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Research article

Bayesian and Non-Bayesian Approaches for Estimating the Extended Exponential Distribution: Applications to COVID-19 and Carbon Fibers

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ABSTRACT

This study focuses on estimating the parameters of the Odd Burr XII–Exponential (OBXII-E) distribution using both Bayesian and classical approaches. For the non-Bayesian framework, seven estimation methods were considered: Maximum Likelihood, Least Squares, Weighted Least Squares, Maximum Product Space, Anderson–Darling, right-Tailed Anderson–Darling, and Kolmogorov estimators. In the Bayesian context, parameter estimation was carried out using the Markov Chain Monte Carlo (MCMC) technique under different loss functions, including Squared Error, General Entropy, and Linear-Exponential. Through extensive simulation experiments, the accuracy and consistency of each estimator were evaluated, revealing that all methods converge toward the true parameter values as sample size increases. The OBXII-E distribution was further applied to two real datasets, where it consistently outperformed competing models based on multiple goodness-of-fit criteria. Overall, the results confirm the robustness and flexibility of the OBXII-E distribution in modeling daily COVID-19 and carbon fibers data.

1. Introduction

The exponential distribution, characterized by its memoryless property and governed by the rate parameter (δ), plays a crucial role in both statistical theory and practical applications. It finds extensive utility in fields such as reliability engineering [1], queueing theory [2], and survival analysis [3]. However, the necessity to expand the parameterization of the exponential distribution has become increasingly evident in modern computational statistics.

A multitude of researchers have explored the extrapolations of probability distributions, drawing insights from a vast body of statistical literature. These investigations have highlighted the limitations of traditional distributions and underscored the importance of developing more flexible models that can accommodate a wider range of data behaviors. For instance, Oguntunde et al. [4] introduced an extension of the exponential distribution using the logistic-x family, while Ademola et al. [5] discussed the Exponentiated Gompertz exponential distribution and its applications. Other studies have focused on different modifications and generalizations of the Burr XII distribution to enhance its flexibility and applicability Alkhazaleh et al., [6].

In this context, we focus on Bayesian and non-Bayesian methods for estimating the parameters of the Odd Burr XII-Exponential (OBXII-E) distribution. Parameter estimation is paramount in statistical inference and data modeling, serving as the foundation for drawing conclusions from data. Bayesian methods offer flexibility and robustness in uncertain environments by incorporating prior information into parameter estimates. Conversely, non-Bayesian approaches provide computationally efficient and straightforward solutions, allowing for ease of interpretation.

Numerous authors have contributed to the study of parameter estimation techniques in this domain, highlighting the strengths and weaknesses of various methodologies. For example, Abdullah et al. [7] utilized optimization methods for parameter estimation, while Al-Harbi et al. [8] proposed a novel discrete linear-exponential distribution for modeling data. The literature demonstrates a clear interest in advancing the methods available for effective parameter estimation, emphasizing the growing relevance of the OBXII-E distribution. Bayesian estimation incorporates prior knowledge with observed data, offering a different perspective compared to frequentist inference that relies solely on observed data. El-Sherpieny et al. [9] studied the bivariate generalized Rayleigh distribution under PrTIICS with random removal, employing Bayesian and non-Bayesian estimation methods. Muhammed expanded the likelihood function to the Marshall–Olkin bivariate class and applied it to the bivariate Dagum distribution under PrTIICS, using MLE and Bayesian techniques. Integrated Bayesian parameter estimation with model-based design of experiments for dynamic processes. Bayesian parameter estimation for dynamical models in systems biology.

Bayesian methods have seen increased use due to their ability to handle complex models and data types, particularly in situations with heterogeneity and sparsity. These methods often involve specifying prior distributions that reflect existing knowledge or beliefs about the parameters. The choice of prior can range from non-informative priors, which allow the data to primarily drive the posterior distribution, to informative priors that incorporate specific knowledge. Can see ref [10-14] Several Bayesian estimation techniques have been developed, including the use of Markov Chain Monte Carlo (MCMC) methods to sample from the posterior distribution. These methods are particularly useful when direct computation of the posterior is challenging. Additionally, various loss functions, such as Squared Error (SE), Linear-Exponential (LINEX), and General Entropy (GE), are used to evaluate the performance of Bayesian estimators.

The structure of the paper is as follows: Section 2 provides a definition of the OBXII-E distribution, followed by a discussion of classical inference techniques in Section 3. Section 4 explores Bayesian estimation approaches that utilize different loss functions, such as Squared Error (SE), Linear-Exponential (LINEX), and General Entropy (GE), while assuming independent priors. Section 5 presents a simulated study to demonstrate the distribution's flexibility, and Section 6 will discuss the applications of the OBXII-E distribution in various fields.

2. The Odd Burr XII Exponential Distribution

According to the study conducted by Khalaf [15]. The cumulative distribution function (CDF) of the OBXII –G Family is

$$K(x)_{OBXII-G} = 1 - \left[1 + \left[\frac{F(x;\xi)^2}{1-F(x;\xi)}\right]^{\rho}\right]^{-\lambda}, \tag{2.1}$$

and probability density function (PDF) of the OBXII –G Family is

$$k(x)_{OBXII-G} = \lambda\rho F(x;\xi) f(x;\xi)\big(2 - F(x;\xi)\big)\big(1 - F(x;\xi)\big)^{-2} \left[\frac{F(x;\xi)^2}{1-F(x;\xi)}\right]^{\rho-1} \left[1 + \left[\frac{F(x;\xi)^2}{1-F(x;\xi)}\right]^{\rho}\right]^{-(\lambda+1)}. \tag{2.2}$$

The Exponential (E) distribution is characterized by its CDF and PDF, which can be expressed as follows:

$$F(x,\delta) = 1 - e^{-\delta x}, \tag{2.3}$$

and

$$f(x,\delta) = \delta e^{-\delta x} \qquad ,x > 0, \delta > 0. \tag{2.4}$$

We define a new distribution known as The Odd Burr XII Exponential (OBXII-E) distribution, obtained by substituting Equation (2.3) into Equation (2.1). This substitution leads to the following expression:

$$K(x)_{\mathrm{OBXII-E}} = 1 - \left[1 + \left[\frac{\left(1-e^{-\delta x}\right)^2}{e^{-\delta x}}\right]^{\rho}\right]^{-\lambda}, \tag{2.5}$$

and by substitution Equations (2.3) and (2.4) into Equation (2.2) we get a PDF of the OBXII-E distribution:

$$k(x)_{\text{OBXII-E}} = \lambda\rho e^{\delta x}(1-e^{-\delta x})(1+e^{-\delta x})\left[\frac{(1-e^{-\delta x})^2}{e^{-\delta x}}\right]^{\rho-1}\left[1+\left[\frac{(1-e^{-\delta x})^2}{e^{-\delta x}}\right]^{\rho}\right]^{-(\lambda+1)}. \tag{2.6}$$

where $x \geq 0$, $\lambda, \rho, \delta > 0$, and the Hazard Function of OBXII- E is

$$hrf(x)_{\text{BXII-E}} = \lambda\rho e^{\delta x}(1-e^{-\delta x})(1+e^{-\delta x})\left[\frac{(1-e^{-\delta x})^2}{e^{-\delta x}}\right]^{\rho-1}\left[1+\left[\frac{(1-e^{-\delta x})^2}{e^{-\delta x}}\right]^{\rho}\right]^{-1}.$$

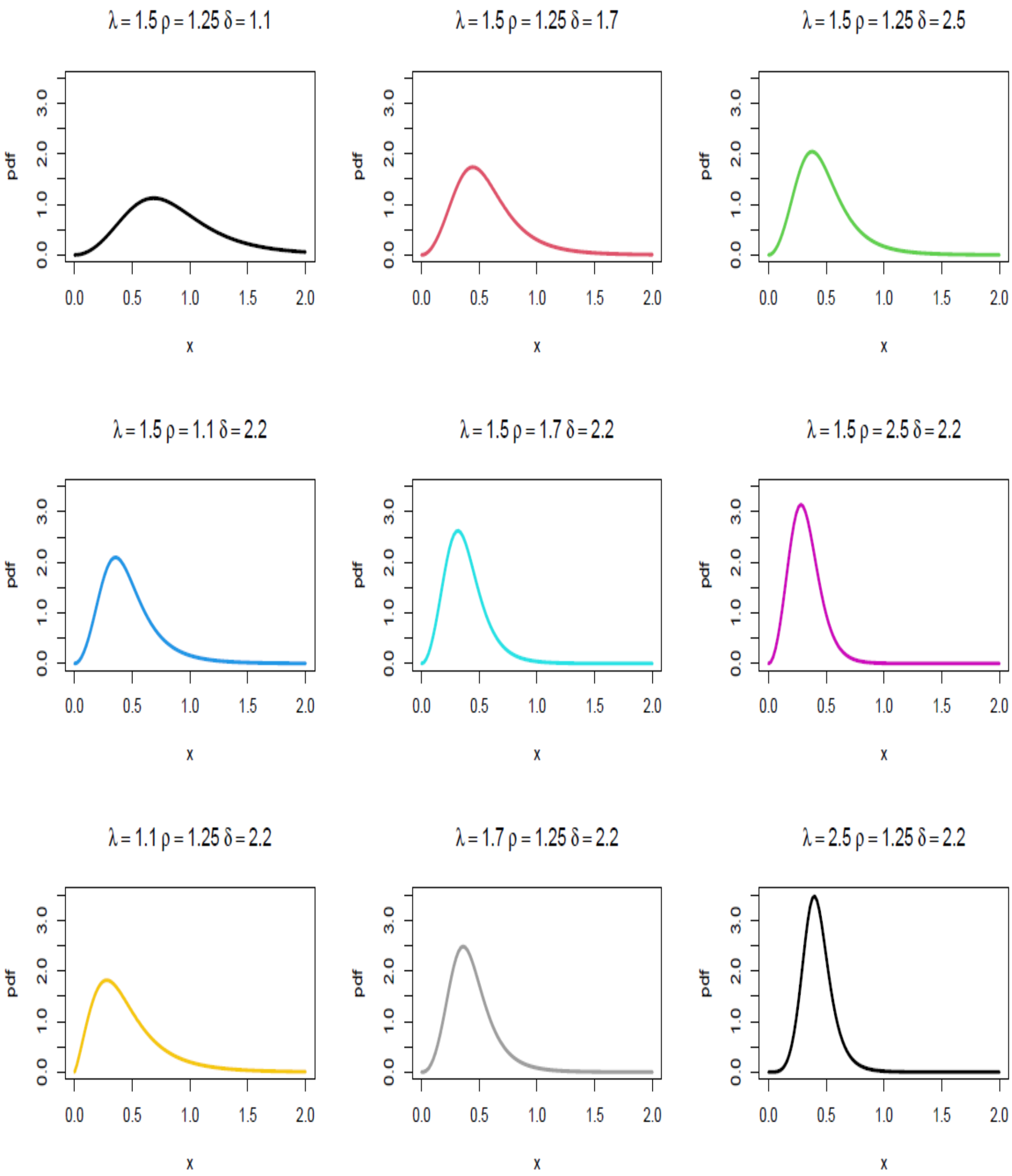

Figure 1: pdf plot for the OBXII-E distribution.

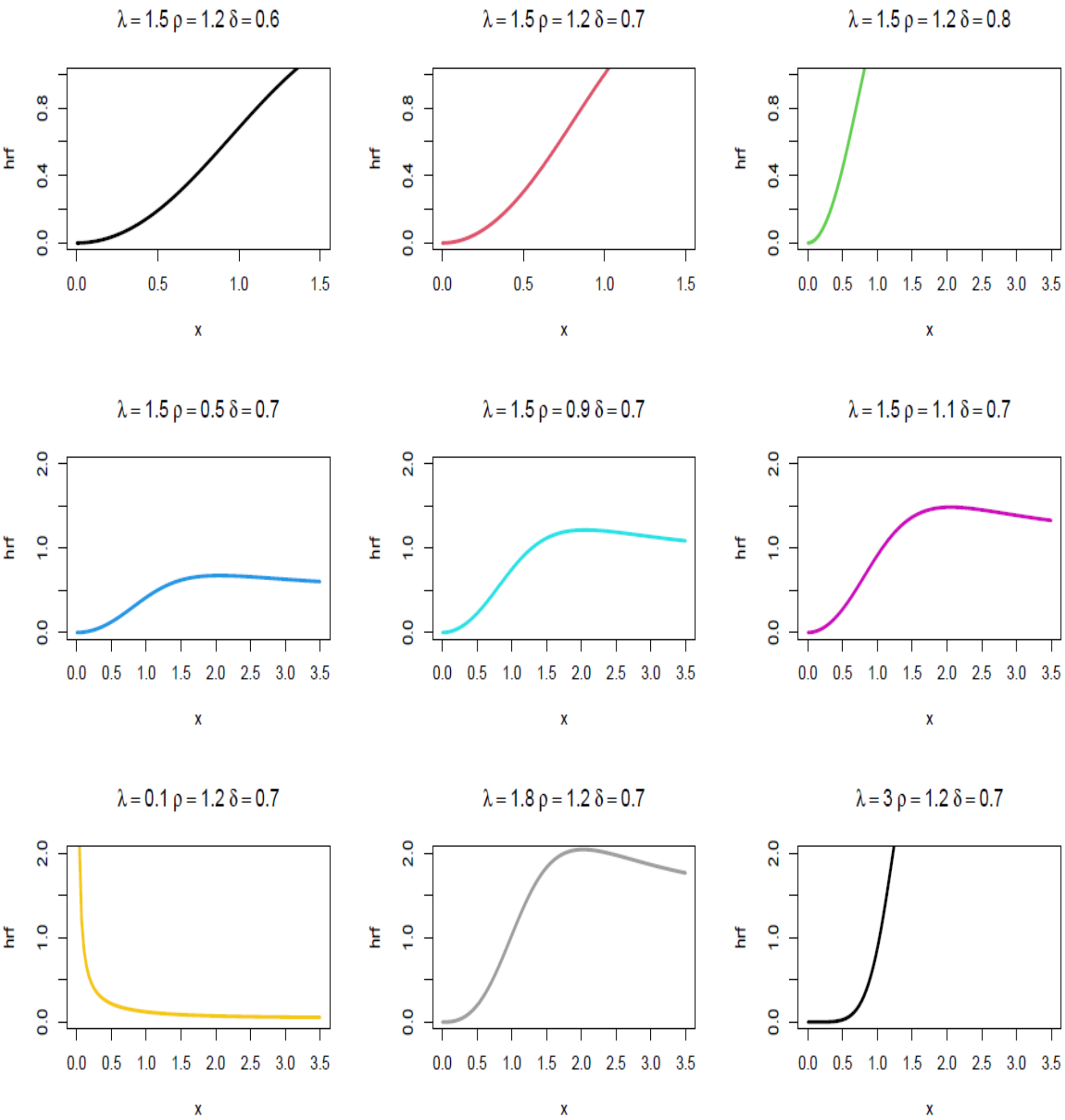

Figure 2: Hazard function plot for the OBXII-E distribution.

3. Non-Bayesian Estimation Methods

3.1 Maximum Likelihood Estimation

The maximum likelihood estimation (MLE) method, as discussed in references [16,17], maximizes the log-likelihood function to estimate λ, ρ and δ

$$l(\psi)=\prod_{i=1}^{n} f(x_i;\lambda,\rho,\delta)$$

$$=\prod_{i=1}^{n}\left(\lambda\rho e^{\delta x}\left(1-e^{-\delta x}\right)\left(1+e^{-\delta x}\right)\left[\frac{\left(1-e^{-\delta x}\right)^2}{e^{-\delta x}}\right]^{\rho-1}\left[1+\left[\frac{\left(1-e^{-\delta x}\right)^2}{e^{-\delta x}}\right]^{\rho}\right]^{-(\lambda+1)}\right). \qquad (3.1)$$

Given (3.1), its natural logarithm gives

$l(\psi)=nlog\lambda+nlog\rho+\sum_{i=1}^{n}\delta x_i+log\sum_{i=1}^{n}\left(1-e^{-bx_i}\right)+log\sum_{i=1}^{n}\left(1+e^{-\delta x_i}\right)$

$$+(\rho-1)\log\sum_{i=1}^{n}\left(\frac{\left(1-e^{-\delta x_i}\right)^2}{e^{-\delta x_i}}\right)-(\lambda+1)\,log\sum_{i=1}^{n}\left(1+\left[\frac{\left(1-e^{-\delta x_i}\right)^2}{e^{-\delta x_i}}\right]^{\rho}\right). \tag{3.2}$$

By computing the first partial derivative of the log-likelihood function concerning the parameters (λ, ρ, δ), we derive:

$$\frac{\partial(l)}{\partial\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}\ln\left(1+\left[\frac{\left(1-e^{-\delta x_i}\right)^2}{e^{-\delta x_i}}\right]^{\rho}\right), \tag{3.3}$$

$$\frac{\partial(l)}{\partial\rho}=\frac{n}{\rho}+\sum_{i=1}^{n}ln\left(\frac{\left(1-e^{-\delta x_i}\right)^2}{e^{-\delta x_i}}\right)-\sum_{i=1}^{n}\frac{(\beta+1)\left[\frac{\left(1-e^{-\delta x_i}\right)^2}{e^{-\delta x_i}}\right]^{\rho}\ln\left(\frac{\left(1-e^{-\delta x_i}\right)^2}{e^{-\delta x_i}}\right)}{1+\left[\frac{(1-e^{-\delta x_i})^2}{e^{-\delta x_i}}\right]^{\rho}}, \tag{3.4}$$

$$\frac{\partial(l)}{\partial\delta}=\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\frac{x_ie^{-\delta x_i}}{1-e^{-\delta x_i}}-\sum_{i=1}^{n}\frac{x_ie^{-\delta x_i}}{1+e^{-\delta x_i}}+\sum_{i=1}^{n}\frac{(\theta-1)e^{-\delta x_i}\left[2x_i\left(1-e^{-\delta x_i}\right)+x_ie^{\delta x_i}\left(1-e^{-\delta x_i}\right)^2\right]}{(1-e^{-\delta x_i})^2}$$

$$-\sum_{i=1}^{n}\frac{(\lambda+1)\rho e^{-\delta x_i}\left[\frac{\left(1-e^{-\delta x_i}\right)^2}{e^{-\delta x_i}}\right]^{\rho}\left[2x_i\left(1-e^{-\delta x_i}\right)+\frac{x_i\left(1-e^{-\delta x_i}\right)^2}{e^{-\delta x_i}}\right]}{(1-e^{-\delta x_i})^2\left(1+\left[\frac{(1-e^{-\delta x_i})^2}{e^{-\delta x_i}}\right]^{\rho}\right)}. \tag{3.5}$$

The maximum likelihood estimates (MLEs) of the parameters ψ = (λ, ρ, δ) are derived by setting the non-linear system of equations to zero: $\frac{\partial(l)}{\partial\lambda}=\frac{\partial(l)}{\partial\rho}=\frac{\partial(l)}{\partial\delta}=0$, then solving them concurrently. Statistical software, such as R, may be used to get the required outcomes.

