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Research article

New Expansion of Chen Distribution According to the Nitrosophic Logic Using the Gompertz Family

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ABSTRACT

The study deals with a new statistical distribution based on the two-parameter chen distribution and its development into a four-parameter distribution by adding the two parameters of the Gompertz-Neutrosophic family, preceded by the development of a Gene distribution model according to the neutrosophic logic, and the new distribution is called the Gompertz-Neutrosophic chen distribution (NGoC). Many basic statistical and mathematical properties of the proposed distribution are presented. The model parameters are estimated using three statistical estimation methods. To determine the efficiency of the NGoC distribution estimation, Monte Carlo simulation is used to determine the bias, squared errors, and root mean square errors. Finally, to determine the efficiency of the proposed distribution, a practical application will be conducted on real neutrosophic data, and a comparison will be made between the NGoC distribution and five other distributions using six statistical measures, where the results showed the flexibility and efficiency of the proposed neutrosophic distribution.

1. Introduction

The T-X method developed by Alzaatreh proposes an innovative approach to generating continuous distributions, aiming to enhance the flexibility of modeling diverse data in statistics. This method is based on transforming known basic distributions into new distributions that are more compatible with complex data that exhibit unusual properties such as skewness or skewness. This method is an effective tool in improving the accuracy of statistical models and increasing the ability to deal with data that do not conform to traditional distributions [1]. Examples of distributions that rely on this method are the NKw-G family [2], GAEPFDs [3], APRAY-G [4], OLE family [5], and NETLE [6].

Neutrosophic logic was first introduced in 1995 as an extension of fuzzy logic, with the aim of dealing with uncertainty and ambiguity in information. This logic consists of vectors representing truth, error, and indeterminacy. The truth vector represents confirmed data, while the error vector represents data that have been proven incorrect. The indeterminate part represents data that cannot be confirmed as correct or incorrect with certainty. Over time, this logic was incorporated into the statistical field, resulting in the development of a family of neutrosophic statistical distributions, such as the NLD [7], the NBL [8], NEIRD [9], NIGD [10], NTLEED [11], and NIPLD [12].

In the context of developing new methods that combine the T-X method with neutrosophic logic, a new family of continuous neutrosophic distributions known as the neutrosophic Gompertz family (NGo-G) was proposed, in which mathematical formulas for the neutrosophic CDF (NCDF) and neutrosophic pdf (Npdf) of these distributions were presented by following:

Definition 1: Let the $X_N = d + tI$, $tI \in [X_L, X_U]$, where X_L, X_U are lower and upper values of the neutrosophic Gompertz family (NGo-G) random variable having determined part d and indeterminate part tI, $tI \in [I_L, I_U]$. Note that the NGo-G reduces to classical Go-G when $\mathrm{X_L} = \mathrm{X_U}$. The neutrosophic cumulative density (NCDF), and neutrosophic probability density (Npdf) of NGo-G has a Neutrosophic shape parameters $r_N \in [r_L, r_U]$, $u_N \in [u_L, u_U]$, has the form:

$$F_{NGo}(x_N, r_N, u_N, \psi_N) = 1 - e^{\frac{r_N}{u_N}\left(1-e^{u_N\frac{G(x_N,\psi_N)}{1-G(x_N,\psi_N)}}\right)}, \tag{1.1}$$

$$f_{NGo}(x_N, r_N, u_N, \psi_N) = \frac{r_N g(x_N,\psi_N) e^{u_N\frac{G(x_N,\psi_N)}{1-G(x_N,\psi_N)}}}{\left(1-G(x_N,\psi_N)\right)^2} e^{\frac{r_N}{u_N}\left(1-e^{u_N\frac{G(x_N,\psi_N)}{1-G(x_N,\psi_N)}}\right)}, \tag{1.2}$$

where $g(x_N, \psi_N)$, and $G(x_N, \psi_N)$ are Npdf, and NCDF of baseline distribution with ψ_N paremeter.

The aim is to propose a continuous statistical distribution with four parameters based on the gene distribution according to the NGo-G neutrosophic family to be the NGoC distribution and obtain the basic functions of the cumulative distribution and probability density functions for the distribution as well as the survival functions and draw those functions with different parameter periods in addition to a number of other properties and then estimate its parameters using three methods: the maximum likelihood method, least squares and weighted least squares and then conduct a Monte Carlo simulation to determine the bias of the estimated parameters and a practical application of neutrosophic data to determine the efficiency of the proposed distribution with a comparison with five other distributions. The study included five parts, the first part contained the proposed neutrosophic distribution, while the second part contained the properties of the proposed distribution, and in the third part the parameters of the neutrosophic model were estimated, the fourth part contained a Monte Carlo simulation of the data resulting from the model, while the last part contained a practical application to determine the efficiency of the proposed distribution.

2. The NGoC Distribution

Definition 2: Let the $X_N = d + tI$, $tI \in [X_L, X_U]$, where X_L, X_U are lower and upper values of the neutrosophic Chen distribution (NC) random variable having determined part d and indeterminate part tI, $tI \in [I_L, I_U]$. Note that the NC reduces to classical Chen when $\mathrm{X_L} = \mathrm{X_U}$. The neutrosophic cumulative density (NCDF), and neutrosophic probability density (Npdf) of NC has a Neutrosophic shape parameters $b_N \in [b_L, b_U]$, $c_N \in [c_L, c_U]$, has the form:

$$G(x_N, b_N, c_N) = 1 - e^{b_N\left(1-e^{x_N^{c_N}}\right)}, \tag{2.1}$$

$$g(x_N, b_N, c_N) = b_N c_N x_N^{c_N-1} e^{x_N^{c_N}} e^{b_N\left(1-e^{x_N^{c_N}}\right)}, \tag{2.2}$$

where $x_N, b_N, c_N \geq 0$, such as x_N is neutrosophic random variable, and b_N, c_N are neutrosophic shape parameters.

The NCDF function of the proposed distribution NGoC is obtained by substituting Equation (2.1) into Equation (1.1) as follows:

$$F_{NGoC}(x_N, r_N, u_N, b_N, c_N) = 1 - e^{\left(\frac{r_N}{u_N}\left[1 - e^{\left[u_N^{\frac{1-e^{b_N\left(1-e^{x_N^{c_N}}\right)}}{e^{b_N\left(1-e^{x_N^{c_N}}\right)}}}\right]}\right]\right)}. \tag{2.3}$$

Figure 1 shows several NCDF plots of the NGoC distribution with different periods for the neutrosophic parameters, while Figure 2 shows several 3D plot of NCDF plots of the NGoC distribution.

