Innovation in Statistics and Probability
1(1), 47–59
DOI: https://doi.org/10.64389/isp.2025.01104
https://sphinxsp.org/journal/index.php/isp/

*Research article*

# Analytical Modeling of Expansion for Odd Lomax Generalized Exponential Distribution in Framework of Neutrosophic Logic: a Theoretical and Applied on Neutrosophic Data

**Nooruldeen A. Noori [1], Mundher A. Khaleel [2], Sara A. Khalaf [3], Subhankar Dutta**

[1] Anbar Education Directorate, Anbar 31001, Iraq; Nooruldeen.a.noori35508@st.tu.edu.iq

[2] Department of mathematics, College of Computer Science and Mathematics, University of Tikrit, Tikrit 28001, Iraq; mun880088@tu.edu.iq

[2] Department of mathematics, College of Computer Science and Mathematics, University of Tikrit, Tikrit 28001, Iraq; sara.a.khalaf@st.tu.edu.iq

[3] Department of Mathematics, Bioinformatics and Computer Applications, Maulana Azad National Institute of Technology, Bhopal 462003, India; subhankar@manit.ac.in

* **Correspondence:** Nooruldeen.a.noori35508@st.tu.edu.iq

## ARTICLE INFO

**Keywords:**

Neutrosophic Logic, Neutrosophic data, Neutrosophic simulation, NPDF, Incomplete Moments.

**Mathematics Subject Classification:**
62A86, 62F10, 62F25

**Important Dates:**
Received: 18 May 2025
Revised: 29 May 2025
Accepted: 23 June 2025
Online: 26 June 2025



## ABSTRACT

This study aims to extend the Odd Lomax Generalized Exponential (OLGE) distribution to include neutrosophic data, which are generalized by uncertainty and ambiguity. It's done formulate a new probabilistic model based on combining Neutrosophic Logic (NL) with Odd Lomax Generalized Exponential to improve the model's flexibility in dealing with data with uncertain and contradictory shapes. The density function and probability distribution of the proposed neutrosophic model are defined, and some mathematical properties are derived as neutrosophic survival, neutrosophic hazard, Incomplete Moments, and Neutrosophic Quantile. In addition, we present new method for parameter estimation using neutrosophic simulation for three techniques (MLE, LSE, WLSE), and compare the model's performance with other model using information criteria and statistical measures. The model is applied to a real neutrosophic data set characterized by uncertainty (the life in 100 hours of 23 batteries), demonstrating its efficiency in analyzing ambiguous data when compared to other neutrosophic distributions.

## 1. Introduction

In recent years, modeling data with ambiguous of uncertain nature has become a major challenge in many fields such as engineering, medical sciences, and economics. Despite significant progress in the development of probability distributions, most current models rely on traditional data assumptions that assume certainty and clarity, limiting their effectiveness in dealing with neutrosophic data characterized by uncertainty and contradiction. Therefore, a number of neutrosophic distributions have recently emerged that address this type of data. Examples include: neutrosophic HWIR [1], Neutrosophic Lindley [2], neutrosophic Generalized pareto [3], neutrosophic exponentiated inverse Rayleigh [4], neutrosophic inverse Gompertz [5], neutrosophic Beta-Lindley [6], neutrosophic Burr XII [7], Neutrosophic inverse power Lindley [8], Neutrosophic Topp-Leone [9], and neutrosophic Topp-Leone [8].

Despite the introduction of these neutrosophic distributions, adding additional parameters to the underlying distribution and integrate it with NL is almost nonexistent. Therefore, the underlying distribution is combined with Odd Lomax family, there the Odd Lomax Generalized Exponential has CDF and PDF functions, respectively, in the form [9]:

$$F(x) = 1 - \left(1 - \frac{(1-e^{-bx})^c . log(1-(1-e^{-bx})^c)}{u}\right)^{-r} \tag{1.1}$$

$$f(x) = \frac{rbce^{-bx}(1-e^{-bx})^{c-1}\left[\frac{(1-e^{-bx})^c}{1-(1-e^{-bx})^c} - \log(1-(1-e^{-bx})^c)\right]}{u\left(1 - \frac{(1-e^{-bx})^c . log(1-(1-e^{-bx})^c)}{u}\right)^{(r+1)}} \tag{1.2}$$

The research gap lies in the lack of probability distributions capable of integrating NL, which addresses ambiguity and uncertainty using concepts such as neutrosophic sets. Therefore, this study aims to develop a theoretical and applied framework for OLGE in neutrosophic context, allowing for more flexible analysis of complex data.

The study aims to develop a neutrosophic version of OLGE distribution by defining the density functions and probability distribution, deriving the mathematical properties of the model, such as quantile function and incomplete Moments, introducing new estimation methods based on neutrosophic simulations, and the comparing the performance of the proposed model with other models using statistical criteria. The model was applied to real data related to the lifetime of 23 batteries over 100 hours, demonstrating its superiority in handling ambiguous data compared to traditional distributions. This study contributes to bridging the gap between probability theory and NL, opening new horizons for applications in multiple fields.

## 2. Neutrosophic Odd Lomax Generalized Exponential

Assume $X_N = d + tI, tI \in [X_L, X_U]$, where $X_L, X_U$ are lower and upper values of the neutrosophic Odd Lomax Generalized Exponential distribution (NOLGE) random variable having determined part $d$ and indeterminate part $tI$, $tI \in [I_L, I_U]$. Note that the NOLGE reduces to classical Chen when $X_L = X_U$. The neutrosophic cumulative density (NCDF), and neutrosophic probability density (NPDF) of NOLGE has a Neutrosophic shape parameters $r_N \in [r_L, r_U]$, $u_N \in [u_L, u_U] b_N \in [b_L, b_U]$, and $c_N \in [c_L, c_U]$, has the form [9]:

$$F(x_N) = 1 - \left(1 - \frac{(1-e^{-b_N x_N})^{c_N} . log(1-(1-e^{-b_N x_N})^{c_N})}{u_N}\right)^{-r_N} \tag{2.1}$$

$$f(x_N) = \frac{r_N b_N c_N e^{-b_N x_N}\left[\frac{\left(1-e^{-b_N x_N}\right)^{c_N}}{1-\left(1-e^{-b_N x_N}\right)^{c_N}} - \log(1-\left(1-e^{-b_N x_N}\right)^{c_N})\right]}{u_N\left(1-e^{-b_N x_N}\right)^{1-c_N}\left(1 - \frac{\left(1-e^{-b_N x_N}\right)^{c_N} . log\left(1-\left(1-e^{-b_N x_N}\right)^{c_N}\right)}{u_N}\right)^{(r_N+1)}} \tag{2.2}$$

The neutrosophic survival function ($S_N$) has a form [1]:

$$S_N(x_N) = \left(1 - \frac{(1-e^{-b_N x_N})^{c_N} . log(1-(1-e^{-b_N x_N})^{c_N})}{u_N}\right)^{-r_N} \tag{2.3}$$

While the neutrosophic hazard function ($h_N$) has a form [11]:

$$h_N(x_N) = \frac{r_N b_N c_N e^{-b_N x_N}\left[\frac{\left(1-e^{-b_N x_N}\right)^{c_N}}{1-\left(1-e^{-b_N x_N}\right)^{c_N}} - \log(1-\left(1-e^{-b_N x_N}\right)^{c_N})\right]}{u_N\left(1-e^{-b_N x_N}\right)^{1-c_N}\left(1 - \frac{\left(1-e^{-b_N x_N}\right)^{c_N} . log\left(1-\left(1-e^{-b_N x_N}\right)^{c_N}\right)}{u_N}\right)} \tag{2.4}$$

Figure 1 shows the NCDF function with varying parameter intervals, while Figure 2 shows the NPDF function with varying parameter values. Figure 3 provides a three-dimensional representation of NCDF, while Figure 4 shows a three-dimensional representation of the NPDF. Figure 5 illustrates the neutrosophic survival function curve with varying parameter intervals.