3.2 Least Squares Estimation (LSE)

The estimation of parameters for the OBXII-E entails the use of the LSE approach. The main objective of this estimating technique is to minimize the following equation:

$$L(\lambda,\rho,\delta)=\sum_{i=1}^{n}\left(\left(1-\left[1+\left[\frac{\left(1-e^{-\delta x_i}\right)^2}{e^{-\delta x_i}}\right]^{\rho}\right]^{-\lambda}\right)-\frac{i}{n+1}\right)^2.$$

3.3 Weighted Least Squares Estimation (WLSE)

The estimation of parameters for the OBXII-E entails the use of the WLSE approach. The main objective of this estimating technique is to minimize the following equation:

$$W(\lambda,\rho,\delta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(\left(\left(1-\left[1+\left[\frac{\left(1-e^{-\delta x_i}\right)^2}{e^{-\delta x_i}}\right]^{\rho}\right]^{-\lambda}\right)\right)-\frac{i}{n+1}\right)^2.$$

3.4 Maximum Product Space Estimators (MPSE)

The estimation of parameters for the OBXII-E entails the use of the MPSE approach. The main objective of this estimating technique is to minimize the following equation:

$$G_s(\lambda,\rho,\delta)=\frac{1}{n+1}\sum_{i=1}^{n}\ln\left(\left(1-\left[1+\left[\frac{\left(1-e^{-\delta x_i}\right)^2}{e^{-\delta x_i}}\right]^{\rho}\right]^{-\lambda}\right)-\left(1-\left[1+\left[\frac{\left(1-e^{-\delta x_{i-1}}\right)^2}{e^{-\delta x_{i-1}}}\right]^{\rho}\right]^{-\lambda}\right)\right).$$

3.5 Anderson-Darling Estimation (ADE)

The estimation of parameters for the OBXII-E entails the use of the ADE approach. The main objective of this estimating technique is to minimize the following equation:

$$A(\lambda,\rho,\delta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2! - 1)\left(\left(1 - \left[1 + \left[\frac{\left(1 - e^{-\delta x_i}\right)^2}{e^{-\delta x_i}}\right]^{\rho}\right]^{-\lambda}\right) + \ln\left[1 + \left[\frac{\left(1 - e^{-\delta x_i}\right)^2}{e^{-\delta x_i}}\right]^{\rho}\right]^{-\lambda}\right).$$

3.6 Right-Tailed Anderson-Darling Estimation (RTADE)

The estimation of parameters for the OBXII-E entails the use of the RTADE approach. The main objective of this estimating technique is to minimize the following equation:

$$R(\lambda,\rho,\delta) = \frac{n}{2} - 2\sum_{i=1}^{n}\left(1 - \left[1 + \left[\frac{\left(1 - e^{-\delta x_i}\right)^2}{e^{-\delta x_i}}\right]^{\rho}\right]^{-\lambda}\right) - \frac{1}{n}\sum_{i=1}^{n}(2! - 1)\ln\left(\left[1 + \left[\frac{\left(1 - e^{-\delta x_i}\right)^2}{e^{-\delta x_i}}\right]^{\rho}\right]^{-\lambda}\right).$$

3.7 Kolmogorov estimation (KE)

The estimation of parameters for the OBXII-E entails the use of the KE approach. The main objective of this estimating technique is to minimize the following equation:

$$KE(\lambda,\rho,\delta) = \underset{1\le i\le s}{\mathrm{MAX}}\left[\frac{i}{s} - \left[1 + \left[\frac{\left(1 - e^{-\delta x_i}\right)^2}{e^{-\delta x_i}}\right]^{\rho}\right]^{-\lambda}, \left[1 + \left[\frac{\left(1 - e^{-\delta x_i}\right)^2}{e^{-\delta x_i}}\right]^{\rho}\right]^{-\lambda} - \frac{i-1}{s}\right].$$

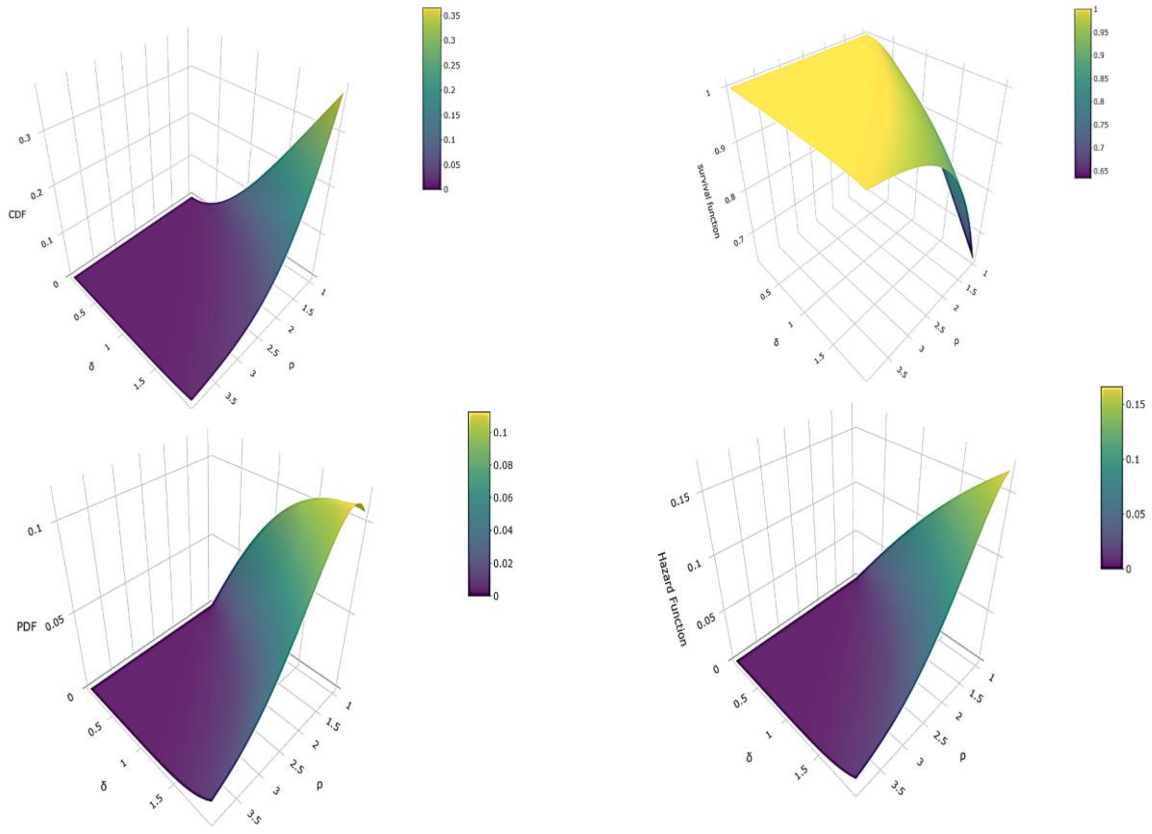

Figure 3. 3D PDF, CDF, hazard function and survival function plots for OBXII-E distribution

4. Bayesian Estimation of BXII-E Distribution Parameters

In this part, the unknown parameters of the OBXII-E distribution are addressed using the Bayesian estimate (BE). Bayesian parameter estimation may consider several loss functions, as pointed out by Albert [18] and Mood [19], including squared error loss (SEL), LINEX, and generalized entropy loss functions (GE). Using different gamma and beta priors might be considered for the λ, ρ, and δ variables. Concerning the past distributions of the OBXII-E parameters, see [20].

$$\pi_1(\lambda) \propto \lambda^{s_1-1} e^{-m_1\lambda} \quad \lambda, s_1, m_1 > 0, \tag{4.1}$$
$$\pi_2(\rho) \propto (\rho-1)^{s_2-1} e^{-m_2(\rho-1)} \quad \rho > 1,, s_2, m_2 > 0, \tag{4.2}$$
$$\pi_3(\delta) \propto \delta^{s_3-1} e^{-m_3\delta} \quad \delta, s_3, m_3 > 0, \tag{4.3}$$

with s_j, m_j, j = 1, 2, and 3 are chosen as hyper-parameters to represent what is known about the unknown parameters. Here is the joint prior to the function $\psi = (\lambda, \rho, \delta)$:

$$\pi(\psi) = \pi_1(\lambda)\pi_1(\rho)\pi_1(c)\pi(\psi) \propto \lambda^{s_1-1}(\rho-1)^{s_2-1}\, \delta^{s_3-1} e^{-m_1\lambda - m_2(\rho-1) - m_3\,\delta.} \tag{4.4}$$

The joint posterior of ψ is expressed as follows:

$$\pi(\psi \backslash x = x_1, x_2, \dots, x_n) = \frac{\pi(\psi) l(\psi)}{\int_0^\infty \int_0^\infty \pi(\psi) l(\psi) d\psi}. \tag{4.5}$$

So

$$\pi(\psi\backslash x) = \lambda^{s_1}(\rho-1)^{s_2}\, \delta^{s_3-1} e^{-m_1\lambda - m_2(\rho-1) - m_3\,\delta} \prod_{i=1}^{n} \left(e^{\delta x}\left(1-e^{-\delta x}\right)\left(1 + e^{-\delta x}\right) \left[\frac{\left(1-e^{-\delta x}\right)^2}{e^{-\delta x}}\right]^{\rho-1} \left[1 + \left[\frac{\left(1-e^{-\delta x}\right)^2}{e^{-\delta x}}\right]^{\rho}\right]^{-(\lambda+1)} \right). \tag{4.6}$$

This study included an examination of three different loss functions, with the SE loss function being mostly used, defined as follows:

$$L_{SE}\left(l(\psi), \hat{l}(\psi)\right) = \left(l(\psi) - \hat{l}(\psi)\right)^2, \tag{4.7}$$

where the function $\hat{l}(\psi)$ is used to estimate $l(\psi)$. And in a Bayesian context, the penalty for under- and over-estimation is the same when using the SE loss function. Applying the LINEX and GE loss functions yields an answer. [21]

The LINEX loss function is defined in the following way:

$$L_{\text{LINEX}}\left(l(\psi), \hat{l}(\psi)\right) = e^{v\left(l(\psi) - \hat{l}(\psi)\right)} - v\left(l(\psi), \hat{l}(\psi)\right) - 1\ , \qquad v \neq 0. \tag{4.8}$$

Define the GE loss function:

$$L_{\text{GE}}\left(l(\psi), \hat{l}(\psi)\right) = \left(\frac{\hat{l}(\psi)}{l(\psi)}\right)^{\omega} - \omega \ln\left(\frac{\hat{l}(\psi)}{l(\psi)}\right) - 1\,, \qquad \omega \neq 0. \tag{4.9}$$

Applying the SE loss function yields the BEs of $l(\psi)$ as follows:

$$\hat{l}_{SEL}(\psi) = E(l(\psi)\backslash x) = \int_{\psi} l(\psi)\, \pi(\psi\backslash x) d\psi \tag{4.10}$$

The LINEX loss function yields the BEs of $l(\psi)$ as follows:

$$\hat{l}_{LINEX}(\psi) = -\frac{1}{v} \ln\left(E_{\psi}\left[e^{-vl(\psi)}\backslash x\right]\right) \tag{4.11}$$

and the BEs of $l(\psi)$ are calculated using the GE loss function as follows:

$$\hat{l}_{GE}(\psi) = \left(E_{\psi}[l(\psi)^{-\omega}\backslash x]\right)^{-\frac{1}{\omega}} \tag{4.12}$$

Since it is difficult to calculate Equations (4.9) and (4.12) analytically, we used MCMC in the R program to estimate the unknown values. Then, to locate appropriate BEs and create posterior samples, we used the MCMC method. If you need to estimate crucial values or sample from posterior distributions, MCMC is a useful simulation strategy. By combining the MCMC method with the three functions, one may get Bayesian estimates. After burn-in is removed from the posterior density's random sample size Q, Bayes estimates may be computed using the remaining samples. We used MCMC under the

SEL, LINEX, and GEL functions to compute the BEs of $\psi^{(i)} = \left(\lambda^{(i)}, \rho^{(i)}, \delta^{(i)}\right)$.

To estimate unknown parameters, we employed MCMC in the R software as Equations (4.9)-(4.12) are difficult to compute analytically. Next, we used the MCMC approach to generate posterior samples and find suitable BEs. MCMC is a helpful simulation approach for estimating important quantities and sampling from posterior distributions. Bayesian estimates may be calculated using the MCMC approach and the three functions. Bayes estimates may be calculated using the remaining samples after eliminating burn-in from the random sample size Q produced from the posterior density. The BEs of $\psi^{(i)} = \left(\lambda^{(i)}, \rho^{(i)}, \delta^{(i)}\right)$ were computed in the following way:

$$\hat{\psi}_{SEL} = \frac{1}{Q - l_B} \sum_{i=l_B}^{Q} \psi^{(i)}, \tag{4.13}$$

$$\hat{\psi}_{LINEX} = -\frac{1}{v} \ln\left(\frac{1}{Q - l_B} \sum_{i=l_B}^{Q} e^{-v\psi^{(i)}}\right), \tag{4.14}$$

and

$$\hat{\psi}_{GE} = \left(\frac{1}{Q - l_B} \sum_{i=l_B}^{Q} \left[\psi^{(i)}\right]^{-\omega}\right)^{-\frac{1}{\omega}}. \tag{4.15}$$

where l_B is the burn-in duration of the MCMC.

5. Simulation Study

In this part, we apply Bayesian and non-Bayesian estimators to the OBXII-E distribution parameters. Seven methods: MLE, LSE, WLSE, MPSE, ADE, RTADE, and KE, were used to perform the non-Bayesian approximate. To generate N=1000 random samples, we use the OBXII-E distribution with initial parameter values of 30, 60, 120, 200, and 300.

- λ= 0.5, ρ=1.3, and δ=2.5
- λ= 0.7, ρ=1.2, and δ=3
- λ= 0.4, ρ=1.4, and δ=3.7.

The Metropolis-Hastings (MH) algorithm and MCMC were used to implement the Bayesian technique with a substantial prior. When determining the informative prior, we assumed that all gamma distribution hyperparameters were set to double the parameter values. We used the SE, GE, and LINEX loss functions to perform the Bayesian estimation. A wide variety of situations with different values for were considered in the simulation.

- λ= 0.5, ρ=2, δ=1.2
- λ= 0.7, ρ=2.3, δ=1.5.

Investigated were the two ρ values of the LINEX loss function, specifically -0.5 and 0.5. We used the GE loss function with $\boldsymbol{s_1 = s_2 = s_3 = p_1 = p_2 = p_3 = 1.5}$ to likewise examine θ = −0.5 and 0.5. Sizes 35, 70, 120, 200, and 300 were included in each of the 1000 samples. To measure how well the estimator worked, we looked at metrics including mean squared error (MSE), root mean squared error (RMSE), bias, $100(1 - \Phi)$ % confidence interval (ACI), Highest Posterior Density Intervals (HPD), average interval length (AIL), and coverage probability (CP). Bayesian findings are provided in Tables 7–10, whereas non-Bayesian estimates are shown in Tables 1– 6. Based on the simulation findings, we may deduce:

- ➢ All estimation techniques, whether non-Bayesian or Bayesian, exhibit the consistency characteristic, guaranteeing that their estimates converge to the true values as the sample size increases.
- ➢ All of our comparison criteria decrease with sample size increases, except CP.
- ➢ Tables 1-10 indicate that all estimate techniques perform very well for the OBXII-E model parameters.
- ➢ The non-Bayesian simulation results indicate that the KE method outperforms all other estimate techniques in every scenario, followed by the AD method. In the Bayesian context, the LN2 approach demonstrates superior performance compared to other estimation methods in the majority of scenarios, followed by the GE method.