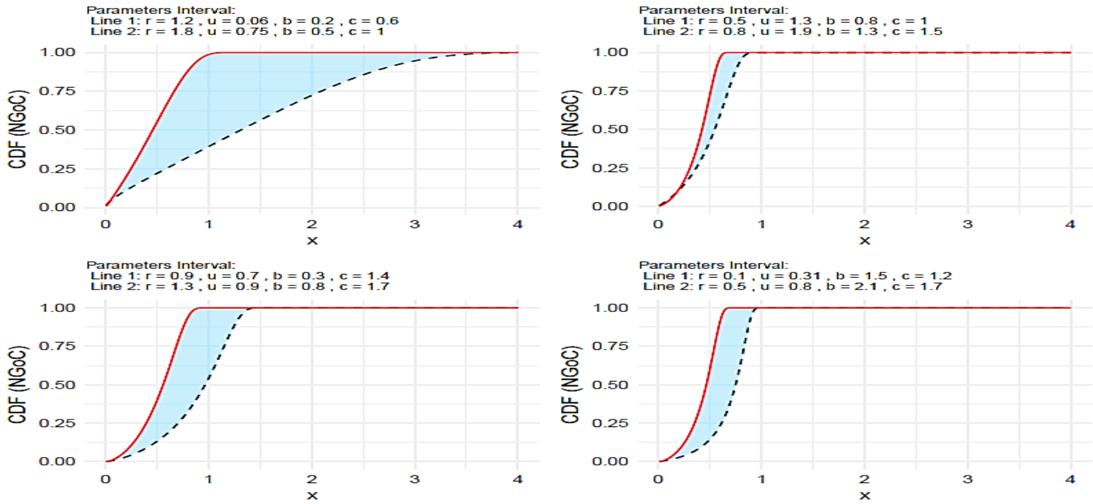

Figure 1: NCDF for NGoC distribution

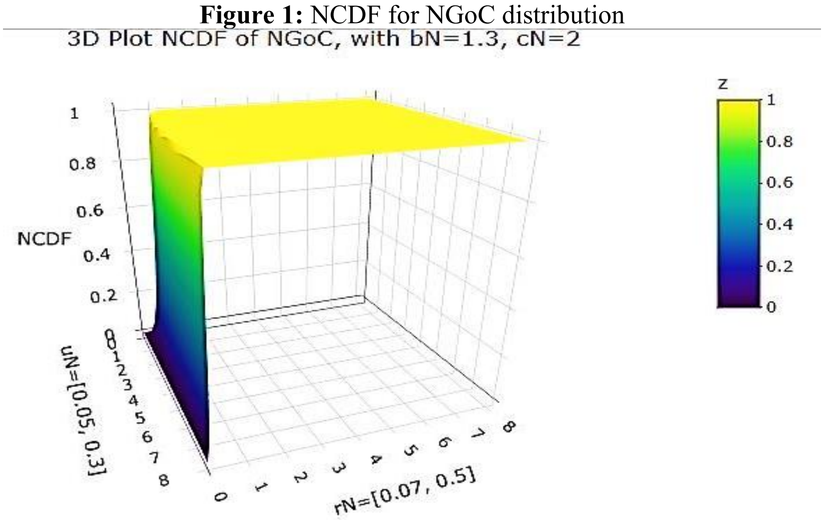

Figure 2: 3D plot of NCDF for NGoC distribution

This Figure 1 represents the Neutrosophic Cumulative Distribution Function (NCDF) for the NGoC distribution under various parameter intervals. The NCDF is a key tool for determining the probability that a random variable will take a value less than or equal to a given point. The neutrosophic parameters (which introduce uncertainty into the model) create intervals rather than exact values, illustrating the flexibility of the NGoC distribution in handling indeterminate data. The varying curves show how different ranges of uncertainty in the parameters can shift the distribution, affecting the probability for any given value. This is especially important when working with data that include vagueness or inexactness, like neutrosophic logic.

While figure 3 plot enhances the understanding of the NCDF by visualizing the distribution in three dimensions. It shows the relationships between the cumulative probability, the value of the random variable, and the neutrosophic parameters. The third dimension allows one to see the effect of changing neutrosophic parameters more clearly. It highlights the non-linear nature of the relationship and how uncertainty in the parameters impacts cumulative probability.

The Npdf function for the NGoC distribution can be obtained by deriving Equation (2.3) or substituting Equation (2.2) into Equation (1.2) as follows:

$$f_{NGoC}(x_N, r_N, u_N, b_N, c_N) = \frac{r_N b_N c_N x_N^{c_N-1} e^{x_N^{c_N}} e^{\left[u_N \frac{1-e^{b_N\left(1-e^{x_N^{c_N}}\right)}}{e^{b_N\left(1-e^{x_N^{c_N}}\right)}}\right]}}{e^{b_N\left(1-e^{x_N^{c_N}}\right)}} e^{\frac{r_N}{u_N}\left(1-e^{\left[u_N \frac{1-e^{b_N\left(1-e^{x_N^{c_N}}\right)}}{e^{b_N\left(1-e^{x_N^{c_N}}\right)}}\right]}\right)}. \quad (3)$$

Figure 3 shows several Npdf plots of the NGoC distribution with different periods for the neutrosophic parameters, while Figure 4 shows several 3D plot of Npdf plots of the NGoC distribution.

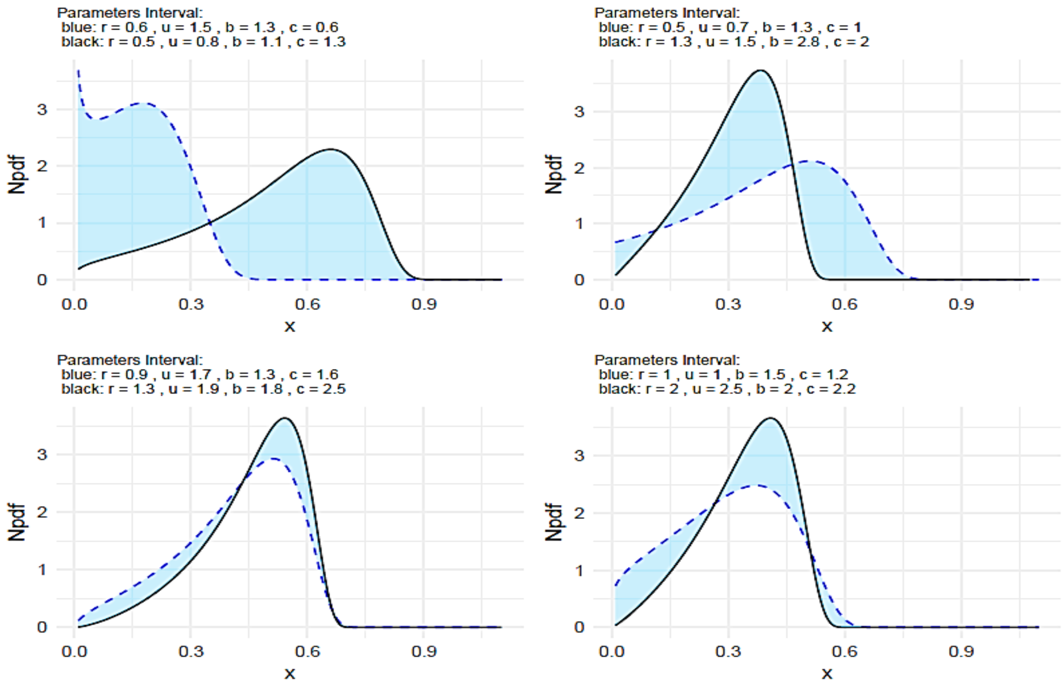

Figure 3: Npdf for NGoC distribution

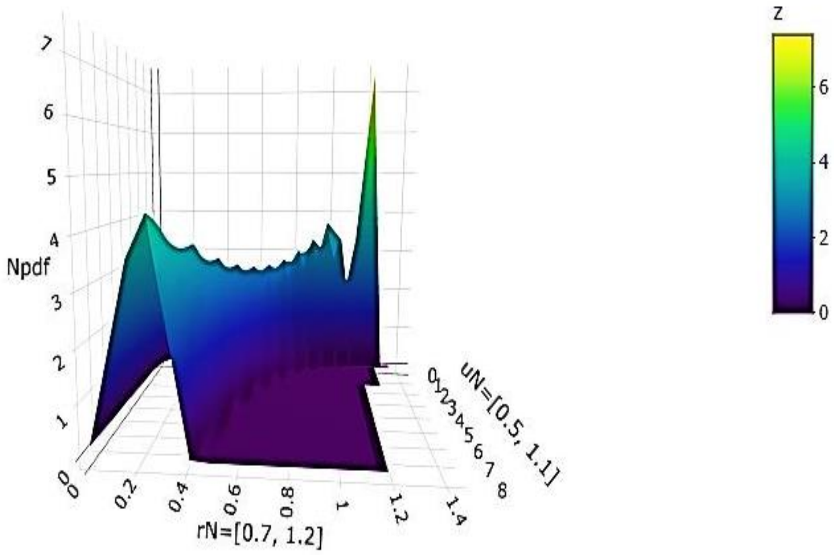

Figure 4: 3D plot of NCDF for NGoC distribution

This figure 3 illustrates Npdf for the NGoC distribution under different parameter intervals. The Npdf shows the likelihood of the random variable taking specific values. It combines the classical probability density with neutrosophic uncertainty, broadening its flexibility. Similar to the NCDF, the variations in the curves indicate how parameter uncertainty changes the probability density. This is vital for applications in reliability and risk analysis, where the exact probability is uncertain.

While The 3D representation of the Npdf provides a comprehensive view of how the density changes with both the random variable's value and the neutrosophic parameters. The plot shows how density values are distributed across different combinations of parameters. This figure illustrates how small variations in the neutrosophic parameters can result in significant changes in the distribution's shape. This is crucial when analyzing datasets where minor changes in the inputs can lead to large shifts in output probabilities.