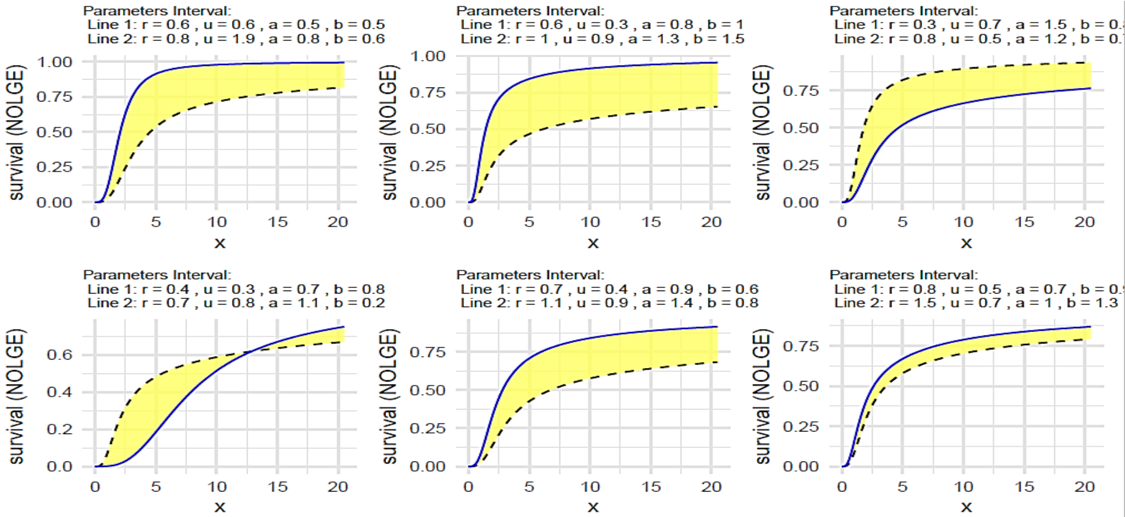



**Figure.1** plot the NCDF function for NOLGE

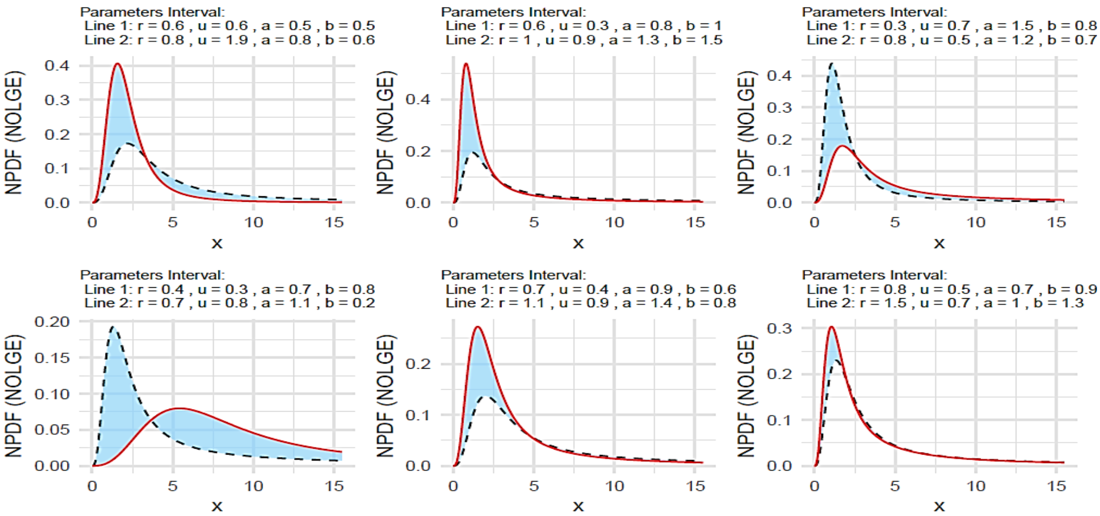



**Figure.2** plot the NPDF function for NOLGE

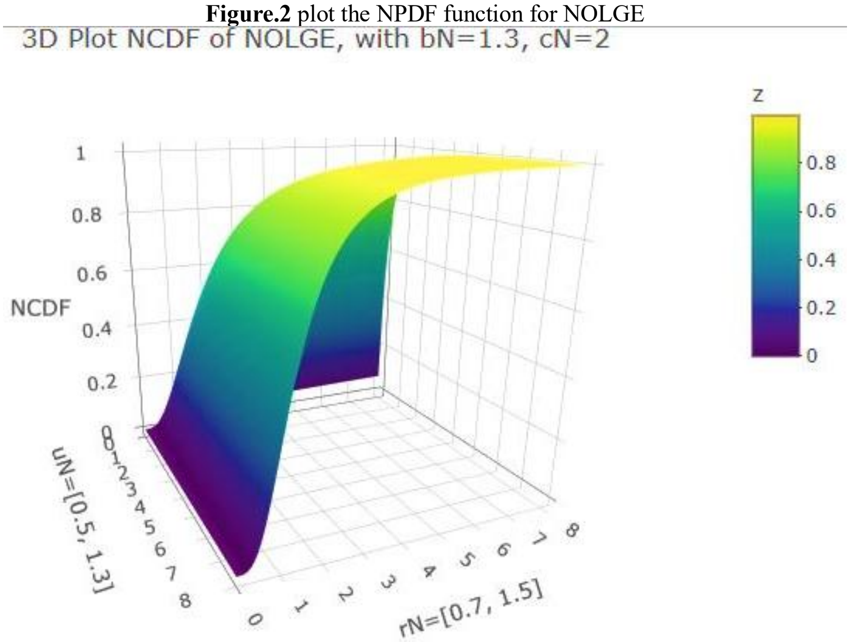



**Figure.3** 3D-plot the NCDF function for NOLGE

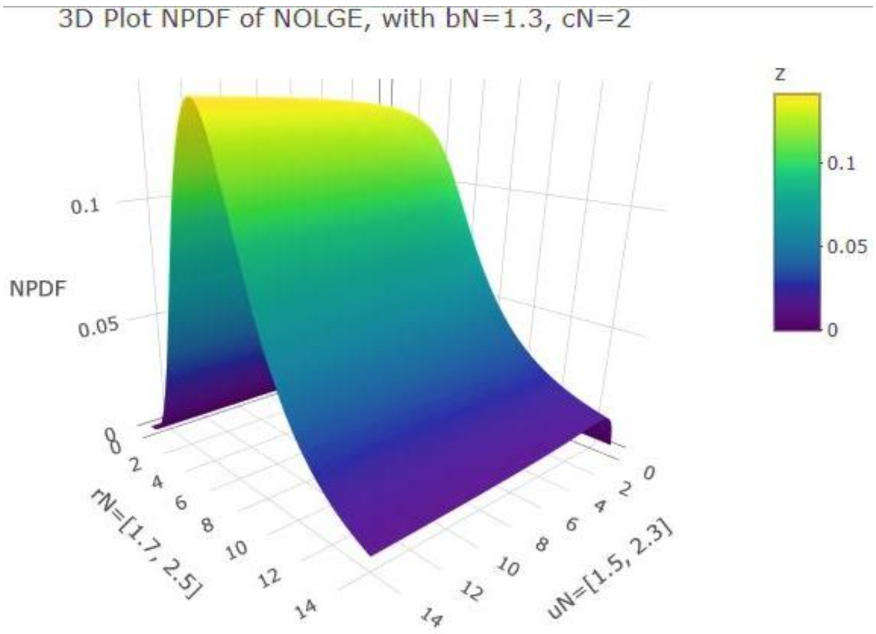



**Figure.4** 3D-plot the NCDF function for NOLGE

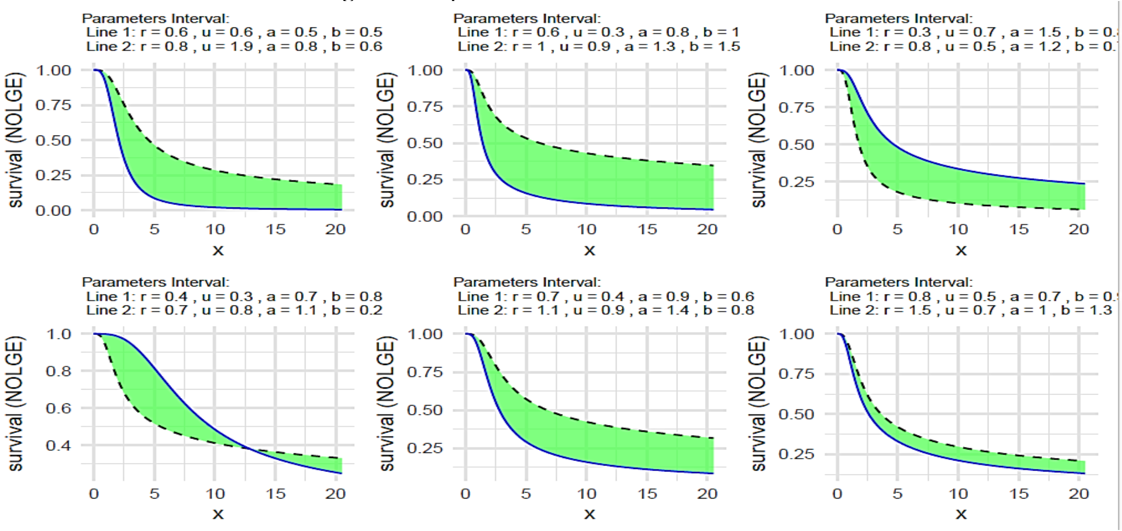



**Figure.5** plot the survival function for NOLGE

Figure 1 shows the curves illustrating how the shape of function changes at the parameters take values within uncertain intervals. The variance in the curves reflects the flexibility of distribution in representing uncertain data, as the distribution can adapt to uncertainties and ambiguities in data.

Figure 2 shows that shape of function changes based on parameters values, demonstrating distribution's ability to model data with divers shapes (such as skewed or multi-peaked data). This flexibility makes the distribution suitable for analyzing complex data characterized by uncertainty.

Figure 3 shows the relationship between random variable and uncertain parameters in three dimensions. The three dimensions representation helps understand how the parameters interact with each other and how they affect the shape of the cumulative function.