Regarding computational efficiency, it was observed that non-Bayesian estimation methods such as MLE, LSE, WLSE, MPSE, ADE, RTADE, and KE required relatively low computation time, as they involve direct analytical or numerical optimization procedures. In contrast, the Bayesian estimation approach, implemented through the Metropolis–Hastings algorithm under the MCMC framework, demanded noticeably higher computation time due to the iterative sampling process and convergence checks. However, despite their higher time cost, the Bayesian estimators generally provided more stable and precise parameter estimates, indicating a clear trade-off between computational speed and estimation accuracy.

Table 1: The non-Bayesian OBXII-E model computes the Mean, RMSE, and Bias for the parameters λ= 0.5, ρ=1.3, δ=2.5

n		E. P.	MLE	LSE	WLSE	MPSE	ADE	RTADE	KE
30	Mean	$\hat{\lambda}$	0.82751{3}	1.32932{7}	1.23097{5}	0.80252{2}	1.08463{4}	1.26205{6}	0.57030{1}
		$\hat{\rho}$	1.82730{5}	1.82541{4}	1.74414{3}	2.07102{7}	1.69188{2}	2.00549{6}	1.38955{1}
		$\hat{\delta}$	2.66204{7}	2.56040{4}	2.50561{2}	2.59589{6}	2.57922{5}	2.55765{3}	2.50305{1}
	RMSE	$\hat{\lambda}$	1.43742{3}	2.12865{7}	1.97497{5}	1.39591{2}	1.66237{4}	2.03092{6}	0.22254{1}
		$\hat{\rho}$	2.06847{6}	1.57654{3}	1.68669{4}	2.17689{7}	1.47393{2}	1.76712{5}	0.38959{1}
		$\hat{\delta}$	1.75108{7}	1.54760{5}	1.51884{4}	1.65911{6}	1.41717{2}	1.44379{3}	0.46126{1}
	Bias	$\hat{\lambda}$	0.32757{3}	0.82932{7}	0.73097{5}	0.30252{2}	0.58463{4}	0.76205{6}	0.07030{1}
		$\hat{\rho}$	0.52735{5}	0.52549{4}	0.44414{3}	0.77102{7}	0.39188{2}	0.70549{6}	0.08955{1}
		$\hat{\delta}$	0.16205{7}	0.06040{5}	0.05261{4}	0.09589{6}	0.02977{1}	0.04234{3}	0.03857{2}
	$\sum Ranks$		46{6.5}	46{6.5}	35{3}	45{5}	26{2}	44{4}	10{1}
60	Mean	$\hat{\lambda}$	0.73665{3}	0.89935{6}	0.78835{4}	0.69847{2}	0.79296{5}	0.95665{7}	0.54132{1}
		$\hat{\rho}$	1.58892{5}	1.54854{4}	1.47779{3}	1.81809{7}	1.45451{2}	1.66072{6}	1.35542{1}
		$\hat{\delta}$	2.61608{6}	2.53253{4}	2.53136{3}	2.65401{7}	2.52445{1}	2.53874{5}	2.52490{2}
	RMSE	$\hat{\lambda}$	1.02805{4}	1.27635{6}	1.00996{2}	1.02531{3}	1.06293{5}	1.47857{7}	0.20331{1}
		$\hat{\rho}$	1.27947{6}	0.87777{4}	0.81558{3}	1.81155{7}	0.55931{2}	0.98395{5}	0.25675{1}
		$\hat{\delta}$	1.15708{5}	1.14098{4}	1.05395{3}	1.23313{7}	1.01819{2}	1.16548{6}	0.44063{1}
	Bias	$\hat{\lambda}$	0.23665{3}	0.39935{6}	0.28835{4}	0.19847{2}	0.29296{5}	0.45665{7}	0.04132{1}
		$\hat{\rho}$	0.28892{5}	0.24854{4}	0.17449{3}	0.51809{7}	0.15451{2}	0.36072{6}	0.05542{1}
		$\hat{\delta}$	0.11608{7}	0.03253{3}	0.03136{2}	0.08540{6}	0.02445{1}	0.03874{5}	0.03490{4}
	$\sum Ranks$		44{5}	41{4}	27{3}	48{6}	25{2}	54{7}	13{1}
120	Mean	$\hat{\lambda}$	0.62596{3}	0.69277{6}	0.64037{5}	0.60091{2}	0.63458{4}	0.75489{7}	0.51576{1}
		$\hat{\rho}$	1.38116{4}	1.38906{5}	1.35440{2}	1.46399{7}	1.35813{3}	1.43837{6}	1.34057{1}
		$\hat{\delta}$	2.52304{1}	2.59582{7}	2.58795{6}	2.57229{4}	2.58278{5}	2.56471{3}	2.53062{2}
	RMSE	$\hat{\lambda}$	0.62303{5}	0.72554{6}	0.55603{3}	0.60604{4}	0.55445{2}	0.89686{7}	0.14748{1}
		$\hat{\rho}$	0.32604{4}	0.40027{5}	0.31504{3}	0.41229{6}	0.30404{2}	0.50716{7}	0.19610{1}
		$\hat{\delta}$	0.74945{4}	0.83202{6}	0.74616{3}	0.78710{5}	0.72784{2}	0.89156{7}	0.37979{1}
	Bias	$\hat{\lambda}$	0.12596{3}	0.19277{6}	0.14037{5}	0.10091{2}	0.13458{4}	0.25489{7}	0.01576{1}
		$\hat{\rho}$	0.08116{4}	0.08906{5}	0.05440{2}	0.16399{7}	0.05813{3}	0.13837{6}	0.04057{1}
		$\hat{\delta}$	0.02304{4}	0.00417{1}	0.01204{2}	0.07229{7}	0.01721{3}	0.03528{6}	0.03062{5}
	$\sum Ranks$		32{4}	47{6}	31{3}	44{5}	28{2}	56{7}	14{1}
200	Mean	$\hat{\lambda}$	0.55997{3}	0.61042{6}	0.57066{5}	0.54588{2}	0.57037{4}	0.63305{7}	0.50576{1}
		$\hat{\rho}$	1.34076{4}	1.34346{5}	1.32789{1}	1.38873{7}	1.32937{2}	1.37207{6}	1.33164{3}
		$\hat{\delta}$	2.51684{1}	2.59176{6}	2.59680{7}	2.55588{3}	2.59146{5}	2.57446{4}	2.53778{2}
	RMSE	$\hat{\lambda}$	0.31040{4}	0.47212{6}	0.30573{3}	0.31379{5}	0.30333{2}	0.52511{7}	0.12318{1}
		$\hat{\rho}$	0.21105{4}	0.20621{2}	0.21455{5}	0.24392{6}	0.20874{3}	0.32325{7}	0.15569{1}
		$\hat{\delta}$	0.58319{3}	0.65393{6}	0.58505{4}	0.60328{5}	0.57616{2}	0.70949{7}	0.34026{1}
	Bias	$\hat{\lambda}$	0.05997{3}	0.11042{6}	0.07066{5}	0.04588{2}	0.07037{4}	0.13303{7}	0.00576{1}
		$\hat{\rho}$	0.04076{4}	0.04346{5}	0.02789{1}	0.08873{7}	0.02937{2}	0.07207{6}	0.03164{3}
		$\hat{\delta}$	0.01689{4}	0.00223{1}	0.00319{2}	0.05588{7}	0.00853{3}	0.02553{6}	0.02078{5}
	$\sum Ranks$		30{3}	43{5}	33{4}	44{6}	27{2}	57{7}	17{1}
300	Mean	$\hat{\lambda}$	0.52245{3}	0.56170{6}	0.53792{5}	0.51081{2}	0.53742{4}	0.57559{7}	0.50457{1}
		$\hat{\rho}$	1.33628{5}	1.33424{4}	1.32678{1}	1.37084{7}	1.32833{2}	1.35538{6}	1.32873{3}
		$\hat{\delta}$	2.53542{3}	2.59939{7}	2.51186{2}	2.56646{5}	2.50896{1}	2.59099{6}	2.53727{4}
	RMSE	$\hat{\lambda}$	0.19039{2}	0.29719{6}	0.21560{5}	0.19272{3}	0.21189{4}	0.34977{7}	0.11767{1}
		$\hat{\rho}$	0.17120{2}	0.22046{6}	0.18197{4}	0.19353{5}	0.17921{3}	0.26876{7}	0.14473{1}
		$\hat{\delta}$	0.45426{2}	0.53642{6}	0.47232{5}	0.47056{4}	0.46841{3}	0.58559{7}	0.30861{1}
	Bias	$\hat{\lambda}$	0.02245{3}	0.06170{6}	0.03792{5}	0.01081{2}	0.03742{4}	0.07559{7}	0.00457{1}
		$\hat{\rho}$	0.03628{5}	0.03424{4}	0.02678{1}	0.07084{7}	0.02833{2}	0.05538{6}	0.02873{3}
		$\hat{\delta}$	0.01142{5}	0.00161{2}	0.00118{1}	0.04646{7}	0.00806{3}	0.00900{4}	0.01727{6}
	$\sum Ranks$		30{4}	47{6}	29{3}	42{5}	26{2}	57{7}	21{1}
Over Ranks			**22.5{4}**	**27.5{6}**	**16{3}**	**27{5}**	**10{2}**	**32{7}**	**5{1}**

Table 2: The non-Bayesian OBXII-E model computes the ACI, AIL, and CP for the parameters λ= 0.5, ρ=1.3, δ=2.5

n		E. P.	MLE	LSE	WLSE	MPSE	ADE	RTADE	KE
30	ACI	$\hat{\lambda}$	0.00000{2} 3.16138{3}	0.00000{2} 4.85184{7}	0.00000{2} 4.70713{6}	0.00000{2} 3.10388{2}	0.00000{2} 3.67185{4}	0.00000{2} 4.41406{5}	0.18446{1} 0.95614{1}
		$\hat{\rho}$	0.00000{2} 5.57618{6}	0.00000{2} 4.34466{3}	0.00000{2} 4.56968{4}	0.00000{2} 5.99339{7}	0.00000{2} 4.08010{2}	0.00000{2} 4.96709{5}	0.68570{1} 2.09340{1}
		$\hat{\delta}$	0.00000{2} 5.65750{7}	0.00000{2} 5.24228{3}	0.00000{2} 5.32659{5}	0.00000{2} 5.33729{6}	0.00000{2} 5.11438{2}	0.00000{2} 5.25436{4}	1.64562{1} 3.36043{1}
	AIL	$\hat{\lambda}$	3.16138{3}	4.85184{7}	4.70713{6}	3.10388{2}	3.67185{4}	4.41406{5}	0.77168{1}
		$\hat{\rho}$	5.57618{6}	4.34466{3}	4.56968{4}	5.99339{7}	4.08010{2}	4.96709{5}	1.40770{1}
		$\hat{\delta}$	5.65750{7}	5.24228{3}	5.32659{5}	5.33729{6}	5.11438{2}	5.25436{4}	1.71474{1}
	CP	$\hat{\lambda}$	95.00{1}	92.30{6}	93.10{5}	93.80{3}	93.30{4}	91.70{7}	94.90{2}
		$\hat{\rho}$	97.30{1}	94.50{6}	95.70{4}	96.70{2}	96.30{3}	93.90{7}	94.60{5}
		$\hat{\delta}$	95.40{4}	94.70{6}	96.00{3}	95.00{5}	96.10{2}	96.70{1}	93.20{7]
	$\sum Ranks$		44{3}	50{7}	48{5}	46{4}	31{2}	49{6}	23{1}
60	ACI	$\hat{\lambda}$	0.00000{2} 2.39595{2}	0.00000{2} 2.94778{7}	0.00000{2} 2.57393{5}	0.00000{2} 2.53414{4}	0.00000{2} 2.78134{6}	0.00000{2} 3.52579{3}	0.19970{1} 0.88294{1}
		$\hat{\rho}$	0.0000{6.5} 3.74147{6}	0.07275{4} 3.02432{4}	0.00788{5} 2.94110{3}	0.00000{6.5} 5.11273{7}	0.42982{3} 2.47921{2}	0.09211{1} 3.22933{5}	0.90207{2} 1.80878{1}
		$\hat{\delta}$	0.67535{3} 4.55682{4}	0.45253{7} 4.61252{6}	0.64534{4} 4.41738{3}	0.53951{5} 4.76850{7}	0.69458{2} 4.35431{2}	0.46696{6} 4.61052{5}	1.67603{1} 3.37377{1}
	AIL	$\hat{\lambda}$	2.39595{2}	2.94778{6}	2.57393{4}	2.53414{3}	2.78134{5}	3.52579{7}	0.68324{1}
		$\hat{\rho}$	3.74147{6}	2.95156{4}	2.93322{3}	5.11273{7}	2.04939{2}	3.13722{5}	0.90671{1}
		$\hat{\delta}$	3.88147{4}	4.15998{6}	3.77203{3}	4.22899{7}	3.65973{2}	4.14355{5}	1.69774{1}
	CP	$\hat{\lambda}$	94.70{5}	93.50{7}	95.20{3}	95.80{2}	95.90{1}	93.80{6}	94.80{4}
		$\hat{\rho}$	98.00{1}	95.30{5}	97.50{3}	97.80{2}	96.10{4}	94.80{6}	93.80{7}
		$\hat{\delta}$	90.20{7}	94.90{1}	92.80{4}	91.10{6}	92.30{5}	94.00{3}	94.20{2}
	$\sum Ranks$		48.5{4}	59{7}	42{3}	58.5{6}	36{2}	54{5}	23{1}
120	ACI	$\hat{\lambda}$	0.00000{2} 1.78817{5}	0.00000{2} 1.92564{6}	0.00000{2} 1.52744{2}	0.00000{2} 1.72580{4}	0.00000{2} 1.62498{3}	0.00000{2} 2.19954{7}	0.24821{1} 0.78331{1}
		$\hat{\rho}$	0.81560{4} 1.94672{4}	0.74439{6} 2.03374{6}	0.87286{2} 1.88094{2}	0.80866{5} 2.11932{5}	0.83218{3} 1.88408{3}	0.53784{7} 2.33890{7}	0.96862{1} 1.71252{1}
		$\hat{\delta}$	1.08543{5} 3.96066{5}	0.92068{6} 4.07097{6}	1.10152{4} 3.87737{4}	1.27399{2} 3.87059{3}	1.26220{3} 3.70336{2}	0.79058{7} 4.13884{7}	1.90198{1} 3.15926{1}
	AIL	$\hat{\lambda}$	1.78817{5}	1.92564{6}	1.52744{2}	1.72580{4}	1.62498{3}	2.19954{7}	0.53510{1}
		$\hat{\rho}$	1.13112{4}	1.28934{5}	1.05308{3}	1.31065{6}	1.05189{2}	1.80106{7}	0.74390{1}
		$\hat{\delta}$	2.87522{5}	3.15028{6}	2.77284{4}	2.59659{3}	2.44115{2}	3.34826{7}	1.25727{1}
	CP	$\hat{\lambda}$	97.30{1}	95.60{5}	96.10{4}	97.20{2}	96.90{3}	94.70{6}	94.30{7}
		$\hat{\rho}$	94.20{4}	93.90{6}	93.10{7}	94.50{3}	94.00{5}	95.40{1}	95.20{2}
		$\hat{\delta}$	94.00{3}	94.30{1}	94.20{2}	90.30{7}	90.90{5}	92.80{4}	90.40{6}
	$\sum Ranks$		47{5}	61{6}	37{3}	46{4}	36{2}	69{7}	24{1}
200	ACI	$\hat{\lambda}$	0.00000{4} 1.14811{5}	0.00000{4} 1.50216{6}	0.01048{2} 1.13085{3}	0.00601{3} 1.08576{2}	0.00000{4} 1.14082{4}	0.00000{4} 1.51582{7}	0.26663{1} 0.74490{1}
		$\hat{\rho}$	0.99444{3} 1.68708{3}	0.85290{6} 1.83402{6}	0.92981{5} 1.72598{4}	1.00502{2} 1.77244{5}	0.98036{4} 1.67839{2}	0.85242{7} 1.89172{7}	1.05000{1} 1.61328{1}
		$\hat{\delta}$	1.48372{4} 3.55006{3}	1.23563{6} 3.74790{6}	1.51711{3} 3.47649{4}	1.42252{5} 3.68924{5}	1.52416{2} 3.45876{2}	1.16963{7} 3.77929{7}	1.89671{1} 3.17886{1}
	AIL	$\hat{\lambda}$	1.14811{5}	1.50216{6}	1.12037{3}	1.07975{2}	1.14082{4}	1.51582{7}	0.47827{1}
		$\hat{\rho}$	0.69264{2}	0.98112{6}	0.79617{5}	0.76742{4}	0.69803{3}	1.03929{7}	0.56328{1}
		$\hat{\delta}$	2.06634{4}	2.51227{6}	1.95938{3}	2.26671{5}	1.93460{2}	2.60966{7}	1.28215{1}
	CP	$\hat{\lambda}$	96.00{2}	96.10{1}	94.30{7}	94.70{5}	95.20{4}	94.50{6}	95.30{3}
		$\hat{\rho}$	92.70{5}	95.50{1}	94.80{3}	93.40{4}	92.40{6.5}	92.40{6.5}	94.90{2}
		$\hat{\delta}$	91.10{5}	93.70{2}	90.60{7}	93.00{4}	90.80{6}	93.10{3}	94.10{1}
	$\sum Ranks$		45{3}	56{6}	49{5}	46{4}	43.5{2}	75.5{7}	15{1}
300	ACI	$\hat{\lambda}$	0.16870{3} 0.87621{2}	0.04324{6} 1.08017{6}	0.13940{5} 0.93645{5}	0.14124{4} 0.88039{3}	0.17933{2} 0.89552{4}	0.00000{7} 1.17249{7}	0.29510{1} 0.71405{1}
		$\hat{\rho}$	1.01389{4} 1.65867{4}	0.91959{6} 1.74890{6}	1.02078{3} 1.63278{2}	1.04399{2} 1.69768{5}	1.00659{5} 1.65006{3}	0.91136{7} 1.79939{7}	1.06641{1} 1.59104{1}
		$\hat{\delta}$	1.72919{3} 3.34164{4}	1.49353{6} 3.50524{7}	1.68760{5} 3.33612{3}	1.78943{2} 3.34349{5}	1.70778{4} 3.31013{2}	1.48234{7} 3.49963{6}	1.97354{1} 3.10100{1}
	AIL	$\hat{\lambda}$	0.70750{2}	1.03692{6}	0.79704{5}	0.73914{4}	0.71619{3}	1.17249{7}	0.41894{1}
		$\hat{\rho}$	0.64477{4}	0.82930{6}	0.61199{2}	0.65369{5}	0.64347{3}	0.88802{7}	0.52463{1}
		$\hat{\delta}$	1.61245{4}	2.01170{6}	1.64852{5}	1.55406{2}	1.60233{3}	2.01728{7}	1.12746{1}
	CP	$\hat{\lambda}$	95.90{3}	95.20{5.5}	96.30{1}	96.10{2}	95.20{5.5}	95.40{4}	94.50{7}
		$\hat{\rho}$	94.30{3}	94.80{1}	92.30{7}	93.60{5}	94.10{4}	93.00{6}	94.50{2}
		$\hat{\delta}$	93.20{3}	93.40{2}	92.30{6}	90.90{7}	91.60{4}	91.50{5}	93.60{1}
	$\sum Ranks$		39{2}	63.5{6}	59{4}	46{5}	42.5{3}	75{7}	19{1}
Over Ranks			**17{3}**	**32{6.5}**	**20{4}**	**23{5}**	**11{2}**	**32{6.5}**	**5{1}**