The following expression can be employed to derive the Neutrosophic Survival Function, as outlined in reference [14].

$$S_{NGoC}(x_N,r_N,u_N,b_N,c_N)=e^{\frac{r_N}{u_N}\left(1-e^{\left[\frac{u_N^{1-e^{b_N\left(1-e^{x_N^{c_N}}\right)}}}{e^{b_N\left(1-e^{x_N^{c_N}}\right)}}\right]}\right)}. \tag{4}$$

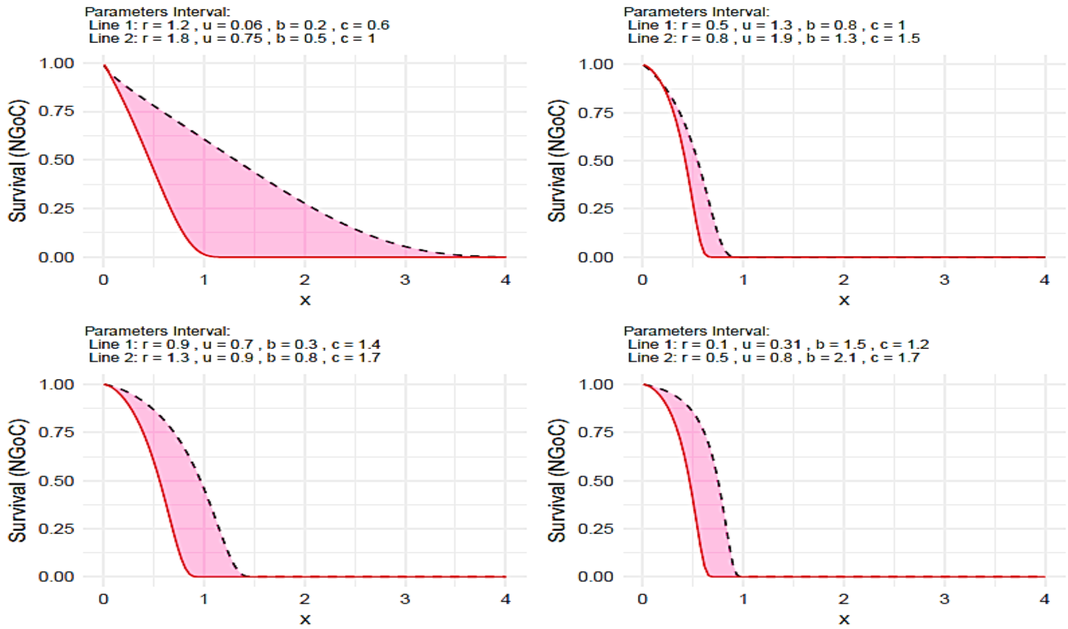

Figure 5. Neutrosophic Survival of NGoC distribution with different interval of parameters

To compute the neutrosophic hazard functions of the NGoC distribution, the following formula is utilized, as presented in reference [14]:

$$h_{NGoC}(x_N,r_N,u_N,b_N,c_N)=\frac{r_N b_N c_N x_N^{c_N-1} e^{x_N^{c_N}} e^{\left[\frac{u_N^{1-e^{b_N\left(1-e^{x_N^{c_N}}\right)}}}{e^{b_N\left(1-e^{x_N^{c_N}}\right)}}\right]}}{e^{b_N\left(1-e^{x_N^{c_N}}\right)}}. \tag{5}$$

3. Mathematical Properties of NGOC Distribution

3.1 Useful Representations Npdf and NCDF of NGOC Distribution

The objective of this component is to enhance and broaden the functionalities of the Neutrosophic Probability Density Function (Npdf) and Neutrosophic Cumulative Distribution Function (NCDF) specifically for the NGoC distribution. Consequently, from Equation (2.3), the NCDF of the NGoC distribution can be determined as follows:

$$G_{NGoC}(x_N,a_N,b_N,c_N)=1-\phi e^{\varpi_N x_N^{c_N}} \tag{3.1}$$

Where $\phi=\sum_{i_N=j_N=k_N=s_N=\nu_N=\varepsilon_N=\varpi_N=0}^{\infty}\frac{(-1)^{i_N+j_N+k_N+\eta_N+\nu_N+\epsilon_N+\varpi_N}\Gamma(k_N+s_N)}{i_N!k_N!s_N!\eta_N!\varepsilon_N!\Gamma(k_N)}\binom{i_N}{j_N}\binom{k_N+s_N}{\nu_N}\binom{\eta_N}{\varpi_N}a_N^{i_N}b_N^{k_N+\varepsilon_N-i_N}j_N^{k_N}\nu_N^{\varepsilon_N}$.

By same way expansion NCDF function, we have get the $G_{NGoC}^{\xi_N}$ by form:

$$G_{NGoC}^{\xi_N}(x_N,r_N,u_N,b_N,c_N)=\mathrm{H}e^{d_N x_N^{c_N}} \tag{3.2}$$

Where $\mathrm{H}=\sum_{l_N=w_N=r_N=z_N=p_N=\mathcal{L}_N=\delta_N=d_N=0}^{\infty}\frac{(-1)^{l_N+w_N+r_N+z_N+\nu_N+\delta_N+\varpi_N}\Gamma(z_N+p_N)}{w_N!z_N!p_N!d_N!\Gamma(z_N)a_N^{-w_N}l_N^{-w_N}}\binom{\xi_N}{l_N}\binom{w_N}{r_N}\binom{z_N+p_N}{\mathcal{L}_N}\binom{\delta_N}{d_N}b_N^{z_N+d_N-w_N}j_N^{z_N}$.

By same way expansion NCDF function, the Npdf from equation (2.2) we get:

$$g_{NGoC}(x_N, r_N, u_N, b_N, c_N) = \mathrm{B} x_N^{c_N-1} e^{(\gamma_N+1)x_N^{c_N}}, \tag{3.3}$$

where $\mathrm{B} = \sum_{i_N=j_N=t_N=v_N=\theta_N=\lambda_N=\gamma_N=0}^{\infty} \frac{(-1)^{i_N+j_N+t_N+\theta_N+\lambda_N+\gamma_N} \Gamma(t_N+2+v_N)}{i_N! b_N^{i_N} t_N! v_N! \lambda_N! \Gamma(t_N+2)(\theta_N+1)^{-\lambda_N}} \binom{i_N}{j_N} \binom{\lambda_N}{\gamma_N} \binom{t_N+v_N}{\theta_N} a_N^{i_N+\lambda_N+1} b_N^{t_N+1} c_N (j_N+1)^{t_N}$.

To get $g_{NGoC}^{\nu_N}(x_N, r_N, u_N, b_N, c_N)$, by same way in above expansion. The function has the form:

$$g_{NGo}^{\nu_N}(x_N, a_N, b_N, \phi_N) = \mathrm{E} x_N^{\nu_N c_N - \nu_N} e^{(\nu_N+\sigma_N)x_N^{c_N}}, \tag{3.4}$$

where $\mathrm{E} = \sum_{d_N=q_N=m_N=\varepsilon_N=\rho_N=\tau_N=\sigma_N=0}^{\infty} \frac{(-1)^{d_N+q_N+m_N+\rho_N+\tau_N+\sigma_N}}{d_N! \tau_N! \, b_N^{d_N-m_N-\tau_N}} \binom{d_N}{q_N} \binom{m_N+\varepsilon_N}{\rho_N} \binom{\tau_N}{\sigma_N} (\rho_N+\nu_N)^{\tau_N} a_N^{d_N+\rho_N} \nu_N^{d_N} b_N^{m_N} (q_N+\nu_N)^{m_N} b_N^{\nu_N} c_N^{\nu_N}$.

3.2 Neutrosophic Quantile Function

The Neutrosophic Quintile Function represents a mathematical framework that partitions a dataset into five equal segments, as referenced in [15]. It is defined as the inverse of the NCDF, which is detailed in Equation (2.3). This function is instrumental in the calculation of statistical measures such as median, skewness, and kurtosis, particularly in distributions characterized by significant skewness or the absence of moments. Moreover, it plays a critical role in generating random numbers for simulation studies. This equation is commonly referred to as:

$$Q_{x_N} = Q_{q_N} \left[\left(\ln \left[1 - \frac{1}{b_N} \ln \left(1 - \frac{ln\left[1 - \frac{b_N}{a_N} ln(1-u_N)\right]}{b_N + ln\left[1 - \frac{b_N}{a_N} ln(1-u_N)\right]} \right) \right] \right)^{\frac{1}{c_N}} \right]. \tag{3.5}$$

The following table shows the values of the quintile neutrosophic function for different parameter intervals.