Figure 4 shows the NCDF in three dimensions, showing how the density changes as the parameters change. This representation is useful for understanding the behavior of distribution when modeling uncertain data, especially in case where the data is uncertain or inconsistent.

Figure 5 depicts the survival function, which is important in analyzing temporal data, such as battery life data mentioned in the study, as it helps estimate the probability of a system surviving for specific period of time.

These figures enhance understanding of the mathematical properties and practical applications of NOLGE distribution, demonstrating its superiority over conventional distributions in analyzing uncertain data.

## 3. Properties for NOLGE distribution

### *3.1 Expansion Basic functions for NOLGE*

The NCDF expansion for NOLGE distribution using binomial series, logarithm expansion and exponential expansion series has a form [11], [12]:

$$F(x_N) = 1 - \mathrm{B}e^{-z_N b_N x_N} \tag{3.1}$$

where $\mathrm{B} = \sum_{k_N=i_N=v_N=j_N=z_N=0}^{\infty} \frac{\Gamma(r_N+k_N) u_N^{-k}(-1)^{i_N+j_N+v_N+z_N} d_{k_N,i_N}}{k_N!\Gamma(r_N)} \binom{2k_N+1}{j_N}\binom{j_N}{v_N}\binom{v_N c_N}{z_N}$

The NPDF expansion for NOLGE distribution by same way in NCDF expansion has a form:

$$f(x_N) = \phi e^{-(p_N+1)b_N x_N} - \psi e^{-(q_N+1)b_N x_N} \tag{3.2}$$

where

$$\phi = \sum_{k_N=i_N=t_N=j_N=z_N=p_N=0}^{\infty} \frac{(-1)^{i_N+t_N+j_N+z_N+p_N} d_{k_N,i_N} r_N\, \Gamma(r_N+1+k_N)}{k_N!\,\Gamma(r_N+1) u_N^{(k_N+1)}} c_N b_N \binom{2k_N+i_N}{j_N}\binom{j_N}{t_N}\binom{(t_N+z_N+1)c_N-1}{p_N}$$

and

$$\psi = \sum_{k_N=l_N=s_N=n_N=q_N=0}^{\infty} \frac{r_N(-1)^{i_N+s_N+n_N+q_N} d_{k_N,i_N} r_N\, \Gamma(r_N+1+k_N)}{k k_N!\,\Gamma(r_N+1) u_N^{(k_N+1)}} c_N b_N \binom{2k_N+l_N+1}{s_N}\binom{s_N}{n_N}\binom{(n_N+1)c_N-1}{q_N}$$

### *3.2 Neutrosophic Moments function for NOLGE*

Let $X_N$ be any neutrosophic random variable with NPDF in equation 8. The neutrosophic $m^{th}$ moment for NOLGE is calculated by form [13]:

$$\mu_{m,N} = E(x_N) = \int_0^\infty x_N^m\, f(x_N) dx_N$$

$$\mu_{m,N} = \phi \int_0^\infty x_N^m\, e^{-(p_N+1)b_N x_N} dx_N - \psi \int_0^\infty x_N^m\, e^{-(q_N+1)b_N x_N} dx_N$$

Then the final form for $\mu_{m,N}$ given as:

$$\mu_{m,N} = \frac{m!}{b_N^{m+1}}\left(\frac{\phi}{(p_N+1)^{m+1}} - \frac{\psi}{(q_N+1)^{m+1}}\right) \tag{3.3}$$

We can get a first four Neutrosophic Moments by forms:

$$\mu_{1,N} = \frac{1}{b_N^2}\left(\frac{\phi}{(p_N+1)^2} - \frac{\psi}{(q_N+1)^2}\right) \tag{3.4}$$

$$\mu_{2,N} = \frac{2}{b_N^3}\left(\frac{\phi}{(p_N+1)^3} - \frac{\psi}{(q_N+1)^3}\right) \tag{3.5}$$

$$\mu_{3,N} = \frac{6}{b_N^4}\left(\frac{\phi}{(p_N+1)^4} - \frac{\psi}{(q_N+1)^4}\right) \tag{3.6}$$

$$\mu_{4,N} = \frac{24}{b_N^5}\left(\frac{\phi}{(p_N+1)^5} - \frac{\psi}{(q_N+1)^5}\right) \tag{3.7}$$

The neutrosophic variance, neutrosophic skewness, and neutrosophic kurtoses respectively has a forms [14], [15]:

$$var_N(x_N) = \frac{2}{b_N^3}\left(\frac{\phi}{(p_N+1)^3} - \frac{\psi}{(q_N+1)^3}\right) - \frac{1}{b_N^4}\left(\frac{\phi}{(p_N+1)^2} - \frac{\psi}{(q_N+1)^2}\right)^2 \tag{3.8}$$

$$SK_N(x_N) = \frac{\frac{6}{b_N^4}\left(\frac{\phi}{(p_N+1)^4} - \frac{\psi}{(q_N+1)^4}\right)}{\left(\frac{2}{b_N^3}\left(\frac{\phi}{(p_N+1)^3} - \frac{\psi}{(q_N+1)^3}\right)\right)^{\frac{3}{2}}} \tag{3.9}$$

$$KU_N(x_N) = \frac{6\left(\frac{\phi}{(p_N+1)^5} - \frac{\psi}{(q_N+1)^5}\right)}{b_N\left(\frac{\phi}{(p_N+1)^3} - \frac{\psi}{(q_N+1)^3}\right)^2} \tag{3.10}$$

The table below provides a comprehensive statistical analysis of NOLGE based on various parameters. The table contains the statistical moments ($\mu_{1,N}, \mu_{2,N}, \mu_{3,N}, \mu_{4,N}$), variance ($var_N$), skewness coefficient ($SK_N$), and kurtosis coefficient ($KU_N$), with all values represented as neutrosophic intervals [lower, upper] that reflect the degree of uncertainty in the results.

**Table.1** statistical moments, variance, skewness, and kurtosis for selected parameter intervals of NOLGE

| $r_N$ | $u_N$ | $b_N$ | $c_N$ | $\mu_{1,N}$ | $\mu_{1,N}$ | $\mu_{1,N}$ | $\mu_{1,N}$ | $var_N$ | $SK_N$ | $KU_N$ |
|---|---|---|---|---|---|---|---|---|---|---|
| **[0.3,1.3]** | **[0.5,1.5]** | **[0.4,1.4]** | **[0.1,1.1]** | [0.058564, 0.299825] | [0.029881, 0.193303] | [0.020049, 0.139349] | [0.015084, 0.10776] | [0.026451, 0.103408] | [1.639623, 3.881547] | [2.883882, 16.89464] |
| | | | **[0.2,1.2]** | [0.097579, 0.299348] | [0.050177, 0.1971] | [0.033553, 0.143782] | [0.025167, 0.111996] | [0.040655, 0.107491] | [1.643135, 2.985262] | [2.882894, 9.995964] |
| | | **[0.6,1.6]** | **[0.3,1.3]** | [0.12956, 0.318846] | [0.067646, 0.207837] | [0.045186, 0.150303] | [0.033813, 0.116271] | [0.05086, 0.106175] | [1.586291, 2.568261] | [2.691684, 7.389141] |
| | | | **[0.4,1.4]** | [0.152708, | [0.083128, | [0.056171, | [0.042213, | [0.059809, | [1.589088, | [2.690553, |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | 0.318653] | 0.211455] | 0.154517] | 0.120304] | 0.109916] | 2.343623] | 6.108697] |
| [0.7,1.7] | [0.8,1.8] | [0.5,1.5] | [0.200816, 0.357159] | [0.106988, 0.23159] | [0.07097, 0.166189] | [0.052653, 0.127607] | [0.066661, 0.104027] | [1.491155, 2.028032] | [2.37923, 4.599972] |
| | | [0.6,1.6] | [0.221018, 0.358536] | [0.123009, 0.236296] | [0.082893, 0.171222] | [0.061916, 0.132276] | [0.074159, 0.107748] | [1.490643, 1.92139] | [2.369009, 4.092004] |
| | [0.9,1.9] | [0.7,1.7] | [0.237972, 0.367066] | [0.135233, 0.241892] | [0.091671, 0.17501] | [0.068581, 0.134959] | [0.078602, 0.107155] | [1.471057, 1.843362] | [2.30652, 3.750054] |
| | | [0.8,1.8] | [0.249138, 0.367714] | [0.147009, 0.245793] | [0.101336, 0.179396] | [0.07645, 0.13912] | [0.084939, 0.110579] | [1.472173, 1.79784] | [2.302782, 3.537481] |

Table 1 shows that the distribution is positive and right-skewed in all cases. Some parameters, particularly the kurtosis coefficient, have high uncertainty, which may indicate that the distribution is suitable for modeling data with extreme values. Increasing some parameters (such as $b_N$ and $c_N$) reduces the variance and skewness, making the distribution more stable. The distribution is also suitable for data with heavy tails (such as financial risk analysis and rare medical data). In short, this table provides a powerful tool for analyzing probability distributions under uncertainty, making it useful in real-world applications where data are uncertain or ambiguous.