Table 3: The non-Bayesian OBXII-E model computes the Mean, RMSE, and Bias for the parameters λ= 0.7, ρ=1.2, δ=3

n		E. P.	MLE	LSE	WLSE	MPSE	ADE	RTADE	KE
30	Mean	$\hat{\lambda}$	1.16363{2}	1.94512{7}	1.85696{6}	1.26770{3}	1.78657{4}	1.82568{5}	0.78348{1}
		$\hat{\rho}$	1.60999{4}	1.61582{5}	1.54537{3}	1.90161{7}	1.49703{2}	1.79073{6}	1.27996{1}
		$\hat{\delta}$	3.21729{6}	3.18160{5}	3.11416{3}	3.14812{4}	3.65864{7}	3.08199{2}	3.03081{1}
	RMSE	$\hat{\lambda}$	1.99462{2}	2.98739{6}	2.99583{7}	2.36729{3}	2.88653{4}	2.93020{5}	0.28965{1}
		$\hat{\rho}$	1.61292{6}	1.28538{3}	1.31130{4}	2.14819{7}	1.08390{2}	1.52816{5}	0.32788{1}
		$\hat{\delta}$	2.46994{7}	2.11721{5}	2.07069{4}	2.36034{6}	1.94147{3}	1.92061{2}	0.51353{1}
	Bias	$\hat{\lambda}$	0.46363{2}	1.25126{7}	1.15696{6}	0.56770{3}	1.08657{4}	1.12568{5}	0.08348{1}
		$\hat{\rho}$	0.40999{4}	0.41582{5}	0.34537{3}	0.70161{7}	0.29703{2}	0.59731{6}	0.07996{1}
		$\hat{\delta}$	0.21729{5}	0.18160{4}	0.11414{3}	0.24812{6}	0.09586{2}	0.31997{7}	0.07081{1}
	$\sum Ranks$		38{3}	47{7}	39{4}	46{6}	28{2}	43{5}	9{1}
60	Mean	$\hat{\lambda}$	1.06432{2}	1.37919{7}	1.29924{5}	1.11800{3}	1.24173{4}	1.36087{6}	0.73452{1}
		$\hat{\rho}$	1.40908{5}	1.38572{4}	1.33074{3}	1.53697{7}	1.32819{2}	1.46648{6}	1.25400{1}
		$\hat{\delta}$	3.17415{6}	3.11422{4}	3.09003{2}	3.19096{7}	3.09105{3}	3.13657{5}	3.06072{1}
	RMSE	$\hat{\lambda}$	1.54854{2}	2.02688{6}	1.98872{5}	1.74154{3}	1.83655{4}	2.10732{7}	0.22465{1}
		$\hat{\rho}$	0.92104{6}	0.63213{4}	0.55425{3}	1.11594{7}	0.45073{2}	0.72745{5}	0.23535{1}
		$\hat{\delta}$	1.60678{6}	1.57412{5}	1.44766{3}	1.69118{7}	1.39799{2}	1.54119{4}	0.49411{1}
	Bias	$\hat{\lambda}$	0.36432{2}	0.67919{7}	0.59924{5}	0.41800{3}	0.54173{4}	0.66087{6}	0.03452{1}
		$\hat{\rho}$	0.20908{5}	0.18572{4}	0.13074{3}	0.33697{7}	0.12841{2}	0.26648{6}	0.05400{1}
		$\hat{\delta}$	0.17415{6}	0.11722{4}	0.09003{2}	0.19096{7}	0.09105{3}	0.13657{5}	0.06072{1}
	$\sum Ranks$		40{4}	45{5}	31{3}	51{6}	26{2}	60{7}	9{1}
120	Mean	$\hat{\lambda}$	0.94004{2}	1.04176{6}	0.96212{5}	0.95362{4}	0.94346{3}	1.07066{7}	0.70956{1}
		$\hat{\rho}$	1.25905{4}	1.26733{5}	1.24294{2}	1.31643{7}	1.24756{3}	1.30256{6}	1.23529{1}
		$\hat{\delta}$	3.01798{3}	3.01460{2}	3.08020{6}	3.05603{4}	3.08742{7}	3.00558{1}	3.05729{5}
	RMSE	$\hat{\lambda}$	0.98754{3}	1.14788{6}	0.99397{4}	1.09427{5}	0.96384{2}	1.26470{7}	0.17260{1}
		$\hat{\rho}$	0.26659{4}	0.32066{6}	0.25976{3}	0.32052{5}	0.25286{2}	0.37577{7}	0.16636{1}
		$\hat{\delta}$	1.04970{4}	1.15115{6}	1.02656{3}	1.10567{5}	1.00127{2}	1.16940{7}	0.43344{1}
	Bias	$\hat{\lambda}$	0.24004{2}	0.34176{6}	0.26212{5}	0.25362{4}	0.24346{3}	0.37066{7}	0.00956{1}
		$\hat{\rho}$	0.05905{4}	0.06733{5}	0.04294{2}	0.11643{7}	0.04756{3}	0.10256{6}	0.03529{1}
		$\hat{\delta}$	0.08798{7}	0.01460{3}	0.00199{1}	0.05603{5}	0.00257{2}	0.05587{4}	0.05729{6}
	$\sum Ranks$		33{4}	45{5}	31{3}	46{6}	27{2}	52{7}	18{1}
200	Mean	$\hat{\lambda}$	0.83963{5}	0.90972{6}	0.83311{3}	0.83738{4}	0.82763{2}	0.90991{7}	0.70238{1}
		$\hat{\rho}$	1.22921{4}	1.23264{5}	1.22146{2}	1.26342{7}	1.22353{3}	1.25354{6}	1.22442{1}
		$\hat{\delta}$	3.01509{4}	3.00994{3}	3.00147{1}	3.04759{5}	3.06822{7}	3.00855{2}	3.06240{6}
	RMSE	$\hat{\lambda}$	0.74038{4}	0.82371{7}	0.54212{3}	0.74326{4}	0.51611{2}	0.79317{6}	0.16888{1}
		$\hat{\rho}$	0.17417{3}	0.21239{6}	0.17691{4}	0.19644{5}	0.17316{2}	0.25162{7}	0.12442{1}
		$\hat{\delta}$	0.80744{4}	0.90693{6}	0.80562{3}	0.83870{5}	0.79113{2}	0.94319{7}	0.40654{1}
	Bias	$\hat{\lambda}$	0.13963{5}	0.20972{6}	0.13311{3}	0.13738{4}	0.12763{2}	0.20991{7}	0.00238{1}
		$\hat{\rho}$	0.02921{4}	0.03264{5}	0.02146{1}	0.06342{7}	0.02353{2}	0.05354{6}	0.02442{3}
		$\hat{\delta}$	0.05509{7}	0.00543{3}	0.00147{1}	0.04759{6}	0.00178{2}	0.00944{4}	0.04240{5}
	$\sum Ranks$		40{4}	47{5.5}	21{2}	47{7}	24{3}	52{5.5}	20{1}
300	Mean	$\hat{\lambda}$	0.74859{3}	0.82148{7}	0.77169{5}	0.74292{2}	0.76822{4}	0.81689{6}	0.70299{1}
		$\hat{\rho}$	1.22809{5}	1.22571{4}	1.22107{2}	1.25288{7}	1.22302{3}	1.24163{6}	1.21898{1}
		$\hat{\delta}$	3.04567{5}	3.00291{1}	3.01897{3}	3.07229{7}	3.01900{4}	3.00820{2}	3.05216{6}
	RMSE	$\hat{\lambda}$	0.31222{2}	0.53162{7}	0.36350{5}	0.32996{3}	0.35007{4}	0.52723{6}	0.14312{1}
		$\hat{\rho}$	0.14526{2}	0.18190{6}	0.15308{4}	0.16074{5}	0.15118{3}	0.21022{7}	0.10934{1}
		$\hat{\delta}$	0.63279{2}	0.75874{6}	0.65805{4}	0.65768{5}	0.65119{3}	0.77537{7}	0.36094{1}
	Bias	$\hat{\lambda}$	0.04859{3}	0.12148{7}	0.07169{5}	0.04292{2}	0.06822{4}	0.11698{6}	0.00199{1}
		$\hat{\rho}$	0.02809{5}	0.02571{4}	0.02107{2}	0.05288{7}	0.02302{3}	0.04163{6}	0.01898{1}
		$\hat{\delta}$	0.04556{7}	0.00291{3}	0.00119{1}	0.03229{6}	0.00121{2}	0.00850{4}	0.02216{5}
	$\sum Ranks$		34{4}	45{6}	31{3}	44{5}	30{2}	50{7}	18{1}
Over Ranks			**19{4}**	**28.5{5}**	**15{3}**	**30{6}**	**11{2}**	**31.5{7}**	**5{1}**

Table 4: The non-Bayesian OBXII-E model computes the ACI, AIL, and CP for the parameters λ= 0.7, ρ=1.2, δ=3

n		E. P.	MLE	LSE	WLSE	MPSE	ADE	RTADE	KE
30	ACI	$\hat{\lambda}$	0.00000{2} 4.39862{2}	0.00000{2} 6.82437{6}	0.00000{2} 7.09270{7}	0.00000{2} 5.14902{3}	0.00000{2} 6.23256{4}	0.00000{2} 6.35538{5}	0.27662{1} 1.29033{1}
		$\hat{\rho}$	0.00000{2} 4.53385{6}	0.00000{2} 3.67718{3}	0.00000{2} 3.74203{4}	0.00000{2} 5.81352{7}	0.00000{2} 3.24911{2}	0.00000{2} 4.36697{5}	0.68967{1} 1.87025{1}
		$\hat{\delta}$	0.00000{2} 7.44384{7}	0.00000{2} 6.83984{4}	0.00000{2} 6.95427{5}	0.00000{2} 7.04700{6}	0.00000{2} 6.67423{2}	0.00000{2} 6.80053{3}	2.04446{1} 4.01716{1}
	AIL	$\hat{\lambda}$	4.39862{2}	6.82437{6}	7.09270{7}	5.14902{3}	6.23256{4}	6.35538{5}	1.01370{1}
		$\hat{\rho}$	4.53385{6}	3.67718{3}	3.74203{4}	5.81352{7}	3.24911{2}	4.36697{5}	1.18058{1}
		$\hat{\delta}$	7.44384{7}	6.83984{4}	6.95427{5}	7.14700{6}	6.67423{2}	6.80053{3}	1.97269{1}
	CP	$\hat{\lambda}$	94.30{2}	92.30{4}	91.90{5}	93.80{3}	92.20{6}	91.00{7}	94.40{1}
		$\hat{\rho}$	97.20{1}	95.20{5}	96.40{4}	96.90{2}	96.00{3}	94.20{7}	95.00{6}
		$\hat{\delta}$	94.90{4}	94.50{6}	95.10{3}	94.80{5}	95.20{2}	95.70{1}	92.10{7}.
	$\sum Ranks$		43{3}	47{4.5}	50{7}	49{6}	33{2}	47{4.5}	23{1}
60	ACI	$\hat{\lambda}$	0.00000{2} 3.56060{2}	0.00000{2} 4.60608{4}	0.00000{2} 4.79743{6}	0.00000{2} 4.20322{3}	0.00000{2} 4.65668{5}	0.00000{2} 5.01651{7}	0.35358{1} 1.11545{1}
		$\hat{\rho}$	0.0000{6.5} 2.95818{6}	0.32648{4} 2.44495{4}	0.33924{3} 2.32225{3}	0.0000{6.5} 3.55609{7}	0.50445{2} 2.15193{2}	0.30664{5} 2.62631{5}	0.83975{1} 1.66825{1}
		$\hat{\delta}$	0.48146{4} 5.86685{4}	0.25443{7} 5.98004{6}	0.50333{3} 5.67673{3}	0.28684{6} 6.09508{7}	0.58322{2} 5.59888{2}	0.40620{5} 5.86694{5}	2.11452{1} 4.00692{1}
	AIL	$\hat{\lambda}$	3.56060{2}	4.60608{4}	4.79743{6}	4.20322{3}	4.65668{5}	5.01651{7}	0.76186{1}
		$\hat{\rho}$	2.95818{5}	2.11846{4}	1.98301{3}	3.55609{7}	1.64747{2}	2.31967{6}	0.82850{1}
		$\hat{\delta}$	5.38539{4}	5.72556{6}	5.17339{3}	5.80824{7}	5.01566{2}	5.46073{5}	1.89240{1}
	CP	$\hat{\lambda}$	95.30{2}	92.70{6}	94.50{4}	95.10{3}	95.40{1}	94.10{5}	92.50{7}
		$\hat{\rho}$	97.90{2}	95.00{5}	96.20{3}	98.20{1}	95.80{4}	94.90{6}	94.70{7}
		$\hat{\delta}$	90.60{7}	95.40{2}	94.20{3}	93.00{6}	93.40{5}	95.60{1}	93.70{4}
	$\sum Ranks$		64.5{7}	54{4}	42{3}	58.5{5}	34{2}	59{6}	27{1}
120	ACI	$\hat{\lambda}$	0.00000{2} 2.76462{4}	0.00000{2} 2.97325{6}	0.00000{2} 2.54293{2}	0.00000{2} 2.95738{5}	0.00000{2} 2.66065{3}	0.00000{2} 3.10215{7}	0.39513{1} 1.02400{1}
		$\hat{\rho}$	0.79343{5} 1.72467{4}	0.74940{6} 1.78525{5}	0.80822{3} 1.67766{2}	0.79909{4} 1.83378{6}	0.80987{2} 1.68524{3}	0.63535{7} 1.96978{7}	0.92013{1} 1.55046{1}
		$\hat{\delta}$	1.00378{5} 5.03218{5}	0.83549{6} 5.19371{6}	1.09032{4} 4.90568{4}	1.22688{3} 4.88518{3}	1.31784{2} 4.67700{2}	0.80803{7} 5.20312{7}	2.04381{1} 3.77076{1}
	AIL	$\hat{\lambda}$	2.76462{4}	2.97325{6}	2.54293{2}	2.95738{5}	2.66065{3}	3.10215{7}	0.62886{1}
		$\hat{\rho}$	0.93124{4}	1.03585{6}	0.86943{3}	1.03469{5}	0.87536{2}	1.33441{7}	0.63032{1}
		$\hat{\delta}$	4.02839{5}	4.35822{6}	3.81536{4}	3.65829{3}	3.35915{2}	4.39508{7}	1.62694{1}
	CP	$\hat{\lambda}$	96.40{3}	94.40{6}	96.00{4}	97.00{1}	96.60{2}	94.50{5}	93.10{7}
		$\hat{\rho}$	94.40{3}	93.80{5}	93.40{7}	93.60{6}	94.10{4}	94.50{2}	96.20{1}
		$\hat{\delta}$	94.80{1}	94.70{2}	93.70{3}	90.80{6}	91.60{5}	93.60{4}	90.00{7}
	$\sum Ranks$		45{4}	62{6}	40{3}	49{5}	32{2}	69{7}	24{1}
200	ACI	$\hat{\lambda}$	0.00000{2} 2.24377{6}	0.00000{2} 2.45720{7}	0.00000{2} 1.82283{3}	0.00000{2} 2.10777{4}	0.00000{2} 1.79448{2}	0.00000{2} 2.23915{5}	0.40421{1} 1.03055{1}
		$\hat{\rho}$	0.94205{3} 1.51637{3}	0.83445{6} 1.63083{6}	0.89285{5} 1.55008{4}	0.94945{2} 1.57740{5}	0.93379{4} 1.51327{2}	0.84809{7} 1.65899{7}	0.99902{1} 1.44982{1}
		$\hat{\delta}$	1.58429{4} 4.44589{4}	1.25719{6} 4.74171{7}	1.65241{3} 4.35054{3}	1.46769{5} 4.62748{5}	1.66986{2} 4.32658{2}	1.25488{7} 4.72622{6}	2.30082{1} 3.82398{1}
	AIL	$\hat{\lambda}$	2.24377{6}	2.45720{7}	1.82283{3}	2.10777{4}	1.79448{2}	2.23915{5}	0.62634{1}
		$\hat{\rho}$	0.57432{2}	0.79638{6}	0.65722{5}	0.62794{4}	0.57948{3}	0.81090{7}	0.45080{1}
		$\hat{\delta}$	2.86160{4}	3.48451{6}	2.69812{3}	3.15970{5}	2.65672{2}	3.47133{7}	1.52315{1}
	CP	$\hat{\lambda}$	97.50{1}	95.90{3}	95.10{5}	96.40{2}	94.90{6}	94.70{7}	95.50{4}
		$\hat{\rho}$	91.70{6}	95.40{1}	94.70{3}	92.80{4}	91.80{5}	91.60{7}	94.80{2}
		$\hat{\delta}$	92.10{5}	93.90{1.5}	90.20{7}	93.30{4}	91.30{6}	93.60{3}	93.90{1.5}
	$\sum Ranks$		46{3.5}	58.5{6}	49{5}	46{3.5}	38{2}	70{7}	16.5{1}
300	ACI	$\hat{\lambda}$	0.17152{3} 1.32567{2}	0.0000{6.5} 1.74448{7}	0.10256{5} 1.44081{5}	0.11457{4} 1.37127{4}	0.17868{2} 1.35776{3}	0.0000{6.5} 1.71554{6}	0.44806{1} 0.95792{1}
		$\hat{\rho}$	0.95347{4} 1.50271{4}	0.88285{7} 1.56857{6}	0.96329{3} 1.47886{2}	0.97740{2} 1.52836{5}	0.91536{5} 1.49469{3}	0.89373{6} 1.58953{7}	1.01984{1} 1.41811{1}
		$\hat{\delta}$	1.92209{3} 4.16925{6}	1.58020{7} 4.42562{7}	1.87071{5} 4.16723{5}	1.98192{2} 4.16266{4}	1.90548{4} 4.13252{3}	1.67291{6} 3.34410{1}	2.39496{1} 3.70936{2}
	AIL	$\hat{\lambda}$	1.15414{2}	1.74448{6}	1.33824{5}	1.25669{4}	1.17908{3}	1.71554{7}	0.50986{1}
		$\hat{\rho}$	0.54924{4}	0.68571{6}	0.51557{2}	0.55096{5}	0.54332{3}	0.69580{7}	0.39826{1}
		$\hat{\delta}$	2.24716{4}	2.84542{6}	2.29652{5}	2.18073{2}	2.22703{3}	2.67119{7}	1.31440{1}
	CP	$\hat{\lambda}$	96.10{3}	96.00{4}	96.40{1}	96.30{2}	95.60{6}	95.70{5}	92.60{7}
		$\hat{\rho}$	94.60{2}	95.70{1}	92.30{7}	93.10{5}	93.60{4}	92.90{6}	94.00{3}
		$\hat{\delta}$	92.20{3}	94.70{1}	92.10{4}	89.90{7}	91.60{6}	91.70{5}	93.40{2}
	$\sum Ranks$		40{2}	64.5{6}	49{4}	50{5}	45{3}	69.5{7}	22{1}
Over ranks			**19.5{3}**	**26.5{6}**	**22{4}**	**24.5{5}**	**11{2}**	**31.5{7}**	**5{1}**