Table 1: Quintile function values for different intervals for a_{Ne}, b_{Ne}, and c_{Ne}

t_N	(r_N, u_N, b_N, c_N)				
	[0.6, 0.9],[0.1, 0.6] , [0.5, 0.8], [0.4, 1.4]	[0.8, 1.2],[0.2, 0.8], [0.7, 1.2], [0.8, 1.8]	[0.3, 0.7],[0.5, 0.9], [0.6, 1.2], [0.2, 1.2]	[0.7, 1.1],[0.4, 0.8], [0.8, 1.2], [0.3, 1.3]	[0.4, 0.8],[0.5, 1] , [0.9, 1.4], [0.7, 1.7]
0.1	[0.04088136, 0.227376]	[0.1019270, 0.2202844]	[0.00826646, 0.15213]	[0.0021152, 0.13068]	[0.1147243, 0.2263868]
0.2	[0.16296595, 0.351155]	[0.2173615, 0.3150734]	[0.0758703, 0.247514]	[0.0149795, 0.21324]	[0.2396136, 0.3223406]
0.3	[0.33797402, 0.445370]	[0.3288875, 0.3854885]	[0.2237501, 0.321359]	[0.0431355, 0.28047]	[0.3489455, 0.3905881]
0.4	[0.5230288, 0.5475968]	[0.4353962, 0.4432676]	[0.38247, 0.44193466]	[0.0875288, 0.33871]	[0.4445618, 0.4447824]
0.5	[0.5903843, 0.7839328]	[0.4935179, 0.5381719]	[0.435555, 0.7200662]	[0.1485093, 0.39132]	[0.4900062, 0.5308202]
0.6	[0.6513476, 1.0464200]	[0.5392182, 0.6395128]	[0.48368, 1.05551921]	[0.2274330, 0.44062]	[0.5302549, 0.6105960]
0.7	[0.7090884, 1.3418554]	[0.5827995, 0.7427121]	[0.529338, 1.4572255]	[0.3281548, 0.48879]	[0.5677885, 0.6875949]
0.8	[0.7673073, 1.6901839]	[0.6270345, 0.8539027]	[0.57560, 1.95639189]	[0.4606280, 0.53877]	[0.6051840, 0.7667761]
0.9	[0.8334402, 2.1536665]	[0.6776714, 0.9888016]	[0.62857, 2.65651185]	[0.59732, 0.6562303]	[0.6473078, 0.8587809]

This table provides quintile function values across different intervals of the neutrosophic parameters $\boldsymbol{r_N}, \boldsymbol{u_N}, \boldsymbol{b_N}$, and $\boldsymbol{c_N}$ .The quintile function divides the data into five equal parts, helping in statistical measures like skewness and kurtosis. These values are essential in understanding the data's distribution at various intervals. The table shows how uncertainty in parameters can shift these quintiles, giving insights into the range and variability of the dataset. This is particularly important for decision-making under uncertainty.

3.3 Neutrosophic Moments

Let x_N be a neutrosophic random variable represented by the neutrosophic probability density function Npdf as defined in Equation (3.3). The neutrosophic moment m^{th} from the NGoC distribution can be expressed using Equation (3.3) as follows [16]:

$$\mu'_{m_N} = \mathrm{B} \int_0^\infty x_N^{c_N+m-1} e^{(\gamma_N+1)x_N^{c_N}} dx_N.$$

Let $u_N = (-\gamma_N - 1)x_N^{c_N} \Rightarrow x_N = \frac{u_N^{\frac{1}{c_N}}}{(-\gamma_N-1)^{\frac{1}{c_N}}} \Longrightarrow dx_N = \frac{u_N^{\frac{1}{c_N}-1} du_N}{c_N(\gamma_N+1)^{\frac{1}{c_N}}}$

$$\mu'_{m_N} = \mathrm{B}\int_0^\infty \left(\frac{u_N^{\frac{1}{c_N}}}{(-\gamma_N - 1)^{\frac{1}{c_N}}}\right)^{c_N+m-1} e^{-u_N}\,\frac{u_N^{\frac{1}{c_N}-1}\,du_N}{c_N(-\gamma_N - 1)^{\frac{1}{c_N}}}$$

$$\mu'_{m_N} = B\int_0^\infty \frac{u_N^{\frac{m}{c_N}-1}}{c_N(-\gamma_N - 1)^{\frac{m}{c_N}+1}} e^{-u_N}\,du_N$$

$$\mu'_{m_N} = \frac{B\Gamma\left(\frac{m}{c_N}\right)}{c_N(-\gamma_N - 1)^{\frac{m}{c_N}+1}}. \tag{3.6}$$

Then the $\mu'_{1_N}, \mu'_{2_N}, \mu'_{3_N}$, and μ'_{4_N} has forms:

$$\mu'_{1_N} = \frac{B\Gamma\left(\frac{1}{c_N}\right)}{c_N(-\gamma_N-1)^{\frac{1}{c_N}+1}}, \tag{3.7}$$

$$\mu'_{2_N} = \frac{B\Gamma\left(\frac{2}{c_N}\right)}{c_N(-\gamma_N - 1)^{\frac{2}{c_N}+1}}, \tag{3.8}$$

$$\mu'_{3_N} = \frac{B\Gamma\left(\frac{3}{c_N}\right)}{c_N(-\gamma_N-1)^{\frac{3}{c_N}+1}}, \tag{3.9}$$

$$\mu'_{4_N} = \frac{B\Gamma\left(\frac{4}{c_N}\right)}{c_N(-\gamma_N-1)^{\frac{4}{c_N}+1}}. \tag{3.10}$$

The neutrosophic variance of the NGoC distribution can be calculated using the following formula: $\sigma_N^2 = \mu'_{2_N} - {\mu'_{1_N}}^2$ [17] to get the form:

$$\sigma_N^2 = \frac{B\Gamma\left(\frac{2}{c_N}\right)}{c_N(-\gamma_N-1)^{\frac{2}{c_N}+1}} - \left[\frac{B\Gamma\left(\frac{1}{c_N}\right)}{c_N(-\gamma_N-1)^{\frac{1}{c_N}+1}}\right]^2. \tag{3.11}$$

Neutrosophic skewness (S_N) and neutrosophic kurtosis (K_N) are defined according to the following mathematical definitions [17]:

$$S_N = \frac{\frac{\mathrm{B}}{3 - b_N(w_N + 1)}}{\left(\frac{B\Gamma\left(\frac{2}{c_N}\right)}{c_N(-\gamma_N - 1)^{\frac{2}{c_N}+1}}\right)^{\frac{3}{2}}}, \tag{3.12}$$

$$K_N = \frac{\frac{B\Gamma\left(\frac{4}{c_N}\right)}{c_N(-\gamma_N-1)^{\frac{4}{c_N}+1}}}{\left(\frac{B\Gamma\left(\frac{2}{c_N}\right)}{c_N(-\gamma_N-1)^{\frac{2}{c_N}+1}}\right)^2} - 3. \tag{3.13}$$

Table 2 shows a range of different intervals for values of neutrosophic moment, neutrosophic variance, neutrosophic skewness, and neutrosophic kurtosis.

Table 2: different intervals for values of neutrosophic moment, neutrosophic variance, neutrosophic skewness, and neutrosophic kurtosis.

r_N	u_N	b_N	c_N	$\acute{\mu}_{1_N}$	$\acute{\mu}_{2_N}$	$\acute{\mu}_{3_N}$	$\acute{\mu}_{4_N}$	σ_N^2	S_N	K_N
[0.2, 0.7]	[1.1, 1.3]	[0.5, 0.6]	0.1	[0.353223, 0.649052]	[0.731786, 1.234462]	[2.610129, 3.409474]	[11.81932, 12.80216]	[0.60702, 0.813193]	[2.485826, 4.16952]	[7.755986, 23.90647]
	[1.2, 1.4]			[0.294321, 0.541413]	[0.50313, 0.858729]	[1.480701, 1.981655]	[5.748562, 5.996134]	[0.416505, 0.565601]	[2.490255, 4.149031]	[7.795552, 23.68702]
	[1.3, 1.6]			[0.026789, 0.384657]	[0.004724, 0.433581]	[0.001494, 0.713783]	[0.000667, 1.481542]	[0.004006, 0.28562]	[2.500119, 4.60254]	[7.880854, 29.88408]
	[1.4, 1.7]			[0.022296, 0.3272]	[0.003243, 0.313867]	[0.000846, 0.440546]	[0.000312, 0.780758]	[0.002746, 0.206807]	[2.505376, 4.583517]	[7.925467, 29.66124]
	[1.5, 1.8]	[0.7, 0.8]		[0.040985, 0.177613]	[0.005321, 0.179008]	[0.00104, 0.310909]	[0.000261, 0.744753]	[0.003641, 0.147462]	[2.679007, 4.105099]	[9.221973, 23.24162]
	[1.6, 1.9]			[0.034833, 0.151893]	[0.003845, 0.130101]	[0.00064, 0.192148]	[0.000137, 0.391721]	[0.002632, 0.10703]	[2.684593, 4.094618]	[9.273619, 23.14263]

	[1.7, 2]			[0.01336, 0.029752]	[0.001139, 0.002807]	[0.000175, 0.0004]	[3.79E-05, 7.35E-05]	[0.000961, 0.001922]	[2.690227, 4.540443]	[9.325467, 29.18133]
	[1.8, 2.1]			[0.01139, 0.025531]	[0.000823, 0.002068]	[0.000107, 0.000254]	[1.97E-05, 4.01E-05]	[0.000693, 0.001416]	[2.695874, 4.529712]	[9.377284, 29.06835]

Table 2 the wide range of values reflects the impact of parameter uncertainty on the distribution's characteristics. It shows how even slight changes in the neutrosophic parameters can drastically alter the distribution, which is crucial when modeling real-world phenomena with inherent uncertainties.