### *3.3 Neutrosophic Quantile function for NOLGE*

The Neutrosophic Quantile function represents in the inverse of NCDF function and it expressed by the relationship [16]:

$$q_N = F(x_N)$$

For each $q_N \in (0,1)$ and $F(x_N)$ is NCDF for NOLGE distribution.

Then a final form for it can be expressed as:

$$x_N = \frac{-1}{b_N}\log\left\{1 - \left[\frac{\Theta}{\Theta + W_{-1}(\Theta e^{-\Theta})}\right]^{\frac{1}{c_N}}\right\}, \Theta = u_N - \frac{u_N}{(1-q_N)^{\frac{1}{\alpha}}} \tag{3.11}$$

The following table presents the quantile values of the NOLGE neutrosophic distribution for a range of different parameters.

**Table.2** The Neutrosophic Quantile for selected parameter intervals of NOLGE distribution

| $q_N$ | $r_N, u_N, b_N, c_N$ | | | | |
|---|---|---|---|---|---|
| | **[0.3,1.3],[0.5,1.5],[0.7,1.7],[0.9,1.9]** | **[0.6,1.6],[0.4,1.4],[0.8,1.8],[0.5,1.5]** | **[0.6,1.6],[0.3,1.3],[0.5,1.5],[0.7,1.7]** | **[0.7,1.7],[0.5,1.5],[0.9,1.9],[0.2,1.2]** | **[0.8,1.8],[0.9,1.9],[0.7,1.7],[0.4,1.4]** |
| 0.1 | [0.3402208, 0.4258823] | [0.1940932, 0.35240] | [0.4947905,0.7349827] | [0.00708521, 0.2578218] | [0.0662213, 0.3178744] |
| 0.2 | [0.5695547,0.5704234] | [0.45907, 0.4977096] | [0.6848535,1.5192220] | [0.0459328, 0.3824720] | [0.17026888, 0.450276] |
| 0.3 | [0.6991737,0.8317391] | [0.6349814,0.8618342] | [0.8642528,2.7368166] | [0.1553307, 0.5028108] | [0.316630,0.57234] |
| 0.4 | [0.8344206,1.1717438] | [0.7839862,1.5409015] | [1.0598887,5.0266177] | [0.4153965, 0.6347834] | [0.525888,0.7007074] |
| 0.5 | [0.9897702,1.6710243] | [0.9620333,2.8525751] | [1.2959503,3.2297234] | [0.7932117.01385864] | [0.839717, 0.8480456] |
| 0.6 | [1.1858321,2.5253364] | [1.1969081,5.8297463] | [1.6119157,3.5344538] | [0.793211,2.443785774] | [1.0323835,1.34996091] |
| 0.7 | [1.4650502,4.3768239] | [1.5492546,3.5982321] | [2.0963920,4.4850896] | [1.0020368,6.084397682] | [1.2896904,2.29675352] |
| 0.8 | [1.9468935,10.1428781] | [2.1963977,4.3031565] | [3.0124143,4.8598500] | [1.3132275,1.878408378] | [1.7167602,4.50057007] |
| 0.9 | [3.1848604,42.2787895] | [3.9448324,4.4141649] | [5.5590296,7.8599556] | [1.8762819,3.800471633] | [2.30976829, 2.7295857 |

From above table, we notice that the values generally increase as $q_N$ increases from 0.1 to 0.9, as the ranges (the difference between the upper and lower limits) widen with increasing $q_N$. Some columns also show very high extreme values at $q_N = 0.9$. third column shows the highest values in general, while the fourth column shows the largest ranges and changes in the values. The fifth column shows an average behavior among the columns.

### *3.4 Neutrosophic Moment Generating Function*

The NMGF for NOLGE distribution from Equation (3.3), and using exponential expansion has a final form [17]:

$$M_{Nx}(t_N) = \sum_{z_N=0}^{\infty} \frac{t_N^r}{z_N!}\left[\frac{m!}{b_N^{m+1}}\left(\frac{\phi}{(p_N+1)^{m+1}} - \frac{\psi}{(q_N+1)^{m+1}}\right)\right] \tag{3.12}$$

### *3.5 Neutrosophic Characteristic function*

The Characteristic function for NOLGE distribution from Equation (3.3) , and using exponential expansion has a final form [18]:

$$Q_{Nx}(t_N) = \sum_{v_N=0}^{\infty} \frac{(i_N t_N)^{v_N}}{v_N!}\left[\frac{m!}{b_N^{m+1}}\left(\frac{\phi}{(p_N+1)^{m+1}} - \frac{\psi}{(q_N+1)^{m+1}}\right)\right] \tag{3.13}$$

### *3.6 Neutrosophic Incomplete Moments*

The $n^{th}$ neutrosophic incomplete moments for NOLGE distribution from Equation (3.3) has a form [19]:

$$M_{m,N}(y_N) = \phi\int_0^{y_N} x_N^m\, e^{-(p_N+1)b_N x_N} dx_N - \psi\int_0^{y_N} x_N^m\, e^{-(q_N+1)b_N x_N} dx_N$$

By same way in integral of neutrosophic moments we can get a final form:

$$M_{m,N}(y_N) = \frac{\phi.\Gamma(m+1,(p_N+1)b_N y_N)}{[(p_N+1)b_N]^{m+1}} - \frac{\psi.\Gamma(m+1,(q_N+1)b_N y_N)}{[(q_N+1)b_N]^{m+1}} \tag{3.14}$$

# 4. Estimation

## 4.1 Maximum likelihood estimation

Let $X \sim NOLGE(r_N, u_N, b_N, c_N)$ and $\Delta = (r_N, u_N, b_N, c_N)^T$ be the parameter vector .The log-likelihood for $\Delta$ can be written as [20], [21]:

$$L(\Theta, x) = \prod_{i=1}^{n} f(x_N)$$

$$L(\Theta, x) = \prod_{i=1}^{n} \frac{r_N b_N c_N e^{-b_N x_{Ni}} \left[ \frac{\left(1-e^{-b_N x_{Ni}}\right)^{c_N}}{1-\left(1-e^{-b_N x_{Ni}}\right)^{c_N}} - \log(1-\left(1-e^{-b_N x_{Ni}}\right)^{c_N}) \right]}{u_N \left(1-e^{-b_N x_{Ni}}\right)^{1-c_N} \left(1-\frac{\left(1-e^{-b_N x_{Ni}}\right)^{c_N} .log\left(1-\left(1-e^{-b_N x_{Ni}}\right)^{c_N}\right)}{u_N}\right)^{(r_N+1)}}$$

Compute the log- likelihood to get a form:

$$\begin{aligned} l = l(\Delta) &= nlogr_N + nlogu_N + nlogb_N - nlogc_N - b_N \sum_{i=1}^{n} x_i \\ &+(c_N - 1) \sum_{i=1}^{n} lo\,g(1 - e^{-b_N x_{Ni}}) \\ &-(r_N + 1) \sum_{i=1}^{n} log\left(1 - \frac{(1 - e^{-b_N x_{Ni}})^{c_N} . log(1 - (1 - e^{-b_N x_{Ni}})^{c_N})}{u_N}\right) \\ &+ \sum_{i=1}^{n} log\left[\frac{(1 - e^{-b_N x_{Ni}})^{c_N}}{1 - (1 - e^{-b_N x_{Ni}})^{c_N}} - \log(1 - (1 - e^{-b_N x_{Ni}})^{c_N})\right] \end{aligned} \quad (4.1)$$

## 4.2 Least square estimation

The Least square estimation (LSE) method can be used to estimate a parameter using following formula [22]:

$$\Theta(\theta_N) = \sum_{i=1}^{m} \left[\left[1 - \left(1 - \frac{(1-e^{-b_N x_N})^{c_N} .log\left(1-(1-e^{-b_N x_N})^{c_N}\right)}{u_N}\right)^{-r_N}\right] - \frac{1}{n+1}\right]^2 \quad (4.2)$$

## 4.3 Weighted Least square estimation

The Weighted Least square estimation (WLSE) method can be used to estimate a parameter using following formula [22]:

$$\begin{aligned} W(\theta_N) = \sum_{i=1}^{m} \frac{(n+1)^2(n+2)}{i(n-i+1)} &\left[\left[1 - \left(1 - \frac{(1 - e^{-b_N x_N})^{c_N} . log(1 - (1 - e^{-b_N x_N})^{c_N})}{u_N}\right)^{-r_N}\right] \right. \\ &\left. - \frac{i}{n+1}\right]^2 \end{aligned} \quad (4.3)$$

The functions in Equations (4.1), (4.2), and (4.3) are derived with respect to the parameters and solved by setting those derivatives equal to zero and then solving the numerical system (often using iterative methods such as Newton-Raphson or BFGS if there is no analytical solution), using the R programming language.