Table 5: The non-Bayesian OBXII-E model computes the Mean, RMSE, and Bias for the parameters λ= 0.4, ρ=1.4, δ=3.7

n		E. P.	MLE	LSE	WLSE	MPSE	ADE	RTADE	KE
30	Mean	$\hat{\lambda}$	0.61950{3}	0.98320{6}	0.94652{5}	0.56415{2}	0.81815{4}	0.98381{6}	0.44529{1}
		$\hat{\rho}$	2.00931{4}	2.03189{5}	1.94428{3}	2.61945{7}	1.91718{2}	2.19981{6}	1.50011{1}
		$\hat{\delta}$	3.90186{7}	3.71155{5}	3.61684{3}	3.83761{6}	3.60806{2}	3.51866{1}	3.71117{4}
	RMSE	$\hat{\lambda}$	1.00910{3}	1.58451{7}	1.52260{5}	0.97555{2}	1.23203{4}	1.57041{6}	0.15007{1}
		$\hat{\rho}$	2.41978{6}	1.85206{2}	1.98622{4}	3.54579{7}	1.95557{3}	2.01658{5}	0.42350{1}
		$\hat{\delta}$	2.23084{7}	2.05643{5}	2.00341{4}	2.17027{6}	1.89945{2}	1.98676{3}	0.62916{1}
	Bias	$\hat{\lambda}$	0.21950{3}	0.58320{6}	0.54652{5}	0.16415{2}	0.41856{4}	0.58381{7}	0.04529{1}
		$\hat{\rho}$	0.60931{4}	0.63189{5}	0.54428{3}	1.21945{7}	0.51718{2}	0.79981{6}	0.10011{1}
		$\hat{\delta}$	0.20186{6}	0.03155{1}	0.08315{3}	0.33761{7}	0.09193{4}	0.18133{5}	0.07117{2}
	$\sum Ranks$		43{5}	42{4}	35{3}	46{7}	27{2}	45{6}	13{1}
60	Mean	$\hat{\lambda}$	0.54105{3}	0.68146{6}	0.61805{5}	0.53658{2}	0.61273{4}	0.74722{7}	0.42000{1}
		$\hat{\rho}$	1.79441{5}	1.69556{4}	1.60343{3}	2.12988{7}	1.57713{2}	1.82934{6}	1.47306{1}
		$\hat{\delta}$	3.86267{6}	3.68839{3}	3.70040{4}	3.92406{7}	3.68323{2}	3.66814{1}	3.75194{5}
	RMSE	$\hat{\lambda}$	0.64969{2}	0.93308{6}	0.83029{5}	0.80245{4}	0.79729{3}	1.12115{7}	0.12450{1}
		$\hat{\rho}$	1.73625{6}	1.05051{4}	0.90794{3}	2.52962{7}	0.66247{2}	1.15653{5}	0.30383{1}
		$\hat{\delta}$	1.55324{5}	1.51632{4}	1.40123{3}	1.64174{7}	1.35722{2}	1.59058{6}	0.59343{1}
	Bias	$\hat{\lambda}$	0.14105{3}	0.28146{6}	0.21805{5}	0.13658{2}	0.21273{4}	0.34772{7}	0.02058{1}
		$\hat{\rho}$	0.39411{5}	0.29556{4}	0.20343{3}	0.72988{7}	0.17713{2}	0.42934{6}	0.07306{1}
		$\hat{\delta}$	0.16267{6}	0.02560{1}	0.04066{2}	0.22406{7}	0.04676{3}	0.09185{5}	0.05194{4}
	$\sum Ranks$		41{5}	38{4}	33{3}	50{6.5}	24{2}	50{6.5}	16{1}
120	Mean	$\hat{\lambda}$	0.48282{3}	0.53188{6}	0.49307{5}	0.46494{2}	0.49167{4}	0.59453{7}	0.40803{1}
		$\hat{\rho}$	1.50071{4}	1.51060{5}	1.46534{2}	1.60859{7}	1.46790{3}	1.57197{6}	1.44974{1}
		$\hat{\delta}$	3.72907{5}	3.67811{4}	3.67243{3}	3.80658{7}	3.66053{2}	3.61112{1}	3.74276{6}
	RMSE	$\hat{\lambda}$	0.44203{4}	0.52801{6}	0.37567{3}	0.46666{5}	0.37931{2}	0.70303{7}	0.10244{1}
		$\hat{\rho}$	0.38305{4}	0.47680{5}	0.36798{3}	0.50699{6}	0.35305{2}	0.62749{7}	0.22481{1}
		$\hat{\delta}$	0.99643{4}	1.10691{6}	0.99272{3}	1.04649{5}	0.96979{2}	1.22129{7}	0.51860{1}
	Bias	$\hat{\lambda}$	0.08282{3}	0.13188{6}	0.09307{5}	0.06494{2}	0.09167{4}	0.19453{7}	0.00805{1}
		$\hat{\rho}$	0.10071{4}	0.11060{5}	0.06534{2}	0.20859{7}	0.06790{3}	0.17197{6}	0.04974{1}
		$\hat{\delta}$	0.02907{3}	0.02088{1}	0.02756{2}	0.10658{7}	0.03943{4}	0.08887{6}	0.04276{5}
	$\sum Ranks$		34{4}	44{5}	28{3}	48{6}	26{2}	54{7}	18{1}
200	Mean	$\hat{\lambda}$	0.43468{3}	0.47112{6}	0.44745{4}	0.41915{2}	0.44800{5}	0.50030{7}	0.40053{1}
		$\hat{\rho}$	1.45128{4}	1.45447{5}	1.43396{1}	1.51173{7}	1.43487{2}	1.49097{6}	1.44606{3}
		$\hat{\delta}$	3.72533{5}	3.68054{3}	3.69028{4}	3.78683{7}	3.68060{2}	3.64359{1}	3.76259{6}
	RMSE	$\hat{\lambda}$	0.20403{3}	0.31303{6}	0.21711{4}	0.20221{2}	0.21754{5}	0.40464{7}	0.10180{1}
		$\hat{\rho}$	0.24550{3}	0.31252{6}	0.25097{4}	0.28821{5}	0.24282{2}	0.39816{7}	0.19024{1}
		$\hat{\delta}$	0.76408{2}	0.86640{6}	0.77694{4}	0.78768{5}	0.76657{3}	0.97010{7}	0.47350{1}
	Bias	$\hat{\lambda}$	0.03468{3}	0.07112{6}	0.04745{4}	0.01915{2}	0.04800{5}	0.10030{7}	0.00537{1}
		$\hat{\rho}$	0.01289{1}	0.05447{5}	0.03396{2}	0.11173{7}	0.03487{3}	0.09097{6}	0.04606{4}
		$\hat{\delta}$	0.02533{3}	0.01945{2}	0.02171{4}	0.08683{7}	0.01939{1}	0.05640{6}	0.03259{5}
	$\sum Ranks$		27{2}	45{6}	31{4}	44{5}	28{3}	54{7}	23{1}
300	Mean	$\hat{\lambda}$	0.41263{3}	0.43999{6}	0.42502{4}	0.40100{1}	0.42524{5}	0.45632{7}	0.40269{2}
		$\hat{\rho}$	1.44404{5}	1.44285{4}	1.43229{1}	1.48726{7}	1.43347{2}	1.46853{6}	1.43563{3}
		$\hat{\delta}$	3.74786{6}	3.69647{2}	3.71380{4}	3.79531{7}	3.70750{3}	3.67019{1}	3.74751{5}
	RMSE	$\hat{\lambda}$	0.13775{2}	0.20744{6}	0.15630{5}	0.13786{3}	0.15470{4}	0.26320{7}	0.08835{1}
		$\hat{\rho}$	0.19611{2}	0.25866{6}	0.20963{4}	0.22551{5}	0.20595{3}	0.32628{7}	0.17356{1}
		$\hat{\delta}$	0.59837{2}	0.70378{6}	0.62419{5}	0.61825{3}	0.62012{4}	0.79524{7}	0.42729{1}
	Bias	$\hat{\lambda}$	0.01263{3}	0.03999{6}	0.02502{4}	0.00199{1}	0.02524{5}	0.05632{7}	0.00269{2}
		$\hat{\rho}$	0.01034{1}	0.04285{5}	0.03228{2}	0.08726{7}	0.03347{3}	0.06853{6}	0.03563{4}
		$\hat{\delta}$	0.01786{4}	0.00352{1}	0.01380{3}	0.06431{7}	0.00750{2}	0.02980{6}	0.02751{5}
	$\sum Ranks$		28{2}	42{6}	32{4}	41{5}	31{3}	54{7}	24{1}
Over Ranks			**18{4}**	**25{5}**	**17{3}**	**29.5{6}**	**12{2}**	**33.5{7}**	**5{1}**

Table 6: The non-Bayesian OBXII-E model calculates the ACI, AIL, and CP with parameters λ= 0.4, ρ=1.4, δ=3.7