Figure 6 presents 3D shapes of moments, skewness, and kurtosis.

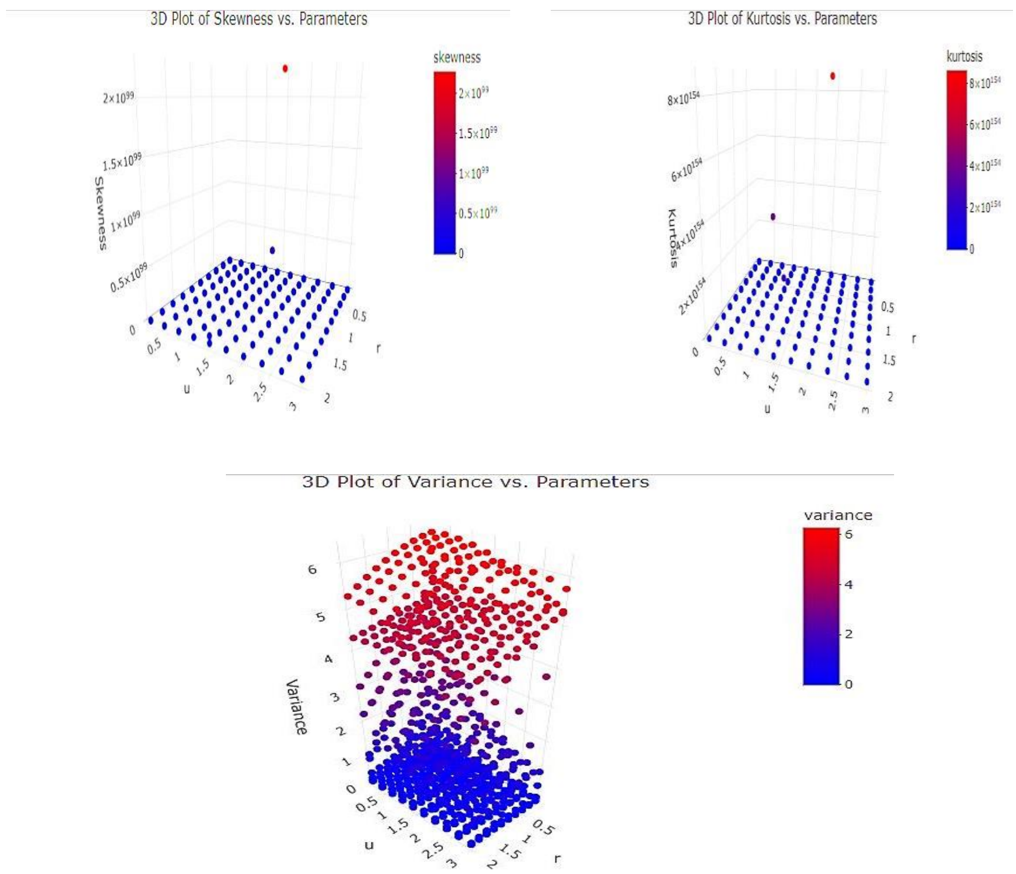

Figure 6. 3D plot of Variance, Skewness, and Kurtosis

Figure 6 the variance, skewness, and kurtosis play a crucial role in understanding the shape of the distribution. High variance indicates greater spread, while positive or negative skewness shows how the distribution leans. Kurtosis informs us whether the distribution is heavy-tailed (more extreme values) or light-tailed. This figure emphasizes how neutrosophic parameters can drastically change the distribution's characteristics.

3.4 Neutrosophic Moment Generating Function

The Neutrosophic moment generating function ($Nmgf$) of NGoC distribution from equation (3.6) given by [18]:

$$M'_{x_N}(y_N)_{NGoC} = \sum_{q=0}^{\infty} \frac{y_N^q}{q!}\left[\frac{B\Gamma\left(\frac{m}{c_N}\right)}{c_N(-\gamma_N-1)^{\frac{m}{c_N}+1}}\right]. \tag{3.14}$$

3.5 Neutrosophic Incomplete Moments

To get the neutrosophic incomplete moments of a random variable X_N, use the following formula. [19]:

$$\grave{\mu}_{N_r}(y_N) = \int_0^{y_N} x_N^r f_{NGoC}(x_N, a_N, b_N, c_N)\, dx_N.$$

Through substituting the NGoC distribution found in Equation (3.3) with $g_{NGoC}(x_N, a_N, b_N, c_N)$ in the preceding equation, we obtain:

$$\grave{\mu}_{N_r}(y_N) = \mathrm{B}\int_0^{y_N} x_N^{c_N+r-1} e^{-(-\gamma_N-1)x_N^{c_N}}\, dx_N.$$

Then we have:

$$\grave{\mu}_{N_r}(y_N) = \frac{B\Gamma\left(\frac{r}{c_N}, (-\gamma_N-1)x_N^{c_N}\right)}{c_N(-\gamma_N-1)^{\frac{r}{c_N}+1}}. \tag{3.15}$$

3.6 Rényi Entropy

One may obtain the Rényi entropy of the NGoC distribution as follows [20] from equation (3.4):

$$I_R(\nu_N)_{NGoC} = \frac{1}{1-\nu_N}\log\left[\mathrm{E}\int_0^{\infty} x_N^{\nu_N c_N-\nu_N} e^{(\nu_N+\sigma_N)x_N^{c_N}}\, dx_N\right].$$

Next, we have the NGoC's Rényi entropy by form:

$$I_R(\nu_N)_{NGoC} = \frac{1}{1-\nu_N}\log\left[\frac{\mathrm{E}\Gamma(\nu_N - \frac{\nu_N-1}{c_n})}{c_N(-\nu_N-\sigma_N)^{\nu_N-\frac{\nu_N}{c_N}+\frac{1}{c_N}}}\right]. \tag{3.16}$$

4. Estimation

4.1 maximum likelihood estimation

The NGoC distribution's parameters are determined using the maximum likelihood estimation method. We calculate the log-likelihood function for a random sample of data points $x_{N_1}, x_{N_2}, \dots, x_{N_m}$. The distribution has the same Npdf as the NGoC distribution [21].

$$L(\theta_N, x_{N_i}) = \prod_{i=1}^{n} \frac{r_N b_N c_N x_{i_N}^{c_N-1} e^{x_{i_N}^{c_N}} e^{\left[u_N \frac{1-e^{b_N\left(1-e^{x_{i_N}^{c_N}}\right)}}{e^{b_N\left(1-e^{x_{i_N}^{c_N}}\right)}}\right]}}{e^{b_N\left(1-e^{x_{i_N}^{c_N}}\right)}}\, e^{\frac{r_N}{u_N}\left(1-e^{\left[u_N \frac{1-e^{b_N\left(1-e^{x_{i_N}^{c_N}}\right)}}{e^{b_N\left(1-e^{x_{i_N}^{c_N}}\right)}}\right]}\right)},$$

where the distribution's parameters are θ_N. Next, the log-likelihood function L can be found as follows:

$$L = m\log(r_N) + m\log(b_N) + m\log(c_N) + (c_N-1)\sum_{i=1}^{m}\log x_{i_N} + u_N\sum_{i=1}^{m}\frac{1-e^{b_N\left(1-e^{x_{i_N}^{c_N}}\right)}}{e^{b_N\left(1-e^{x_{i_N}^{c_N}}\right)}}$$
$$+\frac{r_N}{u_N}\sum\nolimits_{i=1}^{m}\left(1-e^{\left[u_N\frac{1-e^{b_N\left(1-e^{x_{i_N}^{c_N}}\right)}}{e^{b_N\left(1-e^{x_{i_N}^{c_N}}\right)}}\right]}\right) - b_N\sum_{i=1}^{m}\left(1-e^{x_{i_N}^{c_N}}\right). \tag{4.16}$$

