\Gamma(2)}{\lambda^2 t (\lambda(l_4+1) + (\lambda(l_3+1))z)^2} {}_2F_1(1, 2; t+1; \frac{l_3+1}{\lambda(l_4+1)+(\lambda(l_3+1))z}). \quad (5.2)$$

Using the Pfaff transform relation for the hyper geometric function we get

$$M(t|z) = K \sum_{r,s}^* \sum_{l_1,l_2}^* \sum_{l_3,l_4}^* \frac{1}{\lambda^2 (\lambda(l_4+1) + (\lambda(l_3+1))z)^2} \sum_{n=0}^{\infty} \binom{-2}{n} \left(\frac{l_3+1}{\lambda(l_4+1)+(\lambda(l_3+1))z}\right)^n \frac{1}{t+n}.$$

Thus from (2.4), pdf as

$$\begin{aligned}
 h(z) &= \sum_{r,s}^* \sum_{l_1,l_2}^* \sum_{l_3,l_4}^* \frac{K}{2\pi i \lambda^2 (\lambda(l_4+1) + (\lambda(l_3+1))z)^2} \sum_{n=0}^{\infty} \binom{-2}{n} \left(\frac{l_3+1}{\lambda(l_4+1)+(\lambda(l_3+1))z}\right)^n \int_{h-i\infty}^{h+i\infty} \frac{z^{-t}}{t+n} dt \\
 &= K \sum_{r,s}^* \sum_{l_1,l_2}^* \sum_{l_3,l_4}^* \frac{1}{\lambda^2 (\lambda(l_4+1) + (\lambda(l_3+1))z)^2} \sum_{n=0}^{\infty} \binom{-2}{n} \left(\frac{l_3+1}{\lambda(l_4+1)+(\lambda(l_3+1))z}\right)^n \\
 &= K \sum_{r,s}^* \sum_{l_1,l_2}^* \sum_{l_3,l_4}^* \frac{1}{\lambda^2 (\lambda(l_4+1) + (\lambda(l_3+1))z)^2} \left(1 + \frac{l_3+1}{\lambda(l_4+1)+(\lambda(l_3+1))z}\right)^{-2}.
 \end{aligned}$$

Thus (5.1) can be derived.

Corollary 5.2. When $\theta = 1$, pdf of Q&P of GOS from the ordinary exp. distribution (OED) are reduced in (4.1) and (5.1), respectively, as

$$g(u) = \frac{c_{j-1}}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{r,s}^* \sum_{l_1,l_2}^* \sum_{l_3,l_4}^* H_{1,2}^{2,0} \left[\lambda^2(l_4+1)(l_4+l_3+2)u \mid \begin{matrix} (1+n, 1) \\ (n, 1), (n, 2) \end{matrix} \right],$$

$$h(z) = \frac{c_{j-1}}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{r,s}^* [\gamma_j + (m+1)s + (m+1)(j+r-i-s)z]^{-2},$$

see [1].

Corollary 5.3. When $\theta = 1$ and $m = 0$ and $k = 1$, pdf of the Q&P of the i th and j th OS from the OED, respectively, are reduced in (4.1) and (5.1) as

$$g(u) = \frac{n!}{(i-1)!(j-i-1)!(n-j)} \sum_{r,s}^* H_{1,2}^{2,0} \left[\lambda^2 u \mid \begin{matrix} (1+n, 1) \\ (n, 1), (n, 2) \end{matrix} \right],$$

$$h(z) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \sum_{r,s}^* [(n-j+s+1) + (j+r-i-s)z]^{-2},$$

see [11].