n		E. P.	MLE	LSE	WLSE	MPSE	ADE	RTADE	KE
30	ACI	$\hat{\lambda}$	0.00000{2} 2.26184{3}	0.00000{2} 3.63037{6}	0.00000{2} 3.63909{7}	0.00000{2} 2.18813{2}	0.00000{2} 2.74504{4}	0.00000{2} 3.42478{5}	0.18384{1} 0.70675{1}
		$\hat{\rho}$	0.00000{2} 6.39875{6}	0.00000{2} 4.98247{2}	0.00000{2} 5.26130{4}	0.00000{2} 9.03445{7}	0.00000{2} 5.08708{3}	0.00000{2} 5.58366{5}	0.73662{1} 2.26400{1}
		$\hat{\delta}$	0.08485{4} 7.71887{7}	0.14524{3} 7.27786{3}	0.0000{6.5} 7.33465{4}	0.25283{2} 7.42239{6}	0.07990{5} 7.13623{2}	0.0000{6.5} 7.35272{5}	2.54187{1} 4.88046{1}
	AIL	$\hat{\lambda}$	2.26184{3}	3.63037{6}	3.63909{7}	2.18813{2}	2.74504{4}	3.42478{5}	0.52291{1}
		$\hat{\rho}$	6.39875{6}	4.98247{2}	5.26130{4}	9.03452{7}	5.08708{3}	5.58366{5}	1.52777{1}
		$\hat{\delta}$	7.63401{7}	7.13262{3}	7.33465{5}	7.16956{4}	7.05633{2}	7.35272{6}	2.33859{1}
	CP	$\hat{\lambda}$	94.80{2}	92.70{6}	93.60{4.5}	95.00{1}	93.60{4.5}	91.30{7}	93.80{3}
		$\hat{\rho}$	97.70{1}	94.30{6}	95.20{4}	96.60{2}	96.10{3}	94.10{7}	94.80{5}
		$\hat{\delta}$	95.50{4}	94.90{5}	96.40{3}	92.00{7}	96.70{2}	97.30{1}	93.30{6}
	$\sum Ranks$		47{5}	46{4}	53{6}	44{3}	36.5{2}	56.5{7}	23{1}
60	ACI	$\hat{\lambda}$	0.00000{2} 1.59290{2}	0.00000{2} 2.18469{6}	0.00000{2} 2.09598{4}	0.00000{2} 1.67957{3}	0.00000{2} 2.10803{5}	0.00000{2} 2.69496{7}	0.20912{1} 0.63088{1}
		$\hat{\rho}$	0.00000{3} 4.71450{6}	0.00000{3} 3.46273{4}	0.00000{3} 3.23233{3}	0.00000{3} 6.72681{7}	0.36026{2} 2.79400{2}	0.00000{3} 3.66943{5}	0.93974{1} 2.00673{1}
		$\hat{\delta}$	1.25867{2} 6.46666{5}	0.92312{6} 6.45366{4}	1.19824{4} 6.20898{3}	1.11324{5} 6.73489{7}	1.24354{3} 6.12291{2}	0.83971{7} 6.49657{6}	2.61126{1} 4.89261{1}
	AIL	$\hat{\lambda}$	1.59290{2}	2.18469{6}	2.09598{4}	1.97957{3}	2.10803{5}	2.69496{7}	0.42176{1}
		$\hat{\rho}$	4.71450{5}	3.46273{4}	3.23233{3}	6.72681{6}	2.43373{2}	3.66943{4}	1.06663{1}
		$\hat{\delta}$	5.20798{4}	5.53053{5}	5.01716{3}	5.62165{6}	4.87936{2}	5.65685{7}	2.28134{1}
	CP	$\hat{\lambda}$	94.70{4}	94.20{5}	96.40{1.5}	95.90{3}	96.40{1.5}	93.90{6}	93.20{7}
		$\hat{\rho}$	97.70{1}	95.00{5}	97.10{3}	97.50{2}	96.10{4}	94.30{6}	93.40{7}
		$\hat{\delta}$	89.80{7}	93.40{2}	92.40{4}	90.60{6}	91.90{5}	92.80{3}	93.80{1}
	$\sum Ranks$		43{4}	52{5}	37.5{3}	53{6}	35.5{2}	63{7}	24{1}
120	ACI	$\hat{\lambda}$	0.00000{2} 1.30987{4}	0.00000{2} 1.43305{6}	0.00000{2} 1.09316{2}	0.00000{2} 1.33482{5}	0.00000{2} 1.16941{3}	0.00000{2} 1.72956{7}	0.22170{1} 0.59436{1}
		$\hat{\rho}$	0.83879{4} 2.16263{4}	0.74441{6} 2.27680{5}	0.85086{3} 2.07982{3}	0.80805{5} 2.40912{6}	0.85730{2} 2.07849{2}	0.45818{7} 2.68576{7}	1.02473{1} 1.87476{1}
		$\hat{\delta}$	1.81763{5} 5.64052{5}	1.58298{6} 5.77325{6}	1.82835{4} 5.51651{3}	2.08212{2} 5.53103{4}	2.03509{3} 5.28597{2}	1.32212{7} 5.90013{7}	2.68450{1} 4.60103{1}
	AIL	$\hat{\lambda}$	1.30987{4}	1.43305{6}	1.09316{2}	1.33482{5}	1.16941{3}	1.72956{7}	0.37266{1}
		$\hat{\rho}$	1.32383{4}	1.53238{5}	1.22895{3}	1.60106{6}	1.22119{2}	2.22758{7}	0.85002{1}
		$\hat{\delta}$	3.82289{5}	4.19026{6}	3.68815{4}	3.44891{3}	3.25087{2}	4.57801{7}	1.91653{1}
	CP	$\hat{\lambda}$	97.30{2}	95.80{4}	95.20{5.5}	97.60{1}	96.60{3}	95.20{5.5}	93.40{7}
		$\hat{\rho}$	94.10{4}	93.80{5.5}	93.50{7}	94.20{3}	93.80{5.5}	95.50{2}	95.60{1}
		$\hat{\delta}$	94.40{1}	93.90{2}	93.70{3}	89.90{7}	90.50{6}	93.20{4}	91.10{5}
	$\sum Ranks$		44{4}	59.5{6}	41.5{3}	49{5}	35.5{2}	69.5{7}	22{1}
200	ACI	$\hat{\lambda}$	0.04639{4} 0.82298{3}	0.0000{5.5} 1.06334{6}	0.04845{3} 0.84645{5}	0.06905{2} 0.76925{2}	0.03777{4} 0.85824{4}	0.0000{5.5} 1.18155{7}	0.24270{1} 0.59836{1}
		$\hat{\rho}$	1.04976{3} 1.85281{3}	0.87058{6} 2.03836{6}	0.96863{5} 1.89928{4}	1.06308{2} 1.96039{5}	1.02903{4} 1.84071{2}	0.85174{7} 2.13020{7}	1.10504{1} 1.78707{1}
		$\hat{\delta}$	2.37188{4} 5.07877{4}	2.01656{6} 5.34453{6}	2.38935{3} 4.99122{3}	2.30972{5} 5.26395{5}	2.39389{2} 4.96731{2}	1.86133{7} 5.42585{7}	2.87280{1} 4.65238{1}
	AIL	$\hat{\lambda}$	0.77659{3}	1.06334{6}	0.79800{4}	0.70019{2}	0.82046{5}	1.18155{7}	0.35566{1}
		$\hat{\rho}$	0.80305{2}	1.16778{6}	0.93064{5}	0.89730{4}	0.81168{3}	1.27845{7}	0.68203{1}
		$\hat{\delta}$	2.70688{5}	3.32797{7}	2.60187{4}	2.95423{6}	2.57341{3}	3.56451{2}	1.77957{1}
	CP	$\hat{\lambda}$	95.30{3}	95.70{2}	94.80{4.5}	93.90{7}	94.80{4.5}	94.60{6}	96.60{1}
		$\hat{\rho}$	93.40{4}	95.40{1}	94.90{3}	93.10{5}	92.90{6}	92.80{7}	95.00{2}
		$\hat{\delta}$	91.30{5}	93.80{1}	90.00{6.5}	93.00{4}	90.00{6.5}	93.10{3}	93.60{2}
	$\sum Ranks$		43{2}	58.5{6}	50{5}	49{4}	46{3}	72.5{7}	14{1}
300	ACI	$\hat{\lambda}$	0.15596{3} 0.66929{3}	0.07699{6} 0.80300{6}	0.13531{5} 0.71473{5}	0.13621{4} 0.66578{2}	0.16318{2} 0.68730{4}	0.00693{7} 0.90570{7}	0.24535{1} 0.56002{1}
		$\hat{\rho}$	1.07581{4} 1.81227{4}	0.95718{6} 1.92852{5}	1.08014{3} 1.78445{2}	1.10988{2} 1.86464{4}	1.06401{5} 1.80293{3}	0.92995{7} 2.00712{7}	1.12150{1} 1.74975{1}
		$\hat{\delta}$	2.68603{3} 4.80968{4}	2.37683{6} 5.01611{6}	2.62443{5} 4.80316{3}	2.77640{2} 4.81422{5}	2.64674{4} 4.76826{2}	2.30125{7} 5.03912{7}	2.96611{1} 4.52891{1}
	AIL	$\hat{\lambda}$	0.51333{2}	0.72600{6}	0.57419{5}	0.52956{4}	0.52412{3}	0.89877{7}	0.31466{1}
		$\hat{\rho}$	0.73646{3}	0.97133{6}	0.70430{2}	0.75475{5}	0.73891{4}	1.07716{7}	0.62825{1}
		$\hat{\delta}$	2.12365{4}	2.63927{6}	2.17873{5}	2.03782{2}	2.12152{3}	2.73787{7}	1.56280{1}
	CP	$\hat{\lambda}$	96.20{2}	95.20{5}	96.00{3}	96.60{1}	95.10{6}	95.40{4}	92.60{7}
		$\hat{\rho}$	94.40{3}	95.10{1}	92.50{7}	93.70{5}	93.60{6}	93.80{4}	94.90{2}
		$\hat{\delta}$	93.50{2.5}	94.70{1}	91.90{4}	91.20{7}	91.50{5.5}	91.50{5.5}	93.50{2.5}
	$\sum Ranks$		37.5{2}	60{6}	49{5}	43{3}	47.5{4}	80.5{7}	21.5{1}
Over Ranks			**17{3}**	**27{6}**	**22{5}**	**21{4}**	**13{2}**	**35{7}**	**5{1}**

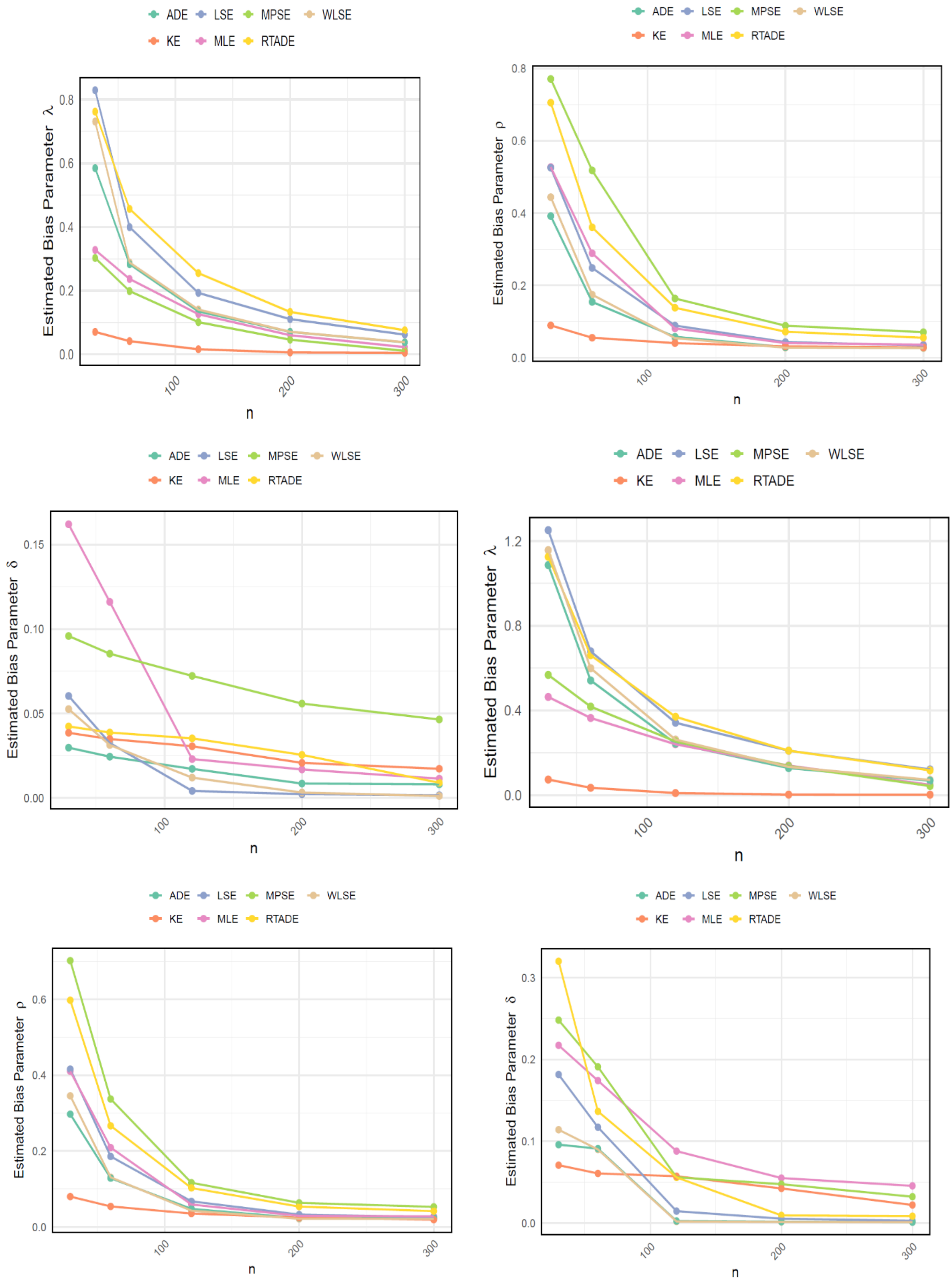

Figure 4: Bias for λ, ρ, δ for the schemes in Tables 1, and 2

Table 7: The MSE, RMSE, and Bias are computed using the Bayesian OBXII-E model with parameters λ= 0.5, ρ=2, δ=1.2

n		E. P.	SEL	LN1 -0.5	LN2 0.5	GE1 -0.5	GE2 0.5
35	MSE	$\hat{\lambda}$	0.616827{4}	0.856571{5}	0.4930051{2}	0.553706{3}	0.451948{1}
		$\hat{\rho}$	1.479347{4}	4.311453{5}	0.818155{1}	1.279568{3}	0.981985{2}
		$\hat{\delta}$	0.090014{2}	0.092041{4}	0.089526{1}	0.090719{3}	0.092744{5}
	RMSE	$\hat{\lambda}$	0.785383{4}	0.925511{5}	0.702143{2}	0.744114{3}	0.672271{1}
		$\hat{\rho}$	1.216284{4}	2.076403{5}	0.904519{1}	1.131180{3}	0.990951{2}
		$\hat{\delta}$	0.300024{2}	0.303383{4}	0.299209{1}	0.301197{3}	0.304539{5}
	Bias	$\hat{\lambda}$	0.400150{4}	0.463184{5}	0.354816{2}	0.366881{3}	0.304412{1}
		$\hat{\rho}$	0.586156{4}	0.846521{5}	0.489457{1}	0.543741{3}	0.496565{2}
		$\hat{\delta}$	0.121042{2}	0.120390{1}	0.127030{4}	0.126325{3}	0.136698{5}
	$\sum Ranks$		30{4}	39{5}	15{1}	27{3}	24{2}
70	MSE	$\hat{\lambda}$	0.315601{4}	0.406861{5}	0.267841{2}	0.287565{3}	0.241262{1}
		$\hat{\rho}$	0.802194{4}	1.633947{5}	0.590603{1}	0.734955{3}	0.629671{2}
		$\hat{\delta}$	0.058460{2}	0.057798{1}	0.059146{3}	0.059382{4}	0.061322{5}
	RMSE	$\hat{\lambda}$	0.561784{4}	0.637857{5}	0.517534{2}	0.536251{3}	0.491184{1}
		$\hat{\rho}$	0.895653{4}	1.278259{5}	0.768507{1}	0.857295{3}	0.793518{2}
		$\hat{\delta}$	0.241786{2}	0.240414{1}	0.243201{3}	0.243686{4}	0.247634{5}
	Bias	$\hat{\lambda}$	0.280904{4}	0.311621{5}	0.257026{2}	0.259700{3}	0.219511{1}
		$\hat{\rho}$	0.526352{4}	0.626317{5}	0.472145{1}	0.504628{3}	0.473669{2}
		$\hat{\delta}$	0.118015{2}	0.114568{1}	0.121451{4}	0.121279{3}	0.127848{5}
	$\sum Ranks$		30{4}	33{5}	19{1}	29{3}	24{2}
120	MSE	$\hat{\lambda}$	0.215009{4}	0.244260{5}	0.191542{2}	0.200504{3}	0.174430{1}
		$\hat{\rho}$	0.586318{4}	0.784249{5}	0.488998{1}	0.556727{3}	0.504032{2}
		$\hat{\delta}$	0.045338{2}	0.044790{1}	0.045897{3}	0.046106{4}	0.047400{5}
	RMSE	$\hat{\lambda}$	0.463690{5}	0.494227{4}	0.437655{2}	0.447777{3}	0.417648{1}
		$\hat{\rho}$	0.765734{4}	0.885578{5}	0.699284{1}	0.746142{3}	0.709952{2}
		$\hat{\delta}$	0.212927{2}	0.211638{1}	0.214236{3}	0.214500{4}	0.217715{5}
	Bias	$\hat{\lambda}$	0.223281{4}	0.239106{5}	0.208787{3}	0.208590{2}	0.180085{1}
		$\hat{\rho}$	0.500236{1}	0.544047{2}	0.462498{4}	0.487036{5}	0.461117{3}
		$\hat{\delta}$	0.113968{2}	0.111486{1}	0.116446{4}	0.116322{3}	0.121054{5}
	$\sum Ranks$		28{3}	29{4}	23{1}	30{5}	25{2}
200	MSE	$\hat{\lambda}$	0.107109{4}	0.115798{5}	0.099421{2}	0.100795{3}	0.089485{1}
		$\hat{\rho}$	0.467008{4}	0.509221{5}	0.429960{1}	0.454386{3}	0.430601{2}
		$\hat{\delta}$	0.031302{2}	0.030902{1}	0.031710{3}	0.031750{4}	0.032674{5}
	RMSE	$\hat{\lambda}$	0.327275{4}	0.340291{5}	0.315312{2}	0.317483{3}	0.299141{1}
		$\hat{\rho}$	0.683380{4}	0.713597{5}	0.655713{1}	0.674082{3}	0.655974{2}
		$\hat{\delta}$	0.176924{2}	0.175791{1}	0.178073{3}	0.178186{4}	0.180760{5}
	Bias	$\hat{\lambda}$	0.153948{4}	0.162724{5}	0.145621{3}	0.143703{2}	0.123732{1}
		$\hat{\rho}$	0.500017{4}	0.535671{5}	0.435807{2}	0.470909{3}	0.423001{1}
		$\hat{\delta}$	0.098770{2}	0.096888{1}	0.100651{4}	0.100531{3}	0.104069{5}
	$\sum Ranks$		30{4}	33{5}	21{1}	28{3}	23{2}
300	MSE	$\hat{\lambda}$	0.056749{4}	0.059776{5}	0.053966{3}	0.053881{2}	0.048752{1}
		$\hat{\rho}$	0.381563{4}	0.406083{5}	0.358961{2}	0.373471{3}	0.357795{1}
		$\hat{\delta}$	0.023022{3}	0.022749{1}	0.023004{2}	0.023310{4}	0.023903{5}
	RMSE	$\hat{\lambda}$	0.238221{4}	0.244491{5}	0.232306{3}	0.232122{2}	0.220726{1}
		$\hat{\rho}$	0.617708{4}	0.637276{5}	0.599133{2}	0.611122{3}	0.598160{1}
		$\hat{\delta}$	0.151731{2}	0.150829{1}	0.152644{3}	0.152679{4}	0.154606{5}
	Bias	$\hat{\lambda}$	0.117464{4}	0.122670{5}	0.112428{3}	0.110376{2}	0.096490{1}
		$\hat{\rho}$	0.499950{4}	0.517615{5}	0.412931{2}	0.453403{3}	0.404395{1}
		$\hat{\delta}$	0.093451{2}	0.092047{1}	0.094855{4}	0.094745{3}	0.097734{5}
	$\sum Ranks$		31{4}	33{5}	24{2}	26{3}	21{1}
Over Ranks			**19{4}**	**24{5}**	**6{1}**	**17{3}**	**9{2}**