4.2 Least Square Estimation

The parameters can be estimated using the Least Squares Estimation (LSE) method using the equation below [22]:

$$\varphi(r_N, u_N, b_N, c_N) = \sum_{i=1}^{m}\left[F_{NGoC}(x_{N_i}, r_N, u_N, b_N, c_N) - \frac{i}{n+1}\right]^2$$

$$\varphi(r_N, u_N, b_N, c_N) = \sum_{i=1}^{n} \left[1 - e^{\frac{r_N}{u_N}\left(1 - e^{\left[u_N \frac{1 - e^{b_N\left(1 - e^{x_{i_N}^{c_N}}\right)}}{e^{b_N\left(1 - e^{x_{i_N}^{c_N}}\right)}}\right]}\right)} - \frac{i}{n+1} \right]^2. \tag{4.2}$$

4.3 Weighted Least Square Estimation

The weighted least squares estimators can be obtained by the equation [23]:

$$\mathrm{W}\,(r_N, u_N, b_N, c_N) = \sum_{i=1}^{m} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F_{NGoC}\left(x_{N_i}, r_N, u_N, b_N, c_N\right) - \frac{i}{n+1} \right]^2$$

$$\mathrm{W}\,(r_N, u_N, b_N, c_N) = \sum_{i=1}^{m} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[1 - e^{\frac{r_N}{u_N}\left(1 - e^{\left[u_N \frac{1 - e^{b_N\left(1 - e^{x_{i_N}^{c_N}}\right)}}{e^{b_N\left(1 - e^{x_{i_N}^{c_N}}\right)}}\right]}\right)} - \frac{i}{n+1} \right]^2. \tag{4.3}$$

By partially differentiating the above equation with respect to the parameters a_N, b_N, and c_N, and by equating the resulting expressions to zero for three different methods (MLE, LSE, and WLSE), numerical techniques were employed to obtain the solutions. Programs like R, MAPLE, SAS, and other similar tools were utilized to carry out these computations.

5. Simulation

The effectiveness of the NGoC distribution estimators was assessed using three methods: maximum likelihood estimation (MLE), least squares estimation (LSE), and weighted least squares estimation (WLSE). These methods were applied within a Monte Carlo simulation using the R programming language. The simulation involved sample sizes of 50, 100, 150, and 200, with 1,000 samples drawn based on the true parameter values provided in Tables 3 and 4. The estimators' mean values were computed by averaging the estimated model parameters, and then bias, mean squared error (MSE), and root mean square error (RMSE) were determined.

From table 3:

- Best Estimation Method: The best estimation method for each parameter is determined by comparing the RMSE and bias across the different methods and sample sizes. Generally, the method with the smallest RMSE and bias is considered the best because it indicates the most accurate and stable parameter estimates.
- Best Parameter and Sample Size: For each parameter, the best sample size is the one that results in the lowest RMSE and bias. Here is the analysis for each parameter:

1. r_N: the best Method is MLE consistently shows the best performance for r_N with the lowest RMSE values. While the Best Parameter Estimate: The estimate of r_N for sample size 100 using MLE.
2. u_N: the best Method is WLSE consistently shows the best performance for u_N with relatively low RMSE and bias compared to MLE and LSE. While the Best Parameter Estimate: The estimate of u_N for sample size 200 using WLSE.
3. b_N: the best Method is WLSE consistently shows the best performance for b_N particularly at higher sample sizes. While the Best Parameter Estimate: The estimate of b_N for sample size 200 using WLSE.
4. c_N: the best Method is WLSE consistently shows the best performance for c_N particularly at higher sample sizes. While the Best Parameter Estimate: The estimate of c_N for sample size 200 using WLSE.

Table 3 : Monte Carlo simulations conducted for the NGoC distribution

$r_N = [0.5,0.9]$, $u_N = [0.5,0.8]$, $b_N = [0.5,1.1]$, $c_N = [0.5,1.3]$					
N	Est.	Est.par	MLE	LSE	WLSE
50	MEAN	$\widehat{r_N}$	[0.6573868,0.6923286]	[0.4222256,0.6862838]	[0.4103989,0.6196573]
		$\widehat{u_N}$	[0.4550581,0.48817528]	[0.4403788,0.45770812]	[0.53119148, 0.6064048]
		$\widehat{b_N}$	[0.51676931, 1.5989660]	[0.51452212, 1.6344543]	[0.51968733, 1.4086956]
		$\widehat{c_N}$	[0.53922882, 1.24438004]	[0.502692797, 0.9531486]	[0.49558728, 0.8613243]
	MSE	$\widehat{r_N}$	[0.3984671,0.4039383]	[0.4245629,0.4446816]	[0.3017732, 0.4752799]
		$\widehat{u_N}$	[0.13353497, 0.2396458]	[0.25265676, 0.3584837]	[0.2113670,0.21406968]
		$\widehat{b_N}$	[0.03717241, 0.5106597]	[0.02451089, 0.6991701]	[0.01974397, 0.3152480]
		$\widehat{c_N}$	[0.02649800, 0.13166018]	[0.041345705, 0.3519297]	[0.04012085, 0.4189753]
	RMSE	$\widehat{r_N}$	[0.6312425,0.6355614]	[0.6515849,0.6668445]	[0.5493388, 0.6894054]
		$\widehat{u_N}$	[0.36542437, 0.4895363]	[0.50264974, 0.5987351]	[0.4597467,0.46267665]
		$\widehat{b_N}$	[0.19280147, 0.7146046]	[0.15655953, 0.8361639]	[0.14051324, 0.5614695]
		$\widehat{c_N}$	[0.16278206, 0.36285008]	[0.203336434, 0.5932366]	[0.20030190, 0.6472831]
	BAIS	$\widehat{r_N}$	[0.1923286, 0.2426132]	[0.1862838, 0.4777744]	[0.1196573, 0.4896011]
		$\widehat{u_N}$	[0.01182472, 0.3449419]	[0.04229188, 0.3596212]	[0.03119148, 0.1935952]
		$\widehat{b_N}$	[0.01676931, 0.4989660]	[0.01452212, 0.5344543]	[0.01968733, 0.3086956]
		$\widehat{c_N}$	[0.03922882, 0.05561996]	[0.002692797, 0.3468514]	[0.00441272, 0.4386757]
100	MEAN	$\widehat{r_N}$	[0.6156806, 0.6174563]	[0.3956621,0.56537552]	[0.4001272,0.6129942]
		$\widehat{u_N}$	[0.3662515,0.52777481]	[0.3530496,0.5777070]	[0.5342626,0.56026246]
		$\widehat{b_N}$	[0.51020409, 1.6871841]	[0.51926799, 1.7574271]	[0.506525799, 1.4822062]
		$\widehat{c_N}$	[0.52311013, 1.25868392]	[0.47571982, 1.0108898]	[0.501073832, 0.9317387]
	MSE	$\widehat{r_N}$	[0.1989800, 0.2332555]	[0.23840953, 0.3485796]	[0.3389510, 0.3587053]
		$\widehat{u_N}$	[0.18002245, 0.2584675]	[0.1998481, 0.3335787]	[0.20925002, 0.2127577]
		$\widehat{b_N}$	[0.02399339, 0.6659458]	[0.01754680, 0.7823143]	[0.018610543, 0.3212667]
		$\widehat{c_N}$	[0.02007337, 0.09173520]	[0.03186424, 0.2425169]	[0.027708850, 0.2971875]
	RMSE	$\widehat{r_N}$	[0.4460717, 0.4829653]	[0.48827198, 0.5904063]	[0.5821949, 0.5989201]
		$\widehat{u_N}$	[0.42429052, 0.5083970]	[0.4470437, 0.5775628]	[0.45743855, 0.4612566]
		$\widehat{b_N}$	[0.15489800, 0.8160550]	[0.13246432, 0.8844853]	[0.136420464, 0.5668040]
		$\widehat{c_N}$	[0.14168052, 0.30287819]	[0.17850557, 0.4924601]	[0.166459755, 0.5451490]
	BAIS	$\widehat{r_N}$	[0.1156806, 0.2825437]	[0.06537552, 0.5043379]	[0.1129942, 0.4998728]
		$\widehat{u_N}$	[0.02777481, 0.4337485]	[0.0777070, 0.4469504]	[0.06026246, 0.2657374]
		$\widehat{b_N}$	[0.01020409, 0.5871841]	[0.01926799, 0.6574271]	[0.006525799, 0.3822062]
		$\widehat{c_N}$	[0.02311013, 0.04131608]	[0.02428018, 0.2891102]	[0.001073832, 0.3682613]
150	MEAN	$\widehat{r_N}$	[0.6075063, 0.6166587]	[0.4240316,0.6031415]	[0.4641645,0.56012606]
		$\widehat{u_N}$	[0.3553297,0.48328167]	[0.3122338,0.5214446]	[0.4360072,0.55900054]
		$\widehat{b_N}$	[0.51226976, 1.6923649]	[0.491341945, 1.8474660]	[0.51629864, 1.5629580]
		$\widehat{c_N}$	[0.52911990, 1.25886934]	[0.51017166, 1.0725798]	[0.48234310, 1.0387190]
	MSE	$\widehat{r_N}$	[0.1663787, 0.2175777]	[0.1625841, 0.3360281]	[0.21723084, 0.3099988]
		$\widehat{u_N}$	[0.10527252, 0.2947929]	[0.1419951, 0.3498539]	[0.13301655, 0.2447740]
		$\widehat{b_N}$	[0.02169005, 0.6325044]	[0.011307771, 0.9521456]	[0.01300212, 0.3611658]
		$\widehat{c_N}$	[0.01622061, 0.06152406]	[0.02560594, 0.1803753]	[0.01805815, 0.2365433]
	RMSE	$\widehat{r_N}$	[0.4078955, 0.4664522]	[0.4032171, 0.5796793]	[0.46608029, 0.5567753]
		$\widehat{u_N}$	[0.32445727, 0.5429484]	[0.3768223, 0.5914845]	[0.36471435, 0.4947464]
		$\widehat{b_N}$	[0.14727541, 0.7953015]	[0.106338003, 0.9757795]	[0.11402683, 0.6009707]
		$\widehat{c_N}$	[0.12736017, 0.24804044]	[0.16001855, 0.4247061]	[0.13438060, 0.4863572]
	BAIS	$\widehat{r_N}$	[0.1075063, 0.2833413]	[0.1031415, 0.4759684]	[0.06012606, 0.4358355]
		$\widehat{u_N}$	[0.01671833, 0.4446703]	[0.0214446, 0.4877662]	[0.05900054, 0.3639928]
		$\widehat{b_N}$	[0.01226976, 0.5923649]	[0.008658055, 0.7474660]	[0.01629864, 0.4629580]
		$\widehat{c_N}$	[0.02911990, 0.04113066]	[0.01017166, 0.2274202]	[0.01765690, 0.2612810]
200	MEAN	$\widehat{r_N}$	[0.5636522,0.6078878]	[0.4054811,0.54881493]	[0.4394459,0.52710797]
		$\widehat{u_N}$	[,0.3441115,0.503200481]	[0.2969170,0.53179663]	[0.4092571,0.59783784]
		$\widehat{b_N}$	[0.504697398, 1.7181177]	[0.506021512, 1.8718031]	[0.508851729, 1.6314746]
		$\widehat{c_N}$	[0.52602308, 1.24129830]	[0.498438208, 1.0738340]	[0.48128351, 1.0581204]
	MSE	$\widehat{r_N}$	[0.1444841, 0.1958855]	[0.08306808, 0.3257807]	[0.12807537, 0.2896013]
		$\widehat{u_N}$	[0.121289410, 0.3098474]	[0.12192944, 0.3631464]	[0.14552498, 0.2784126]
		$\widehat{b_N}$	[0.020287666, 0.6510553]	[0.00945492, 0.9522372]	[0.010049585, 0.4647088]
		$\widehat{c_N}$	[0.01406573, 0.05831673]	[0.016778, 0.1555466]	[0.01572286, 0.2152323]
	RMSE	$\widehat{r_N}$	[0.3801107, 0.4425896]	[0.28821534, 0.5707720]	[0.35787619,, 0.5381461]
		$\widehat{u_N}$	[0.348266292, 0.5566394]	[0.34918397, 0.6026163]	[0.38147736, 0.5276481]
		$\widehat{b_N}$	[0.142434778, 0.8068800]	[0.097236411, 0.9758264]	[0.100247617, 0.6816955]
		$\widehat{c_N}$	[0.11859901, 0.24148858]	[0.12952992, 0.3943939]	[0.12539083, 0.4639313]
	BAIS	$\widehat{r_N}$	[0.1078878, 0.3363478]	[0.04881493, 0.4945189]	[0.02710797, 0.4605541]
		$\widehat{u_N}$	[0.003200481, 0.4558885]	[0.03179663, 0.5030830]	[0.09783784, 0.3907429]
		$\widehat{b_N}$	[0.004697398, 0.6181177]	[0.006021512, 0.7718031]	[0.008851729, 0.5314746]
		$\widehat{c_N}$	[0.02602308, 0.0587017]	[0.001561792, 0.2261660]	[0.01871649, 0.2418796]