# 5. Neutrosophic Simulation

To illustrate the accuracy of the estimation of NOLGE distribution, a Monte Carlo neutrosophic simulation is performed for three approaches (MLE, LSE, WLSE) discussed in fifth section. The neutrosophic simulation algorithm is:

1. Define the statistical model NOLGE, then determine the actual parameters to be estimated, represented, when $r_N = [0.5,1.5], u_N = [0.3,1.3], b_N = [0.2,1.2], c_N = [1.1,2.1]$.
2. Use the different estimation technique.
3. Select multiple sample sizes (n=30, 60, 90, 120) to test the effect of sample size. Then, generate N iteration (e.g., 1000 samples) using a random number generator based on chosen distribution.
4. Apply the estimation technique to each sample. Parameter estimates are calculated using all the selected estimation techniques.
5. Compute the statistical accuracy criteria for each technique across all iterations (mean square error (MSE), its root (RMSE), and bias) [12], [23], and then the values are compared between techniques to determine the best performer.
6. Rank techniques based on their performance according to three statistical criteria. Then, identify the technique with the lowest MSE, RMSE, and Bias values as a measure of performance advantage.
7. Present the results and make recommendations.

The following table shows the result of neutrosophic simulation for NOLGE

**Table 3 :** Monte Carlo simulations conducted for the NOLGE

| N | Est. | Ess. | MLE | LSE | WLSE |
|---|---|---|---|---|---|

| | | Par. | | | |
|---|---|---|---|---|---|
| 30 | Mean | $\widehat{r_N}$ | [0.7684220,1.3782479] | [0.7256307,1.878254] | [0.8107461,1.6774104] |
| | | $\widehat{u_N}$ | [0.4255054,1.1655208] | [0.25003486,0.9294016] | [0.32136101,1.1585889] |
| | | $\widehat{b_N}$ | [0.206263996,1.6004971] | [0.216034072,1.2406271] | [0.205126612,1.3025457] |
| | | $\widehat{c_N}$ | [1.12297389, 3.667120] | [1.2003628, 2.303705] | [1.14682594, 2.519430] |
| | MSE | $\widehat{r_N}$ | [0.5611748, 0.7723860] | [0.4115597,1.188257] | [0.4354975,0.8530636] |
| | | $\widehat{u_N}$ | [0.3754524,2.5376444] | [0.15581210,0.6674667] | [0.14539049,0.9619678] |
| | | $\widehat{b_N}$ | [0.003005731,0.5913223] | [0.005269381,0.2007503] | [0.004068094,0.1616277] |
| | | $\widehat{c_N}$ | [0.17577498, 45.179781] | [0.1412719, 2.435437] | [0.11178179, 8.590642] |
| | RMSE | $\widehat{r_N}$ | [0.749116, 0.8788549] | [0.6415292,1.090072] | [0.6599224,0.9236144] |
| | | $\widehat{u_N}$ | [0.6127417,1.5929985] | [0.39473041,0.8169863] | [0.38130104,0.9807996] |
| | | $\widehat{b_N}$ | [0.054824549,0.7689748] | [0.072590504,0.4480517] | [0.38130104,0.4020294] |
| | | $\widehat{c_N}$ | [0.41925527, 6.721591] | [0.3758616, 1.560589] | [0.33433784, 2.930980] |
| | Bias | $\widehat{r_N}$ | [0.1217521, 0.268422] | [0.2256307,0.378254] | [0.1774104, 0.3107461] |
| | | $\widehat{u_N}$ | [0.1255054,0.1344792] | [0.04996514,0.3705984] | [0.02136101,0.1414111] |
| | | $\widehat{b_N}$ | [0.006263996,0.4004971] | [0.016034072,0.0406271] | [0.005126612,0.1025457] |
| | | $\widehat{c_N}$ | [0.02297389, 1.567120] | [0.1003628, 0.203705] | [0.04682594, 0.419430] |
| 60 | Mean | $\widehat{r_N}$ | [0.6455354,1.42217682] | [0.6423265,1.7989833] | [0.7008391,1.6609149] |
| | | $\widehat{u_N}$ | [0.4580180,1.1157927] | [0.3569932,0.9855108] | [0.39685327,0.9557224] |
| | | $\widehat{b_N}$ | [0.198722701,1.5014615] | [0.1997717,1.27372395] | [0.194966094,1.3394544] |
| | | $\widehat{c_N}$ | [1.14432764, 3.195213] | [1.13445143, 2.2378522] | [1.11220108, 2.3716367] |
| | MSE | $\widehat{r_N}$ | [0.25821676, 0.2445797] | [0.1168153,0.5741687] | [0.2105605,0.4267738] |
| | | $\widehat{u_N}$ | [0.3201380,1.5947631] | [0.1089898,0.4422803] | [0.12215183,0.3930340] |
| | | $\widehat{b_N}$ | [0.00147877,0.4184299] | [0.00251044,0.21085239] | [0.001901072,0.1650236] |
| | | $\widehat{c_N}$ | [0.17885977, 20.497538] | [0.13072780, 1.784905] | [0.09582104, 4.5432516] |
| | RMSE | $\widehat{r_N}$ | [0.4945499,0.50815033] | [0.3417826,0.7577392] | [0.4588687,0.6532793] |
| | | $\widehat{u_N}$ | [0.5658074,1.2628393] | [0.330136,0.6650416] | [0.34950226,0.6269243] |
| | | $\widehat{b_N}$ | [0.0384548,0.6468616] | [0.05010430,0.45918666] | [0.0436012,0.4062309] |
| | | $\widehat{c_N}$ | [0.422918164.527421,] | [0.36156299, 1.3360034] | [0.30954974, 2.1314905] |
| | Bias | $\widehat{r_N}$ | [0.07782318, 0.1455354] | [0.1423265,0.2989833] | [0.1609149, 0.2008391] |
| | | $\widehat{u_N}$ | [0.158018,0.1842073] | [0.0569932,0.3144892] | [0.0968532,0.3442776] |
| | | $\widehat{b_N}$ | [0.00127729,0.3014615] | [0.00022828,0.07372395] | [0.0050339,0.1394544] |
| | | $\widehat{c_N}$ | [0.04432764, 1.095213] | [0.03445143, 0.1378522] | [0.01220108, 0.2716367] |
| 90 | Mean | $\widehat{r_N}$ | [0.6162185,1.41549816] | [0.6536816,1.57877112] | [0.6468753,1.6171214] |
| | | $\widehat{u_N}$ | [0.361018,1.27882503] | [0.3344813,1.0098268] | [0.3840868,1.0678597] |
| | | $\widehat{b_N}$ | [0.2054801,1.5431271] | [0.2050540,1.3065827] | [0.1985389,1.3019430] |
| | | $\widehat{c_N}$ | [1.05448471, 2.8368908] | [1.13540953, 2.2520674] | [1.12840014, 2.2327755] |
| | MSE | $\widehat{r_N}$ | [0.2369878,0.26604037] | [0.2678811,0.28823410] | [0.1278475,0.3886124] |
| | | $\widehat{u_N}$ | [0.1204802,1.59561304] | [0.0718905,0.3956973] | [0.09984225,0.5128128] |
| | | $\widehat{b_N}$ | [0.0012963,0.4840597] | [0.0027755,0.1437240] | [0.0013455,0.1486854] |
| | | $\widehat{c_N}$ | [0.12591376, 9.0372132] | [0.08801489, 2.9331265] | [0.06714572, 3.0834354] |
| | RMSE | $\widehat{r_N}$ | [0.4868139,0.51579101] | [0.5175724,0.53687438] | [0.357557,0.6233879] |
| | | $\widehat{u_N}$ | [0.3471025,1.26317578] | [0.26812413,0.6290448] | [0.315978,0.7161095] |
| | | $\widehat{b_N}$ | [0.0360052,0.6957440] | [0.052683,0.3791095] | [0.0366822,0.3855974] |
| | | $\widehat{c_N}$ | [0.35484329, 3.0061958] | [0.29667304, 1.712637] | [0.25912492, 1.7559714] |
| | Bias | $\widehat{r_N}$ | [0.1162185,0.08450184] | [0.07877112, 0.1536816] | [0.1171214, 0.1468753] |
| | | $\widehat{u_N}$ | [0.02117497, 0.0610182] | [0.034481,0.2901732] | [0.08408681,0.2321403] |
| | | $\widehat{b_N}$ | [0.0054801,0.3431271] | [0.0050540,0.1065827] | [0.0014610,0.1019430] |
| | | $\widehat{c_N}$ | [0.04551529, 0.7368908] | [0.03540953, 0.1520674] | [0.02840014, 0.1327755] |
| 120 | Mean | $\widehat{r_N}$ | [0.6088066,1.43542573] | [0.6170890,1.6056755] | [0.6798458,1.52847868] |
| | | $\widehat{u_N}$ | [0.37131488,1.36276409] | [0.35682408,1.0426991] | [0.39421229,1.0386712] |
| | | $\widehat{b_N}$ | [0.203115990,1.4732599] | [2.000039e-01,1.27294] | [0.195334361,1.3647878] |
| | | $\widehat{c_N}$ | [1.101154562, 2.705355] | [1.13603778, 2.311424] | [1.0859956, 2.16066186] |
| | MSE | $\widehat{r_N}$ | [0.1781467,0.21948012] | [0.120401,0.2529646] | [0.1330275,0.17951959] |
| | | $\widehat{u_N}$ | [0.13330633,1.89577521] | [0.0640675,0.4142318] | [0.082044,0.3504012] |