Table 8: The HPD, AIL, and CP are computed using the Bayesian OBXII-E model with parameters λ= 0.5, ρ=2, δ=1.2

n		E. P.	SEL	LN1 -0.5	LN2 0.5	GE1 -0.5	GE2 0.5
35	HPD	$\hat{\lambda}$	0.067147{2} 2.503323{4}	0.067332{1} 3.006783{5}	0.066964{3} 2.133159{2}	0.064648{4} 2.284688{3}	0.059906{5} 2.043212{1}
		$\hat{\rho}$	1.432408{2} 5.921148{4}	1.335345{5} 8.249691{5}	1.436694{1} 4.482167{1}	1.376764{3} 4.982697{3}	1.367197{4} 4.817694{2}
		$\hat{\delta}$	0.548433{2} 1.619676{4}	0.550601{1} 1.603875{3}	0.545173{3} 1.601786{2}	0.541646{4} 1.593853{1}	0.520841{5} 1.639920{5}
	AIL	$\hat{\lambda}$	2.436175{4}	2.939450{5}	2.066195{2}	2.220040{3}	1.983305{1}
		$\hat{\rho}$	4.488740{3}	6.914346{5}	3.045472{1}	3.605933{4}	3.450496{2}
		$\hat{\delta}$	1.071243{4}	1.053273{2}	1.056612{3}	1.052207{1}	1.119079{5}
	CP	$\hat{\lambda}$	96.10{2}	96.60{1}	95.40{4}	95.60{3}	95.30{5}
		$\hat{\rho}$	96.90{3}	96.40{4}	97.10{1}	95.80{5}	97.00{2}
		$\hat{\delta}$	96.60{2.5}	96.20{5}	96.60{2.5}	96.30{4}	97.50{1}
	$\sum Ranks$		36.5{4}	42{5}	25.5{1}	33{3}	29{2}
70	HPD	$\hat{\lambda}$	0.137193{3} 1.872553{3}	0.138236{2} 2.222278{5}	0.145111{1} 1.931436{4}	0.110091{4} 1.794140{2}	0.094366{5} 1.720605{1}
		$\hat{\rho}$	1.541623{2} 3.832342{3}	1.555087{1} 3.968184{5}	1.535513{3} 3.647312{1}	1.535337{4} 3.874380{4}	1.518946{5} 3.713608{2}
		$\hat{\delta}$	0.635624{4} 1.500236{3}	0.656353{1} 1.502733{4}	0.644186{3} 1.475588{1}	0.645855{2} 1.515475{5}	0.631525{5} 1.479724{2}
	AIL	$\hat{\lambda}$	1.735359{3}	2.084041{5}	1.786324{4}	1.684048{2}	1.632693{1}
		$\hat{\rho}$	2.290719{3}	2.413096{5}	2.111799{1}	2.339047{4}	2.194662{2}
		$\hat{\delta}$	0.865611{4}	0.846373{3}	0.831402{1}	0.869619{5}	0.848199{2}
	CP	$\hat{\lambda}$	95.70{4}	96.80{2}	97.00{1}	95.50{5}	95.80{3}
		$\hat{\rho}$	96.20{2}	96.00{3}	95.80{4.5}	96.70{1}	95.80{4.5}
		$\hat{\delta}$	97.50{2.5}	97.50{2.5}	96.90{5}	98.00{1}	97.10{4}
	$\sum Ranks$		36.5{2.5}	38.5{4}	29.5{1}	39{5}	36.5{2.5}
120	HPD	$\hat{\lambda}$	0.125849{5} 1.625708{5}	0.169966{2} 1.572102{4}	0.169465{3} 1.498583{3}	0.194938{1} 1.483010{2}	0.142141{4} 1.461274{1}
		$\hat{\rho}$	1.604620{4} 3.681863{4}	1.609494{3} 3.688363{5}	1.654858{1} 3.529478{2}	1.601052{5} 3.575281{3}	1.626038{2} 3.496206{1}
		$\hat{\delta}$	0.707439{1} 1.450990{4}	0.676555{4} 1.453088{5}	0.671087{5} 1.448866{1}	0.705850{2} 1.449512{3}	0.684185{3} 1.449093{2}
	AIL	$\hat{\lambda}$	1.499859{5}	1.402136{4}	1.329117{3}	1.288071{1}	1.319132{2}
		$\hat{\rho}$	2.077316{4}	2.078869{5}	1.874619{2}	1.974229{3}	1.870168{1}
		$\hat{\delta}$	0.743551{1}	0.776533{4}	0.777779{5}	0.743661{2}	0.764908{3}
	CP	$\hat{\lambda}$	96.80{1}	95.70{5}	96.40{3}	95.90{4}	96.50{2}
		$\hat{\rho}$	97.20{1}	96.70{3}	96.60{4}	96.80{2}	96.20{5}
		$\hat{\delta}$	98.10{4.5}	98.10{4.5}	98.30{2}	98.40{1}	98.20{3}
	$\sum Ranks$		39.5{4}	48.5{5}	34{3}	29{1.5}	29{1.5}
200	HPD	$\hat{\lambda}$	0.203221{3} 1.211355{2}	0.203434{2} 1.315706{5}	0.203008{4} 1.214429{3}	0.211065{1} 1.289341{4}	0.200108{5} 1.200815{1}
		$\hat{\rho}$	1.768628{1} 3.517301{3}	1.744957{2} 3.583679{5}	1.706595{4} 3.550759{2}	1.707815{3} 3.552266{4}	1.704383{5} 3.415038{1}
		$\hat{\delta}$	0.784733{2} 1.422896{4}	0.783569{3} 1.423915{5}	0.779904{1} 1.381720{1}	0.773750{5} 1.382003{2}	0.775213{4} 1.409346{3}
	AIL	$\hat{\lambda}$	1.008123{2}	1.112271{5}	1.011420{3}	1.078276{4}	1.000707{1}
		$\hat{\rho}$	1.748673{3}	1.838722{4}	1.844164{5}	1.084445{1}	1.710656{2}
		$\hat{\delta}$	0.638162{4}	0.640345{5}	0.601816{2}	0.608252{1}	0.634132{3}
	CP	$\hat{\lambda}$	95.50{5}	96.70{2}	96.10{4}	97.10{1}	96.50{3}
		$\hat{\rho}$	96.90{4}	97.00{3}	97.50{1}	97.20{2}	96.20{5}
		$\hat{\delta}$	98.50{1.5}	98.50{1.5}	97.20{4.5}	97.20{4.5}	97.90{3}
	$\sum Ranks$		33.5{2}	42.5{5}	34.5{3}	32.5{1}	36{4}
300	HPD	$\hat{\lambda}$	0.268789{4} 1.046425{2}	0.269542{2} 1.080514{5}	0.269090{3} 1.028397{1}	0.268124{5} 1.075231{4}	0.275507{1} 1.048386{3}
		$\hat{\rho}$	1.900589{2} 3.252750{2}	1.893499{4} 3.267022{3}	1.903704{1} 3.329987{5}	1.882581{5} 3.286623{4}	1.899323{3} 3.236701{1}
		$\hat{\delta}$	0.879023{1} 1.348087{5}	0.869605{2} 1.345010{4}	0.859271{3} 1.325525{1}	0.856319{5} 1.325706{2}	0.858135{4} 1.336193{3}
	AIL	$\hat{\lambda}$	0.777635{3}	0.810971{5}	0.759307{1}	0.807106{4}	0.772879{2}
		$\hat{\rho}$	1.352160{3}	1.137352{1}	1.426282{5}	1.404041{4}	1.337377{2}
		$\hat{\delta}$	0.469063{2}	0.475404{4}	0.466254{1}	0.469386{3}	0.478058{5}
	CP	$\hat{\lambda}$	95.60{4}	96.20{3}	95.50{5}	96.80{2}	97.20{1}
		$\hat{\rho}$	96.50{4}	96.20{5}	98.10{1}	97.00{2}	96.80{3}
		$\hat{\delta}$	98.40{1}	98.10{2}	96.70{4.5}	96.70{4.5}	97.50{3}
	$\sum Ranks$		33{3}	40{4}	31.5{2}	44{5}	31{1}
Over Ranks			**15.5{3.5}**	**23{5}**	**10{1}**	**15.5{3.5}**	**11{2}**

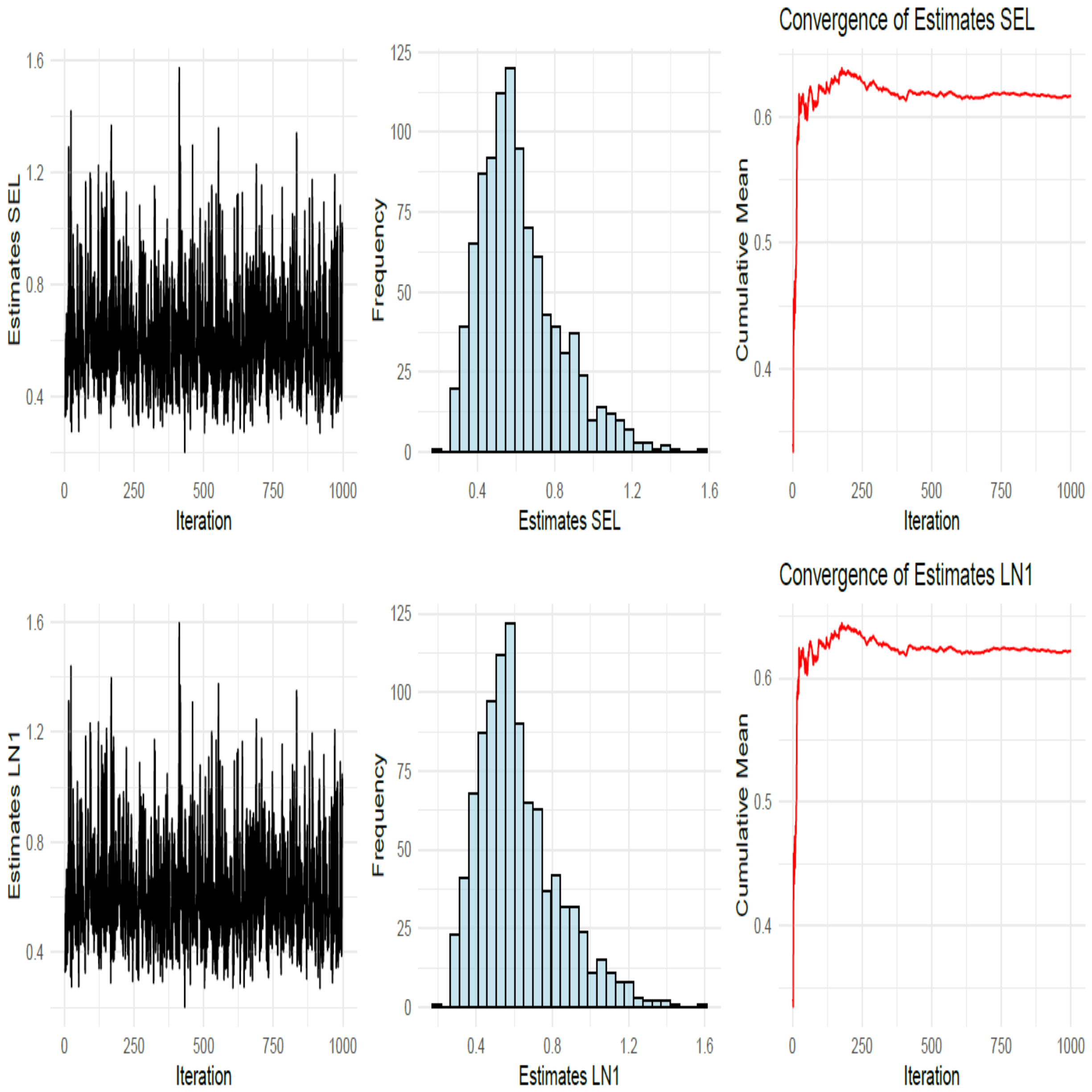

Figure 5: MCMC plots of the OBXII-E distribution with parameters λ= 0.5, ρ=2, δ=1.2 for SEL, LN1, and loss functions.

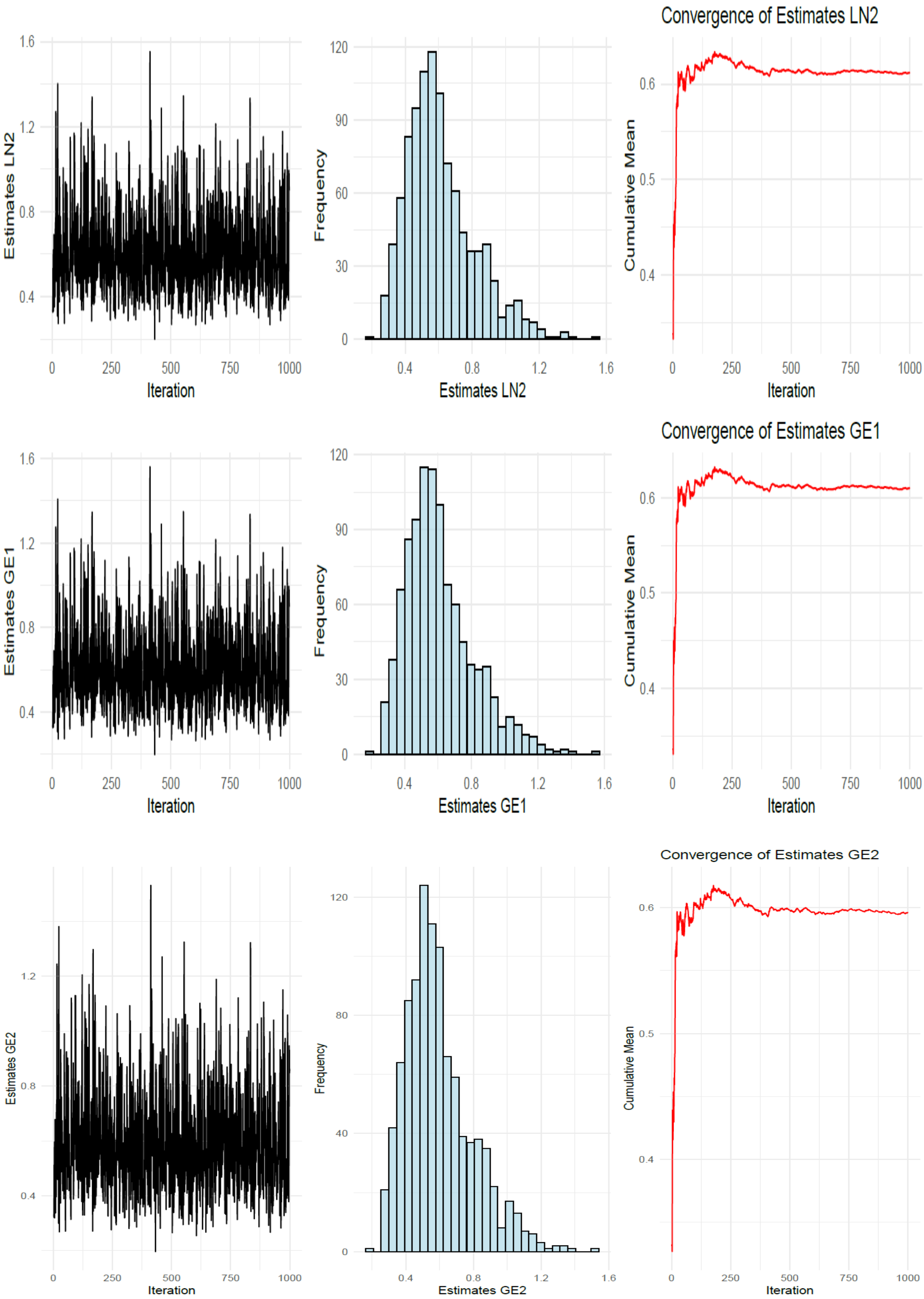

Figure 6: MCMC plots of the OBXII-E distribution with parameters λ= 0.5, ρ=2, δ=1.2 for LN2, GE1, and GE2 loss functions.