The Best Overall Method: Weighted Least Squares Estimation (WLSE) is generally the most reliable method for parameter estimation, especially for larger sample sizes. Best Sample Size: 200 is the optimal sample size for most parameters, yielding the lowest RMSE and bias. These estimates, along with their respective methods and sample sizes, provide the most accurate reflection of the NGoC distribution's parameters based on the simulation results.

6. Application

We propose a practical approach that incorporates two types of data to evaluate the performance of the NGoC distribution in fitting the data. The results highlight the benefits of using the NGoC distribution and show its strong fit to the data. The comparison relies on eight different criteria. Among these are two statistical metrics: the Kolmogorov-Smirnov (KS) statistic and the Anderson-Darling (A) statistic. Additionally, the Cramer-von Mises (W) statistic is included, along with several information criteria such as HQIC, BIC, AIC, and CAIC. The p-value from the Kolmogorov-Smirnov test is also used to assess the accuracy of the fit. These measures are standard in evaluating goodness-of-fit.

- **Neotrosophic Beta Chen (NBeC)**
- **Neotrosophic Kumaraswamy Chen (NKuC)**
- **Neotrosophic [0,1] Trucked Exponential Exponential Chen (NTEEC)**
- **Neotrosophic Lomax Chen (NLC)**
- **Neotrosophic [0,1] Nadarajah Haghighi Chen (N[0,1] NHC)**

The Dataset I: real dataset related to the monthly low and high temperatures of Lahore, Pakistan for the last five years (2016–2020). The data observations are collected from World Weather Online [27].

Table 4: Descriptive statistics for the data

Var	N	Mean	SD	Median	Trimmed	Mad	Min	Max	Range	SK	KU	Se
1	60	70.45, 91.62	13.35, 14.71	72, 95.5	70.92, 92.27	14.08, 18.53	43, 64	94, 114	50, 51	-0.38, -0.23	-1.33, -1.22	1.72, 1.9

Table 4 provides basic descriptive statistics (mean, standard deviation, median, skewness, kurtosis, etc.) for a real dataset of monthly temperatures. These statistics describe the distribution and variability of the temperature data. The data's mean, standard deviation, and range indicate its central tendency and variability. The skewness and kurtosis values suggest whether the data is normally distributed or if it has heavier tails or asymmetry, which is important for understanding extreme temperature behavior.

Table 5: Estimates of models for data

Dist.	-2L	AIC	CAIC	BIC	HQIC
NGoC	[239.492, 245.3568]	[486.984, 498.7136]	[487.7112, 499.4409]	[495.3613, 507.091]	[490.2608, 501.9905]
NBEC	[241.0583, 246.305]	[490.1165, 500.6099]	[490.8438, 501.3372]	[498.4939, 508.9873]	[493.3934, 503.8868]
NKuC	[239.5136, 246.4272]	[487.0271, 500.8544]	[487.7544, 501.5817]	[495.4045, 509.2318]	[490.304, 504.1313]
NTEEC	[248.6309, 260.3303]	[505.2618, 528.6606]	[505.989, 529.3879]	[513.6392, 537.038]	[508.5386, 531.9374]
NLC	[244.345, 249.0665]	[496.69, 506.1331]	[497.4173, 506.8603]	[505.0674, 514.5104]	[499.9669, 509.4099]
Ne[0,1] NHC	[269.6427, 278.4263]	[547.2854, 564.8526]	[548.0126, 565.5798]	[555.6628, 573.2299]	[550.5622, 568.1294]

Table 5 compares various models (NGoC, NBEC, NKuC, etc.) by fitting them to the dataset using metrics like the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and log-likelihood (-2L). Lower AIC and BIC values indicate a better fit. From this table, the NGoC model appears to have a good fit for the data compared to other models. This is crucial for selecting the best statistical model for forecasting and decision-making.