| | | | | | |
|---|---|---|---|---|---|
| | | $\widehat{b_N}$ | [0.0010468,0.4175756] | [2.288731e-03,0.098167] | [0.0011985,0.1565051] |
| | | $\widehat{c_N}$ | [0.157561444, 6.767972] | [0.07026270, 1.646679] | [0.06288197, 2.3363673] |
| | **RMSE** | $\widehat{r_N}$ | [0.4220743,0.46848706] | [0.3469885,0.5029559] | [0.3647293,0.42369752] |
| | | $\widehat{u_N}$ | [0.3651114,1.37687153] | [0.2531157,0.6436084] | [0.286433,0.5919470] |
| | | $\widehat{b_N}$ | [0.0323547,0.6462009] | [4.784069e-02,0.313316] | [0.034619,0.3956073] |
| | | $\widehat{c_N}$ | [0.396940101, 2.601532] | [0.26507113, 1.28323] | [0.25076278, 1.528518] |
| | **Bias** | $\widehat{r_N}$ | [0.06276409, 0.1088066] | [0.06457427, 0.117089] | [0.02847868, 0.1798458] |
| | | $\widehat{u_N}$ | [0.0713148,0.2732599] | [0.056824,0.2573009] | [0.0942122,0.2613288] |
| | | $\widehat{b_N}$ | [0.001154562, 0.003115] | [3.937511e-06,0.072945] | [0.0046656,0.1647878] |
| | | $\widehat{c_N}$ | [0.1056755, 0.6053554] | [0.03603778, 0.211424] | [0.0140044, 0.06066186] |

The results shown in table 3 show a clear contrast in the performance of three methods. In general, WLSE trends to outperform in many scenarios, especially as the sample size increases, with lower MSE, RMSE, and Bias values compared to MLE and LSE. For example, at a sample size of 120, WLSE estimates of the parameter $r_N$ were more accurate, with an MSE of [0.1330275, 0.17951959] compared to [0.1781467, 0.21948012] for MLE and [0.120401, 0.2529646] for LSE. This suggests that WLSE better balances precision and reliability, especially in neutrosophic environments where uncertainty is a fundamental part of data. Additionally, its noted that increasing the sample size generally leads to improved accuracy for all methods, underscoring the importance of sufficient sample size in statistical analysis. For example, the MSE for $c_N$ decreased from [0.17577498, 45.179781] at a sample size of 30 to [0.157561444, 6.767972] at a sample size of 120 for MLE, showing a significant improvement with increasing data. In terms of bias, WLSE demonstrated balanced performance, with bias values being relatively small across most parameters. Conversely, MLE exhibited higher bias in some cases, such as the estimation of $u_N$ at a sample 30, where bias was [0.1255054, 0.1344792] compared to [0.02136101, 0.1414111] for WLSE.

Based on these results, it can be concluded that WLSE is the most reliable method for estimating NOLGE distribution parameters in neutrosophic context, especially when dealing with large samples. However, the choice of the optimal method may also depend on the nature of data and the available computational resources, therefore, WLSE is recommended when sufficient data are available, taking into account the practical feasibility of each method.

## 6. Applications

In this section, we illustrate the effectiveness of NOLGE distribution in fitting data using a real-world scenario. The data set used is lifetime in 100 hours of 23 batteries is given as [24]:

[2.9,3.99], [5.24,7.2],[6.56,9.02], [7.14,9.82], [11.6,15.96], [12.14,16.69], [12.65,17.4], [13.24,18.21],[13.67,18.79], [13.88,19.09], [15.64,21.51], [17.05,23.45], [17.4,23.93], [17.8,24.48], [19.01,26.14], [19.34,26.59], [23.13,31.81], [23.34,32.09],[26.07,35.84], [30.29,41.65], [43.97,60.46], [48.09,66.13], [73.48,98.04].

To illustrate advantages of NOLGE distribution and its data-fitting capabilities. Eight measurement employed in this comparison are: Cramer-von Mises statistic, the Anderson-Darling statistic, and the Kolmogorov-Smirnov statistic (KS), statistic (W), the p-value that corresponds to the KS-test [25], [26], and the information criterion HQIC, BIC, AIC, and CAIC [27], [28]. These are typical goodness of fit measures. Contrasting the outcomes of suggested distribution with these of six other distributions, which are:

- Neutrosophic Beta exponential generalized exponential distribution (NBeGE).
- Neutrosophic Kumaraswamy Exponential Generalized exponential distribution (NKuGE).
- Neutrosophic Exponential generalized exponential distribution (NEGGE).
- Neutrosophic Log-gamma exponential generalized exponential distribution (NLGamGE).
- Neutrosophic Trunked exponential exponential generalized exponential distribution (NTEEGE).
- Neutrosophic Generalized exponential distribution (NGE).

Table 4 shows the results of the criteria for the neutrosophic distributions, table 5 shows the value of the statistical measures, and table 6 shows the estimator value interval for parameters by MLE

**Table 4.** results of the criteria for the distributions

| Dist. | -L | AIC | CAIC | BIC | HQIC |
|---|---|---|---|---|---|
| **NOLGE** | [88.43435,95.60102] | [184.8687,199.202] | [187.0909,201.4243] | [189.4107,203.744] | [186.011,200.3443] |
| **NBeGE** | [93.16967,95.94182] | [194.3393,199.9018] | [196.5616,202.124] | [198.8813,204.4438] | [195.4816,201.0441] |
| **NKuGE** | [93.94476,96.32557] | [195.8895,200.6671] | [198.1117,202.8893] | [200.4315,205.2091] | [197.0318,201.8094] |
| **NEGGE** | [88.86065,96.01458] | [185.7234,200.0295] | [187.9456,202.2517] | [190.2654,204.5714] | [186.8657,201.1718] |
| **NLGamGE** | [88.96139,97.23907] | [185.9233,202.5317] | [188.1455,204.7539] | [190.4653,207.0737] | [187.0656,203.674] |
| **NTEEGE** | [88.96954,96.19195] | [186.5684,200.3989] | [188.7906,202.6211] | [191.1104,204.9409] | [187.7107,201.5412] |
| **NGE** | [92.09421,99.87432] | [188.2608,203.7663] | [188.8608,204.3663] | [190.5318,206.0373] | [188.832,204.3375] |

Table 4 compares the NOLGE distribution with sex other neutrosophic distributions using information criteria and negative logarithmic function (-L). the results shows NOLGE distribution performed exceptionally well, recording the lowest information criteria values in most cases compared to the other distributions. this suggests that NOLGE distribution is more efficient at modeling data while reducing model complexity. These results reinforce NOLGE's superiority in providing a better balance between accuracy and simplicity, making it a strong candidate for data analysis under uncertainty.

**Table 5 .** value of the statistical measures

| **Dist.** | **W** | **A** | **K-S** | **p-value** |
|---|---|---|---|---|
| **NOLGE** | [0.04494521,0.04979327] | [0.2624134,0.28484] | [0.1176604,0.1253289] | [0.8197149, 0.8713342] |
| **NBeGE** | [0.06792112,0.08741665] | [0.3771706,0.4937468] | [0.1435358, 0.268247] | [0.05956844,0.6781621] |
| **NKuGE** | [0.07413739,0.09087818] | [0.4153534,0.5157557] | [0.1522436, 0.2950746] | [0.02861457,0.607189] |
| **NEGGE** | [0.07159541,0.07320745] | [0.4007221,0.4092349] | [0.1282813, 0.1295094] | [0.7890527,0.7982154] |
| **NLGamGE** | [7.227293, 7.227405] | [45.82877, 45.85284 | [0.9841392, 0.9946239] | 4.440892e-16 |
| **NTEEGE** | [0.06795516,0.0700356] | [0.3778144, 0.3893318] | [0.1588924, 0.1611754] | [0.5358761,0.553846] |
| **NGE** | [0.08148869,0.08332876] | [0.4577023,0.4677389] | [0.2444697,0.261] | [0.07171139, 0.1073506] |

Table 5 evaluate the performance of the distributions using some statistical measures. NOLGE distribution performed better on the K-S statistics than other distributions such us NKuGE demonstrating a better convergence between the data and the model. The p-value for NOLGE was high, confirming that there was no evidence to reject the goodness-of-fit hypothesis, unlike distributions such as NKuGE, which recorded low p-values. These results confirm the NOLGE provides a more reliable statistical fit to the real data than other distributions.

**Table 6.** Estimator value interval for parameters by MLE

| **Dist.** | $\widehat{r_N}$ | $\widehat{u_N}$ | $\widehat{b_N}$ | $\widehat{c_N}$ |
|---|---|---|---|---|
| **NOLGE** | [0.06044682,5.35617856] | [1.56437111,5.53381522] | [0.01762876,0.06342953] | [1.26461437,1.51203034] |
| **NBeGE** | [1.16508722,1.6517759] | [0.09953987,0.4698984] | [0.1095705,0.47460313] | [0.55594414,1.6978604] |
| **NKuGE** | [1.0817318,1.40013684] | [0.1038297,0.89767281] | [0.05793171,0.4670658] | [0.4686354,1.46052338] |
| **NEGGE** | [0.8377522,0.8954365] | [1.5989327, 1.7174486] | [0.0673655,0.0986491] | [1.5520771,1.6442189] |
| **NLGamGE** | [1.47187623,1.6799803] | [0.7201890,0.90885039] | [0.05764093,0.1330494] | [1.02119949, 1.5765075] |
| **NTEEGE** | [0.88082429,0.88654756] | [1.54093053,1.59309664] | [0.04644511,0.05870377] | [1.35292695,1.43715812] |
| **NGE** | --- | --- | [0.03528371,0.0501071] | [0.98810296,1.0637076] |

Table 6 shows the parameter estimation intervals for studied distributions. for NOLGE, parameter estimates were within reasonable ranges. Compared to the other distributions, NOLGE estimates showed better consistency, with no extreme or unstable values, as in case of NBeGE or NKuGE. This reflects NOLGE's ability to accurately estimate parameters even in the presence of neutrosophic uncertainty.