Corollary 5.4. Consider $j = n$, $i = 1$ in (4.1) and (5.1) we obtain PDFs of the Q&P of extreme GOS respectively as

$$g(u) = \frac{\theta^2 c_n}{(n-2)!(m+1)^{n-2}} \sum_{r,s}^* \sum_{l_1,l_2}^{**} \sum_{l_3,l_4}^* H_{1,2}^{2,0} \left[\lambda^2(l_4+1)(l_4+l_3+2)u \mid \begin{matrix} (1+n, 1) \\ (n, 1), (n, 2) \end{matrix} \right],$$

$$h(z) = \frac{\theta^2 c_n}{(n-2)!(m+1)^{n-2}} \sum_s^{**} \sum_{l_1,l_2}^{**} \sum_{l_3,l_4}^* (l_4+1+(l_3+1)z)^{-2}.$$

See [1] and [11].

where

$$\sum_s^* = \sum_{s=0}^{n-2} (-1)^s \binom{n-2}{s},$$

$$\sum_{l_1,l_2}^{**} = \sum_{l_1=0}^{(m+1)(n-s-1)-1} \sum_{l_2=0}^{(m+1)s+k-1} (-1)^{l_1+l_2} \binom{(m+1)(n-s-1)-1}{l_1} \binom{(m+1)s+k-1}{l_2}$$

Corollary 5.5. Let $j = i + 1$ in (4.1) and (5.1) we obtain PDF of Q&P of the consecutive GOS ($i < j$) respectively as

$$g(u) = \frac{\theta^2 c_i}{(i-1)!(m+1)^{i-1}} \sum_r^{**} \sum_{l_1,l_2}^{***} \sum_{l_3,l_4}^* H_{1,2}^{2,0} \left[\lambda^2(l_4+1)(l_4+l_3+2)u \mid \begin{matrix} (1+n, 1) \\ (n, 1), (n, 2) \end{matrix} \right],$$

$$h(z) = \frac{\theta^2 c_i}{(i-1)!(m+1)^{i-1}} \sum_r^{**} \sum_{l_1,l_2}^{***} \sum_{l_3,l_4}^* (l_4+1+(l_3+1)z)^{-2},$$

See [1] and [11]. Where $\sum_r^{**} = \sum_{r=0}^{i-1} (-1)^r \binom{i-1}{r}$, and $\sum_{l_1,l_2}^{***} = \sum_{l_1=0}^{(m+1)(r+1)-1} \sum_{l_2=0}^{\gamma_{i+1}-1} (-1)^{l_1+l_2} \binom{(m+1)(r+1)-1}{l_1} \binom{\gamma_{i+1}-1}{l_2}$.

6. Conclusion

In this paper, we derived explicit probability density functions for the Quotient-Product (Q&P) of two Generalized Order Statistics (GOS) drawn from the Generalized Exponential Distribution (GED). By employing the Mellin transform and its inverse, along with the Fox H-function, we established closed-form expressions for both the product (Theorem 4.1) and the quotient (Theorem 5.1) of two GOS. The results were further validated through several corollaries that recover known results in the literature as special cases, including the ordinary exponential distribution (Corollaries 5.2 and 5.3), as well as the extreme and consecutive GOS cases (Corollaries 5.4 and 5.5). The consistency of our findings with those of [1] and [11] confirms the correctness and generality of the derived formulas. These results enrich the theoretical foundation of order statistics and provide useful tools for professionals in reliability engineering, risk analysis, survival analysis, and stochastic modeling.

6.1. Limitations

Despite the contributions of this work, several limitations should be acknowledged. First, the derived results are specific to the two-parameter Generalized Exponential Distribution; extension to other distributions may require substantially different analytical frameworks. Second, the derivations assume the independence of the two GOS, whereas many real-world applications involve dependence structures. Third, the use of the Fox H-function, while mathematically elegant, may pose computational challenges to practitioners who lack specialized software to evaluate such special functions. Finally, the paper does not include numerical simulations or empirical studies to validate the theoretical results, which would strengthen the practical relevance of the findings.

6.2. Future Research Directions

Several directions are identified for future research. (i) The results can be extended to other flexible lifetime distributions, such as the exponentiated Weibull, beta-generated, or Kumaraswamy families, where similar Mellin transform techniques may apply. (ii) The assumption of independence may be relaxed by studying the Q&P of GOS under bivariate or multivariate dependent models, particularly using copula-based frameworks. (iii) Numerical algorithms and software implementations for computing the Fox H-function should be developed to facilitate practical application of the derived formulas. (iv) A simulation study that compares the empirical distributions of Q&P with the theoretical PDFs derived here would provide further validation. (v) Bayesian estimation of the parameters of Q&P distributions under various loss functions is another promising avenue, based on previous work such as [14] for the GED.

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