Table 6: Evaluate statistical metrics for the data

Dist.	W	A	K-S	p-value
NGoC	[0.1481045, 0.1732993]	[0.9930809, 1.081373]	[0.101182, 0.1108151]	[0.4526935, 0.5707895]
NBEC	[0.2411097, 0.3246123]	[1.42744, 1.859337]	[0.1460358, 0.1510966]	[0.1291546, 0.1546656]
NKuC	[0.2433492, 0.2786144]	[1.441059, 1.600843]	[0.1412629, 0.1461473]	[0.1540631, 0.1822754]
NTEEC	[0.2160051, 0.3692855]	[1.289485, 2.115399]	[0.1506204, 0.2632144]	[0.0004902127, 0.131399]
NLC	[0.1743729, 0.2666631]	[1.022327, 1.547068]	[0.1580346, 0.1580848]	[0.09986171, 0.09967188]
Ne[0,1] NHC	[0.2159803, 0.2837328]	[1.287564, 1.628228]	[0.2791387, 0.3672622]	[1.869112e-07, 0.0001738941]

This table lists statistical metrics such as the Kolmogorov-Smirnov (K-S) statistic and Anderson-Darling (A) statistic for evaluating the goodness-of-fit of different models to the data. The smaller values of W, A, and K-S suggest a better fit. A p-value closer to 1 indicates that the model fits the data well. The NGoC model shows relatively good performance across these metrics, making it a strong candidate for further analysis.

Table 7: parameter estimators by MLE for the data

Dist.	$\hat{r}_N$	$\hat{u}_N$	$\hat{b}_N$	$\hat{c}_N$
NGoC	[0.00525928, 0.01450011]	[0.27956978, 2.17720096]	[0.05963716, 0.11778285]	[0.21104274, 0.28738177]
NBEC	[9.81920689, 34.9337914]	[2.76577423, 6.04046707]	[0.01604209, 0.03160685]	[0.31491079, 0.35945136]
NKuC	[26.93315570, 34.611994]	[7.5249652, 20.10717927]	[0.06064647, 0.1537193]	[0.2520067, 0.28257767]
NTEEC	[0.026674116, 70.336943]	[2.176309473, 120.3337805]	[0.002169374, 0.00384889]	[0.24555331, 0.439039440]
NLC	[0.6613639, 10.34333054]	[0.106353563, 1.637366311]	[0.001289686, 0.00322373]	[0.403930976, 0.50048886]
Ne[0,1] NHC	[0.998908228, 0.9989559]	[0.07353235, 0.166810506]	[0.002096039, 0.00312738]	[0.403167218, 0.40983715]

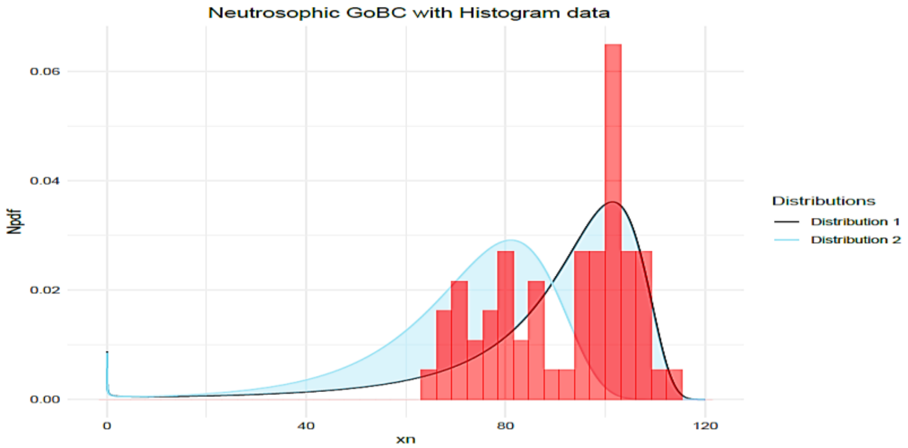

Figure 7: Fitting pdfs NGoC with histogram data set

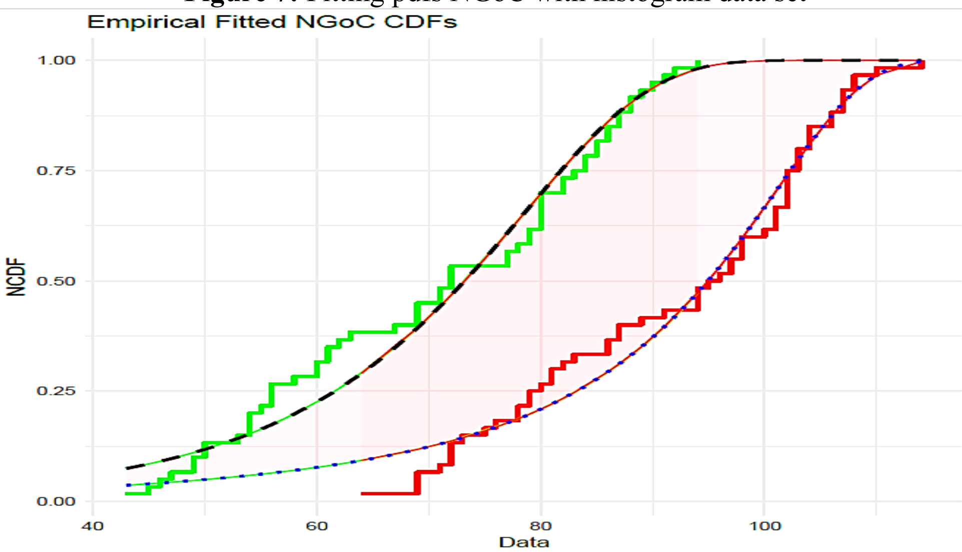

Figure 8: Empirical Fitted CDFs NGoC with histogram data set

Figure 7 compares the Probability Density Function (PDF) of the NGoC distribution with an actual dataset. The histogram represents the observed data, while the fitted PDF shows how well the NGoC distribution models it. A close fit between the histogram and the NGoC PDF suggests that the NGoC distribution is appropriate for modeling the dataset. Any significant deviations might indicate that the model needs further refinement or that additional parameters must be adjusted. While figure 8 shows the empirical cumulative distribution function (CDF) fitted to a histogram of the actual dataset. Like Figure 7, this figure assesses how well the NGoC distribution models the cumulative probabilities of the data. The closeness of the empirical CDF to the fitted CDF reveals the accuracy of the NGoC model. Any significant gaps between the two could suggest that the model requires adjustments to better capture the data's cumulative behavior.

7. Conclusions

This research introduces a new four-parameter distribution named the Neutrosophic Gompertz-Chen (NGoC) distribution. This distribution expands the classical two-parameter Chen distribution by incorporating elements from the Gompertz family and neutrosophic logic. The integration of neutrosophic logic allows for the inclusion of indeterminate values in the model, making it suitable for handling uncertain or ambiguous data, which is common in real-world applications. the study thoroughly examines the mathematical properties of the NGoC distribution, including the Neutrosophic Cumulative Distribution Function (NCDF), Neutrosophic Probability Density Function (Npdf), survival functions, moments, variance, skewness, and kurtosis. The flexibility of the distribution is demonstrated by its ability to model data with varying levels of uncertainty in the parameters, allowing it to adapt to a wide range of applications that deal with vague or imprecise data. The parameters of the NGoC distribution were estimated using three different methods: Maximum Likelihood Estimation (MLE), Least Squares Estimation (LSE), and Weighted Least Squares Estimation (WLSE). A Monte Carlo simulation was conducted to compare these methods based on their Mean Squared Error (MSE), Root Mean Squared Error (RMSE), and bias for different sample sizes (50, 100, 150, and 200). WLSE consistently performed better than MLE and LSE, particularly for larger sample sizes, with lower RMSE and bias values. This makes WLSE the most reliable method for parameter estimation in the NGoC distribution. The NGoC distribution was compared to five other neutrosophic distributions (NBEC, NKuC, NTEEC, NLC, and Ne[01]NHC) using various statistical measures, including the Kolmogorov-Smirnov (K-S) test, Anderson-Darling (A) test, and Cramer-von Mises (W) test. The NGoC distribution showed a strong fit to the data, with smaller values for these tests, indicating better performance in terms of goodness-of-fit. By incorporating neutrosophic logic, this research enhances the ability of statistical models to handle uncertainty, ambiguity, and incomplete information. This is particularly important in fields like engineering, medicine, and social sciences, where data often contain indeterminate values. The NGoC distribution, with its flexibility, is well-suited for such applications.

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