In addition to the three previous tables, a visual test is conducted by drawing the fitting NPDFs of NOLGE with histogram data set, in addition to drawing the empirical fitted NCDFs with data set used for the proposed distribution and other comparative distributions, as shown in figures 6 and 7 as follows.

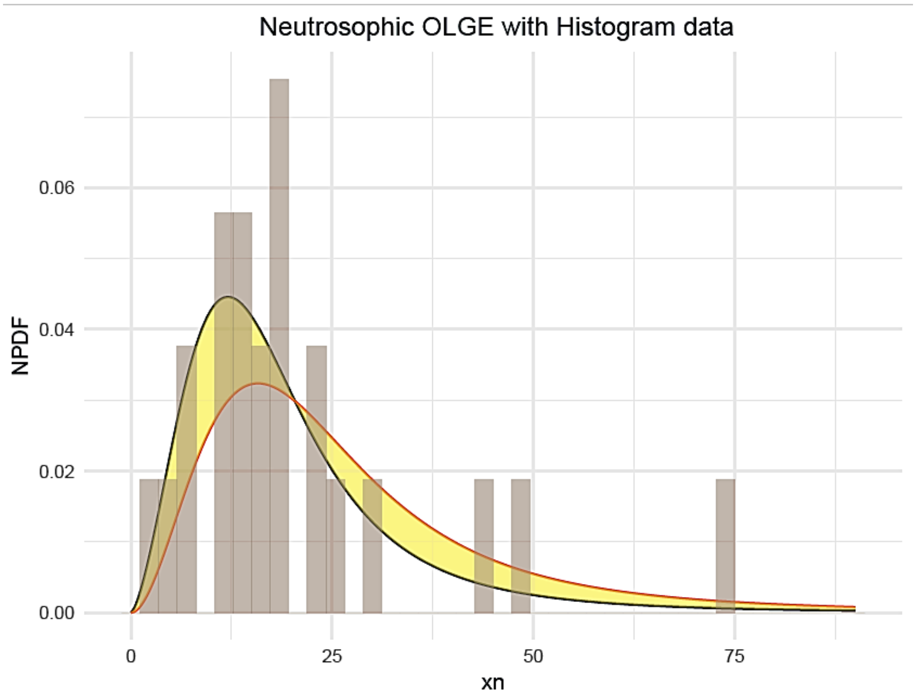
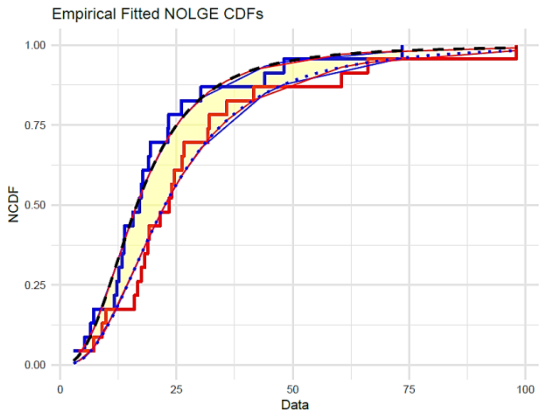
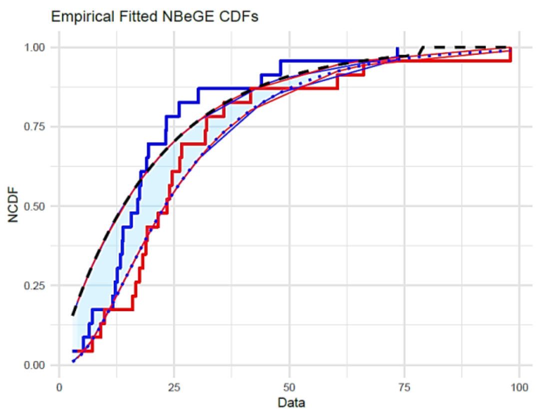


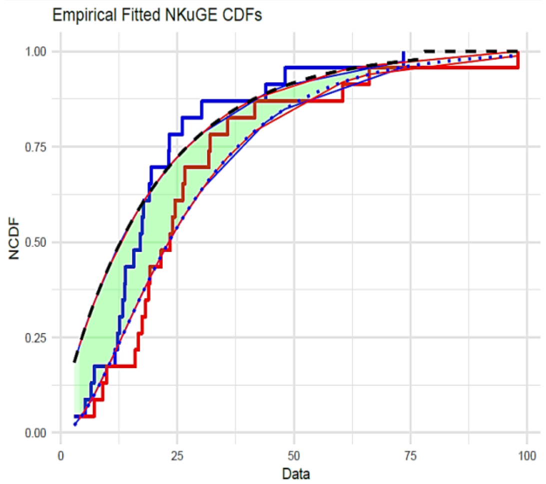

**Figure 6:** Fitting NPDFs of NOLGE with histogram data set



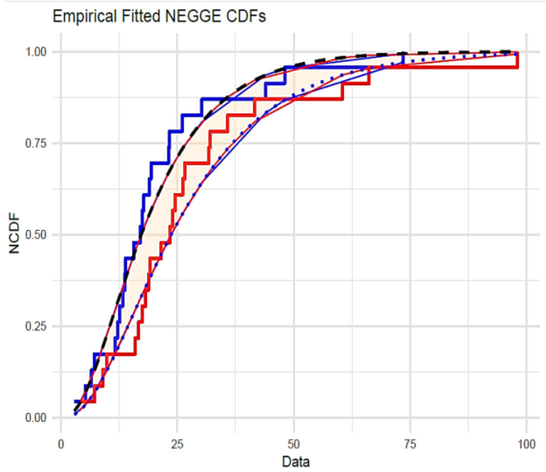

Fitted NOLGE



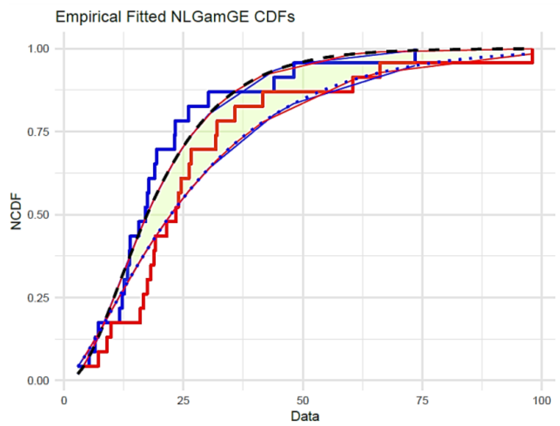

Fitted NBeGE



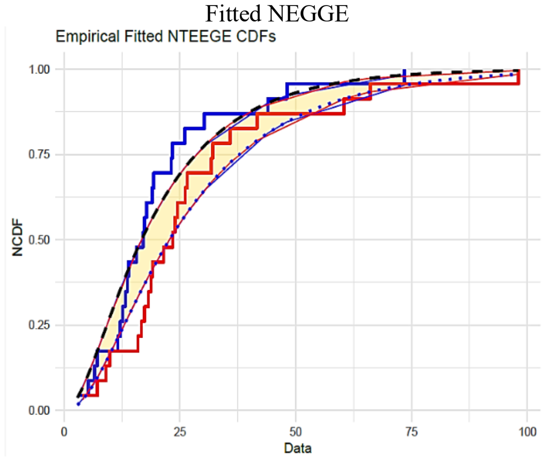

Fitted NKuGE



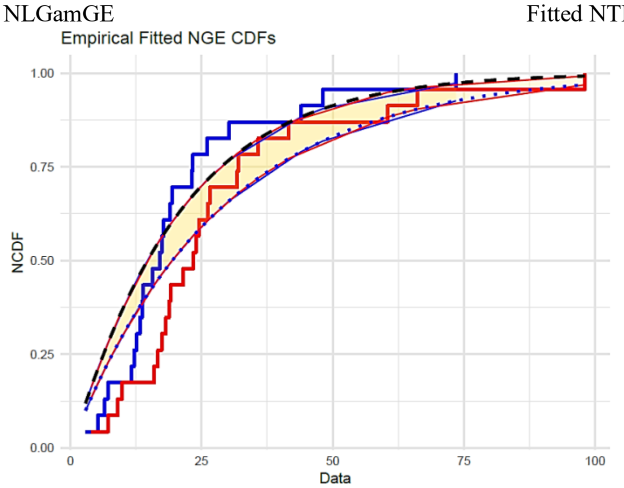

Fitted NEGGE




Fitted NLGamGE




Fitted NTEEGE



Fitted NGE

**Figure 7:** Empirical Fitted CDFs with data set used

Figure 6 shows the fit of NOLGE's NPDF to the data histogram. It can be seen that the NOLGE curve closely follows the data distribution, confirming its ability to accurately represent the data. Figure 7 compares the NCDFs with theoretical NCDFs of various distributions, such as NBeGE or NKuGE, which exhibit significant deviations.

These visual results support the quantitative conclusions from the tables, confirming NOLGE's superiority in modeling data.

## Conclusion

The proposed distribution demonstrated outstanding performance when compared to six other neutrosophic distributions (e.g., NBeGE, NKuGE, and NEGGE). It recorded the lowest information criterion values, indicating its high efficiency in achieving a balance between accuracy and model simplicity. This makes it an ideal choice for analyzing complex data involving a high degree of uncertainty. Through neutrosophic Monte Carlo simulations, the weighted least squares (WLSE) method outperformed other estimation methods (MLE and LSE) in most scenarios, especially as the sample size increased, reflecting its high accuracy in parameter estimation under uncertain conditions. The study provides an integrated theoretical and practical framework for integrating neutrosophic logic with probability distributions, opening new horizons for data analysis under uncertain conditions. This represents an important advance in fields such as engineering, medical sciences, and economics, where data are often incomplete or inconsistent. While NOLGE distribution offers a powerful framework for analyzing data under uncertainty, it may have limitations, such as the small size of the data studied or the limited study of battery life, which may not fully represent the behavior of data in other fields such as medicine or economics. These limitations highlight the need for further research to deepen its understanding and improve its applications. Overcoming these limitations could make it a more powerful tool in fields such as artificial intelligence and decision science.

### References


1. Abd El-latif, A. M., Almulhim, F. A., Noori, N. A., Khaleel, M. A., & Alsaedi, B. S. (2025). Properties with application to medical data for new inverse Rayleigh distribution utilizing neutrosophic logic. *Journal of Radiation Research and Applied Sciences*, *18*(2), 101391.
2. Alanaz, Mazin M., Marwah Yahya Mustafa, and Zakariya Yahya Algamal. "Neutrosophic Lindley distribution with application for Alloying Metal Melting Point." *International Journal of Neutrosophic Science (IJNS)* 21.4 (2023).
3. Khan, Z., Almazah, M. M., Hamood Odhah, O., & Alshanbari, H. M. (2022). Generalized pareto model: properties and applications in neutrosophic data modeling. *Mathematical Problems in Engineering*, *2022*(1), 3686968.
4. Alanaz, M. M., & Algamal, Z. Y. (2023). Neutrosophic exponentiated inverse Rayleigh distribution: Properties and Applications. *International Journal of Neutrosophic Science (IJNS)*, *21*(4).
5. Al-Saqal, O. E., Hadied, Z. A., & Algamal, Z. Y. (2025). Modeling bladder cancer survival function based on neutrosophic inverse Gompertz distribution. *International Journal of Neutrosophic Science (IJNS)*, *25*(1).
6. Algamal, Z. Y., Alobaidi, N. N., Hamad, A. A., Alanaz, M. M., & Mustafa, M. Y. (2024). Neutrosophic Beta-Lindley distribution: mathematical properties and modeling bladder Cancer data. *Int J Neutrosophic Sci*, *23*, 186-6.
7. Al-Essa, L. A., Jamal, F., Shafiq, S., Khan, S., Abbas, Q., Khan Sherwani, R. A., & Aslam, M. (2025). Properties and Applications of Neutrosophic Burr XII Distribution. *International Journal of Computational Intelligence Systems*, *18*(1), 10.
8. Mustafa, M. Y., & Algamal, Z. Y. (2023). Neutrosophic inverse power Lindley distribution: A modeling and application for bladder cancer patients. *International Journal of Neutrosophic Science (IJNS)*, *21*(2).
9. Ahsan-ul-Haq, M., Zafar, J., Aslam, M., & Tariq, S. (2024). Neutrosophic topp-leone distribution for interval-valued data analysis. *Journal of Statistical Theory and Applications*, *23*(2), 164-173.
10. Alsaab, N. (2025). Data Modelling and Analysis Using Odd Lomax Generalized Exponential Distribution: an Empirical Study and Simulation. *Iraqi Statisticians journal*, 146-162.
11. Odeyale, A. B., Gulumbe, S. U., Umar, U., & Aremu, K. O. (2023). New New Odd Generalized Exponentiated Exponential-G Family of Distributions. *UMYU Scientifica*, *2*(4), 56-64.
12. Khalaf, A. A., Ibrahim, M. Q., & Noori, N. A. (2024). [0, 1] Truncated Exponentiated Exponential Burr type X Distributionwith Applications. *Iraqi Journal of Science*, 4428-4440.
13. Noori, N. A., Khalaf, A. A., & Khaleel, M. A. (2024). A new expansion of the Inverse Weibull Distribution: Properties with Applications. *Iraqi Statistians Journal*, *1*(1), 52-62.
14. Bhatti, F. A., Hamedani, G. G., Korkmaz, M. C., Cordeiro, G. M., Yousof, H. M., & Ahmad, M. (2019). On Burr III Marshal Olkin family: development, properties, characterizations and applications. *Journal of Statistical Distributions and Applications*, *6*(1), 1-21.
15. Gómez, H. J., Santoro, K. I., Barranco-Chamorro, I., Venegas, O., Gallardo, D. I., & Gómez, H. W. (2023). A Family of Truncated Positive Distributions. *Mathematics*, *11*(21), 4431.
16. Ishaq, A. I., Panitanarak, U., Abiodun, A. A., Suleiman, A. A., & Daud, H. (2024). The generalized odd maxwell-kumaraswamy distribution: Its properties and applications. *Contemporary Mathematics*, 711-742.
17. Al-Habib, K. H., Khaleel, M. A., & Al-Mofleh, H. (2023). A New Family of Truncated Nadarajah-Haghighi-G Properties with Real Data Applications. *Tikrit Journal of Administrative and Economic Sciences*, *19*(2), 311-333.
18. Khaleel, M. A., Oguntunde, P. E., Al Abbasi, J. N., Ibrahim, N. A., & AbuJarad, M. H. (2020). The Marshall-Olkin Topp Leone-G family of distributions: A family for generalizing probability models. *Scientific African*, *8*, e00470.
19. Abdelall, Y. Y., Hassan, A. S., & Almetwally, E. M. (2024). A new extention of the odd inverse Weibull-G family of

distributions: Bayesian and non-Bayesian estimation with engineering applications. *Computational Journal of Mathematical and Statistical Sciences*, *3*(2), 359-388.

20. Alsaab, N. (2024). Estimation and Some Statistical Properties of the hybrid Weibull Inverse Burr Type X Distribution with Application to Cancer Patient Data. *Iraqi Statisticians journal*, 8-29.
21. Oguntunde, P. E., Khaleel, M. A., Okagbue, H. I., & Odetunmibi, O. A. (2019). The Topp–Leone Lomax (TLLo) distribution with applications to airbone communication transceiver dataset. *Wireless Personal Communications*, *109*, 349-360.
22. Abid, S., & Abdulrazak, R. (2018). [0, 1] truncated Frechet-Weibull and Frechet distributions. *International Journal of Research in Industrial Engineering*, *7*(1), 106-135.
23. Hussain, S., Ul Hassan, M., Rashid, M. S., & Ahmed, R. (2023). Families of Extended Exponentiated Generalized Distributions and Applications of Medical Data Using Burr III Extended Exponentiated Weibull Distribution. *Mathematics*, *11*(14), 3090.
24. Noori, N. A. (2023). Exploring the Properties, Simulation, and Applications of the Odd Burr XII Gompertz Distribution. *Advances in the Theory of Nonlinear Analysis and its Application*, *7*(4), 60-75.
25. Aslam, M. (2021). A new goodness of fit test in the presence of uncertain parameters. *Complex & Intelligent Systems*, *7*(1), 359-365.
26. Noori, N. A., Khalaf, A. A., & Khaleel, M. A. (2023). A New Generalized Family of Odd Lomax-G Distributions: Properties and Applications. *Advances in the Theory of Nonlinear Analysis and its Applications*, *7*(4), 01-16.
27. Mahdi, G. A., Khaleel, M. A., Gemeay, A. M., Nagy, M., Mansi, A. H., Hossain, M. M., & Hussam, E. (2024). A new hybrid odd exponential-Φ family: Properties and applications. *AIP Advances*, *14*(4).
28. Cordeiro, G. M., Alizadeh, M., Ramires, T. G., & Ortega, E. M. (2017). The generalized odd half-Cauchy family of distributions: Properties and applications. *Communications in Statistics-Theory and Methods*, *46*(11), 5685-5705.
29. Rahman, M. M., Gemeay, A. M., Islam Khan, M. A., Meraou, M. A., Bakr, M. E., Muse, A. H., ... & Balogun, O. S. (2023). A new modified cubic transmuted-G family of distributions: Properties and different methods of estimation with applications to real-life data. *AIP Advances*, *13*(9).