Table 9: The MSE, RMSE, and Bias are computed using the Bayesian OBXII-E model with parameters λ= 0.7, ρ =2.3, δ=1.5

n		E. P.	SEL	LN1 -0.5	LN2 0.5	GE1 -0.5	GE2 0.5
35	MSE	$\hat{\lambda}$	0.986879{4}	1.613010{5}	0.746098{2}	0.877698{3}	0.709345{1}
		$\hat{\rho}$	0.957315{4}	2.845551{5}	0.576457{1}	0.844464{3}	0.676943{2}
		$\hat{\delta}$	0.088782{2}	0.087927{1}	0.089856{3}	0.090083{4}	0.092926{5}
	RMSE	$\hat{\lambda}$	0.993418{4}	1.270043{5}	0.863770{2}	0.936855{3}	0.842226{1}
		$\hat{\rho}$	0.978425{4}	1.686876{5}	0.759248{1}	0.918947{3}	0.822766{2}
		$\hat{\delta}$	0.297963{2}	0.296526{1}	0.299760{3}	0.300139{4}	0.304837{5}
	Bias	$\hat{\lambda}$	0.569284{4}	0.695637{5}	0.490126{2}	0.522715{3}	0.437185{1}
		$\hat{\rho}$	0.462455{4}	0.633201{5}	0.365141{1}	0.432007{3}	0.375349{2}
		$\hat{\delta}$	0.138917{2}	0.132748{1}	0.144982{4}	0.143419{3}	0.152456{5}
	$\sum Ranks$		30{4}	33{5}	19{1}	29{3}	24{2}
70	MSE	$\hat{\lambda}$	0.653858{4}	0.908515{5}	0.530684{2}	0.596474{3}	0.501848{1}
		$\hat{\rho}$	0.616541{4}	1.437479{5}	0.441951{1}	0.562379{3}	0.479244{2}
		$\hat{\delta}$	0.065272{2}	0.064360{1}	0.066218{4}	0.066175{3}	0.068062{5}
	RMSE	$\hat{\lambda}$	0.808615{4}	0.953160{5}	0.728480{2}	0.772317{3}	0.708412{1}
		$\hat{\rho}$	0.785187{4}	1.198949{5}	0.664794{1}	0.749919{3}	0.692274{2}
		$\hat{\delta}$	0.255484{2}	0.253693{1}	0.257329{4}	0.257146{3}	0.260888{5}
	Bias	$\hat{\lambda}$	0.460159{4}	0.524488{5}	0.413229{2}	0.429962{3}	0.373046{1}
		$\hat{\rho}$	0.460300{4}	0.538286{5}	0.407548{1}	0.443401{3}	0.411062{2}
		$\hat{\delta}$	0.136687{2}	0.132816{1}	0.140549{4}	0.139575{3}	0.145382{5}
	$\sum Ranks$		30{4}	33{5}	21{1}	27{3}	24{2}
120	MSE	$\hat{\lambda}$	0.449987{4}	0.541029{5}	0.387683{2}	0.418178{3}	0.361724{1}
		$\hat{\rho}$	0.386783{4}	0.434393{5}	0.346502{1}	0.374389{3}	0.350746{2}
		$\hat{\delta}$	0.051677{2}	0.050892{1}	0.052481{4}	0.052377{3}	0.0538151{5}
	RMSE	$\hat{\lambda}$	0.670811{4}	0.735546{5}	0.622642{2}	0.646667{3}	0.601435{1}
		$\hat{\rho}$	0.621919{4}	0.659085{5}	0.588644{1}	0.611873{3}	0.592238{2}
		$\hat{\delta}$	0.227327{2}	0.225593{1}	0.229088{4}	0.228861{3}	0.231980{5}
	Bias	$\hat{\lambda}$	0.399555{4}	0.434399{5}	0.369272{2}	0.377780{3}	0.335560{1}
		$\hat{\rho}$	0.426534{4}	0.455877{5}	0.398878{2}	0.417006{3}	0.398155{1}
		$\hat{\delta}$	0.138758{2}	0.135878{1}	0.141634{4}	0.140915{3}	0.145240{5}
	$\sum Ranks$		30{4}	33{5}	22{1}	27{3}	23{2}
200	MSE	$\hat{\lambda}$	0.299099{4}	0.336327{5}	0.267424{2}	0.281758{3}	0.250188{1}
		$\hat{\rho}$	0.333357{4}	0.359544{5}	0.309693{1}	0.325625{3}	0.310723{2}
		$\hat{\delta}$	0.039700{2}	0.039120{1}	0.040291{4}	0.040188{3}	0.041188{5}
	RMSE	$\hat{\lambda}$	0.546900{4}	0.579937{5}	0.517131{2}	0.530809{3}	0.500188{1}
		$\hat{\rho}$	0.577371{2}	0.599620{5}	0.556500{1}	0.570636{4}	0.557426{3}
		$\hat{\delta}$	0.199249{2}	0.197789{1}	0.200726{4}	0.200471{3}	0.202949{5}
	Bias	$\hat{\lambda}$	0.327890{4}	0.348797{5}	0.308432{2}	0.312233{3}	0.281730{1}
		$\hat{\rho}$	0.434848{4}	0.454928{5}	0.415596{2}	0.428081{3}	0.414687{1}
		$\hat{\delta}$	0.128269{2}	0.126082{1}	0.130450{4}	0.129890{3}	0.133144{5}
	$\sum Ranks$		30{4}	33{5}	22{1}	28{3}	24{2}
300	MSE	$\hat{\lambda}$	0.183486{4}	0.198142{5}	0.170349{2}	0.174294{3}	0.157227{1}
		$\hat{\rho}$	0.282782{4}	0.299275{5}	0.267387{1}	0.277636{3}	0.267612{2}
		$\hat{\delta}$	0.031831{2}	0.031419{1}	0.032249{4}	0.032167{3}	0.032848{5}
	RMSE	$\hat{\lambda}$	0.428353{4}	0.445132{5}	0.412734{2}	0.417485{3}	0.396518{1}
		$\hat{\rho}$	0.531773{4}	0.547061{5}	0.517095{1}	0.526912{3}	0.517312{2}
		$\hat{\delta}$	0.178414{2}	0.177255{1}	0.179582{4}	0.179351{3}	0.181242{5}
	Bias	$\hat{\lambda}$	0.283161{3}	0.295845{4}	0.412734{5}	0.272335{2}	0.251010{1}
		$\hat{\rho}$	0.427910{4}	0.441914{5}	0.414303{2}	0.423077{3}	0.413484{1}
		$\hat{\delta}$	0.125569{2}	0.123969{1}	0.127167{4}	0.126748{3}	0.129111{5}
	$\sum Ranks$		29{4}	32{5}	25{2}	26{3}	23{1}
Over Ranks			**20{4}**	**25{5}**	**6{1}**	**15{3}**	**9{2}**

Table 10: The HPD, AIL, and CP are computed using the Bayesian OBXII-E model with parameters λ= 0.7, ρ=2.3, δ=1.5

n		E. P.	SEL	LN1 -0.5	LN2 0.5	GE1 -0.5	GE2 0.5
35	HPD	$\hat{\lambda}$	0.108358{5} 3.069537{4}	0.108984{4} 3.917466{5}	0.199953{1} 2.719697{2}	0.123671{3} 2.878168{3}	0.122243{4} 2.664315{1}
		$\hat{\rho}$	1.740561{1} 4.406172{4}	1.709747{5} 4.706492{5}	1.726458{3} 4.149576{1}	1.732452{2} 4.218135{2}	1.716501{4} 4.285632{3}
		$\hat{\delta}$	0.836710{3} 1.876800{2}	0.895357{1} 1.916019{5}	0.834513{4} 1.871019{1}	0.871789{2} 1.895051{3}	0.811475{5} 1.906366{4}
	AIL	$\hat{\lambda}$	2.961178{4}	3.808481{5}	2.519744{1}	2.754497{3}	2.542072{2}
		$\hat{\rho}$	2.665611{4}	2.996745{5}	2.423311{1}	2.485682{2}	2.569130{3}
		$\hat{\delta}$	1.040090{4}	1.020661{1}	1.036506{3}	1.023262{2}	1.094890{5}
	CP	$\hat{\lambda}$	96.10{2.5}	96.60{1}	96.10{2.5}	95.80{5}	95.90{4}
		$\hat{\rho}$	96.70{3}	96.40{4}	96.80{2}	95.80{5}	97.00{1}
		$\hat{\delta}$	96.50{5}	97.20{2}	96.60{4}	97.00{3}	97.80{1}
	$\sum Ranks$		51.5{5}	43{4}	25.5{1}	35{2}	37{3}
70	HPD	$\hat{\lambda}$	0.201605{5} 2.588388{2}	0.246755{4} 3.173580{5}	0.295925{1} 2.627982{4}	0.265679{2} 2.597857{3}	0.259328{3} 2.502859{1}
		$\hat{\rho}$	1.883547{4} 3.921070{3}	1.910401{2} 4.009824{5}	1.891876{3} 3.768181{1}	1.913478{1} 3.973907{4}	1.882559{5} 3.803329{2}
		$\hat{\delta}$	0.897837{3} 1.788719{4}	0.899050{2} 1.767453{2}	0.931205{1} 1.786263{3}	0.896475{4} 1.792534{5}	0.890748{5} 1.757608{1}
	AIL	$\hat{\lambda}$	2.386794{4}	2.926824{5}	2.332056{1}	2.332178{2}	2.243530{3}
		$\hat{\rho}$	2.037523{3}	2.099423{5}	1.876307{1}	2.060429{4}	1.920769{2}
		$\hat{\delta}$	0.890882{4}	0.868402{3}	0.855058{1}	0.896059{5}	0.866860{2}
	CP	$\hat{\lambda}$	95.70{5}	96.90{2}	97.30{1}	96.00{4}	96.30{3}
		$\hat{\rho}$	96.30{2.5}	96.30{2.5}	96.00{5}	97.40{1}	96.10{4}
		$\hat{\delta}$	97.60{2.5}	97.00{4.5}	97.60{2.5}	97.70{1}	97.00{4.5}
	$\sum Ranks$		42{4.5}	42{4.5}	24.5{1}	36{3}	35.5{2}
120	HPD	$\hat{\lambda}$	0.335670{1} 2.430806{5}	0.246320{4} 2.205083{4}	0.245079{5} 2.078100{1}	0.333125{2} 2.094448{2}	0.263303{3} 2.101465{3}
		$\hat{\rho}$	1.939829{4} 3.705289{4}	1.975487{1} 3.777020{5}	1.964436{2} 3.568768{1}	1.937312{5} 3.654945{3}	1.961558{3} 3.572869{2}
		$\hat{\delta}$	0.944264{4} 1.702164{1}	0.950993{1} 1.728568{5}	0.942275{5} 1.721619{4}	0.947005{3} 1.703537{2}	0.947494{2} 1.719239{3}
	AIL	$\hat{\lambda}$	2.095135{5}	1.958763{4}	1.833021{2}	1.761322{1}	1.838161{3}
		$\hat{\rho}$	1.765459{4}	1.801533{5}	1.604331{1}	1.717632{3}	1.611210{2}
		$\hat{\delta}$	0.757899{2}	0.777575{4}	0.779344{5}	0.756532{1}	0.771745{3}
	CP	$\hat{\lambda}$	97.30{1}	95.40{5}	96.10{3}	95.70{4}	96.40{2}
		$\hat{\rho}$	97.10{1}	96.90{2}	96.30{4}	96.70{3}	96.00{5}
		$\hat{\delta}$	97.20{5}	97.90{2.5}	98.00{1}	97.40{4}	97.90{2.5}
	$\sum Ranks$		37{4}	42.5{5}	34{3}	33{1}	33.5{2}
200	HPD	$\hat{\lambda}$	0.400745{1} 1.892389{3}	0.349292{5} 2.030853{5}	0.374450{2} 1.855118{2}	0.358560{3} 1.974792{4}	0.357539{4} 1.832551{1}
		$\hat{\rho}$	2.075035{1} 3.524417{2}	2.050682{2} 3.561773{5}	2.022201{5} 3.534944{4}	2.024706{4} 3.530917{3}	2.035022{3} 3.461164{1}
		$\hat{\delta}$	1.080018{2} 1.271421{1}	1.081020{1} 1.728140{5}	1.079616{3} 1.686853{2}	1.079125{4} 1.687658{3}	1.060947{5} 1.693602{4}
	AIL	$\hat{\lambda}$	1.491643{3}	1.681560{5}	1.480667{2}	1.616232{4}	1.475012{1}
		$\hat{\rho}$	1.449381{2}	1.511091{4}	1.512742{5}	1.506210{3}	1.426141{1}
		$\hat{\delta}$	0.641402{4}	0.647120{5}	0.607236{1}	0.608533{2}	0.632654{3}
	CP	$\hat{\lambda}$	96.30{5}	96.80{2}	96.60{4}	97.10{1}	96.70{3}
		$\hat{\rho}$	96.70{3.5}	96.70{3.5}	97.30{1}	97.00{2}	96.20{5}
		$\hat{\delta}$	99.20{2}	99.30{1}	98.20{4.5}	98.20{4.5}	98.60{3}
	$\sum Ranks$		29.5{1}	43{5}	35.5{3}	37.5{4}	34{2}
300	HPD	$\hat{\lambda}$	0.448895{1} 1.650610{4}	0.409873{4} 1.693808{5}	0.407951{3} 1.563655{2}	0.418927{2} 1.248552{1}	0.402544{5} 1.603699{3}
		$\hat{\rho}$	2.157747{2} 3.367426{2}	2.162630{1} 3.391121{4}	2.146762{4} 3.402469{5}	2.148436{3} 3.387483{3}	2.145513{5} 3.343767{1}
		$\hat{\delta}$	1.139516{2} 1.640198{3}	1.142547{1} 1.654132{5}	1.139093{3} 1.637816{1}	1.137222{4} 1.638334{2}	1.130607{5} 1.642147{4}
	AIL	$\hat{\lambda}$	1.201714{3}	1.283935{5}	1.155703{1}	1.248552{4}	1.201154{2}
		$\hat{\rho}$	1.209679{2}	1.228490{3}	1.255726{5}	1.239246{4}	1.198254{1}
		$\hat{\delta}$	0.500681{2}	0.511585{5}	0.498780{1}	0.501112{3}	0.511539{4}
	CP	$\hat{\lambda}$	96.10{4}	96.30{3}	95.50{5}	96.90{1}	96.60{2}
		$\hat{\rho}$	96.20{3}	96.10{4.5}	97.30{1}	96.60{2}	96.10{4.5}
		$\hat{\delta}$	98.40{4.5}	99.00{1}	98.40{4.5}	98.60{2}	98.70{3}
	$\sum Ranks$		32.5{2}	41.5{5}	35.5{3}	31{1}	39.5{4}
Over Ranks			**16.5{4}**	**23.5{5}**	**11{1.5}**	**11{1.5}**	**14{3}**

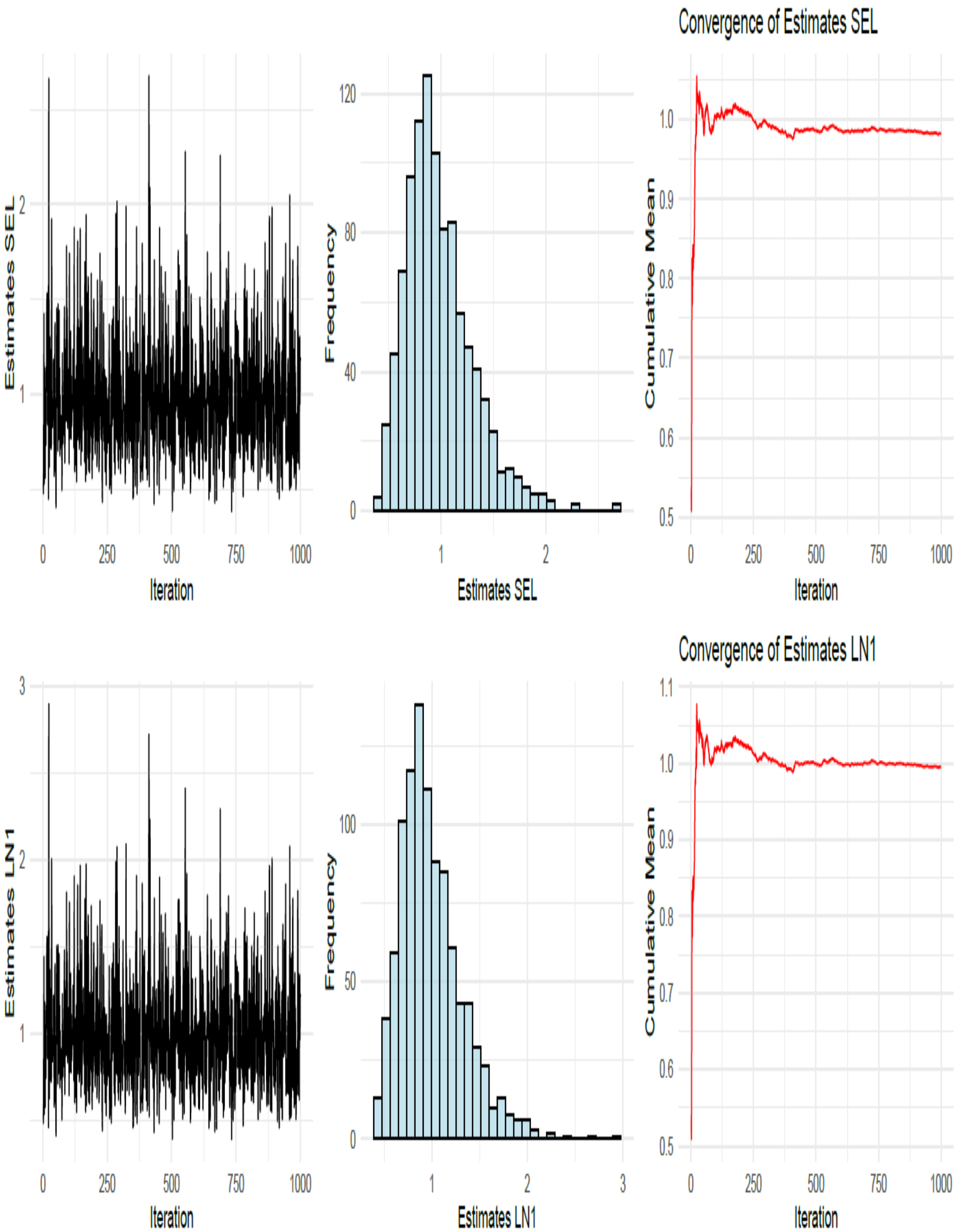

Figure 7: MCMC plots of the OBXII-E distribution with parameters λ= 0.7, ρ=2.3, δ=1.5 for SEL, and LN1 loss functions.

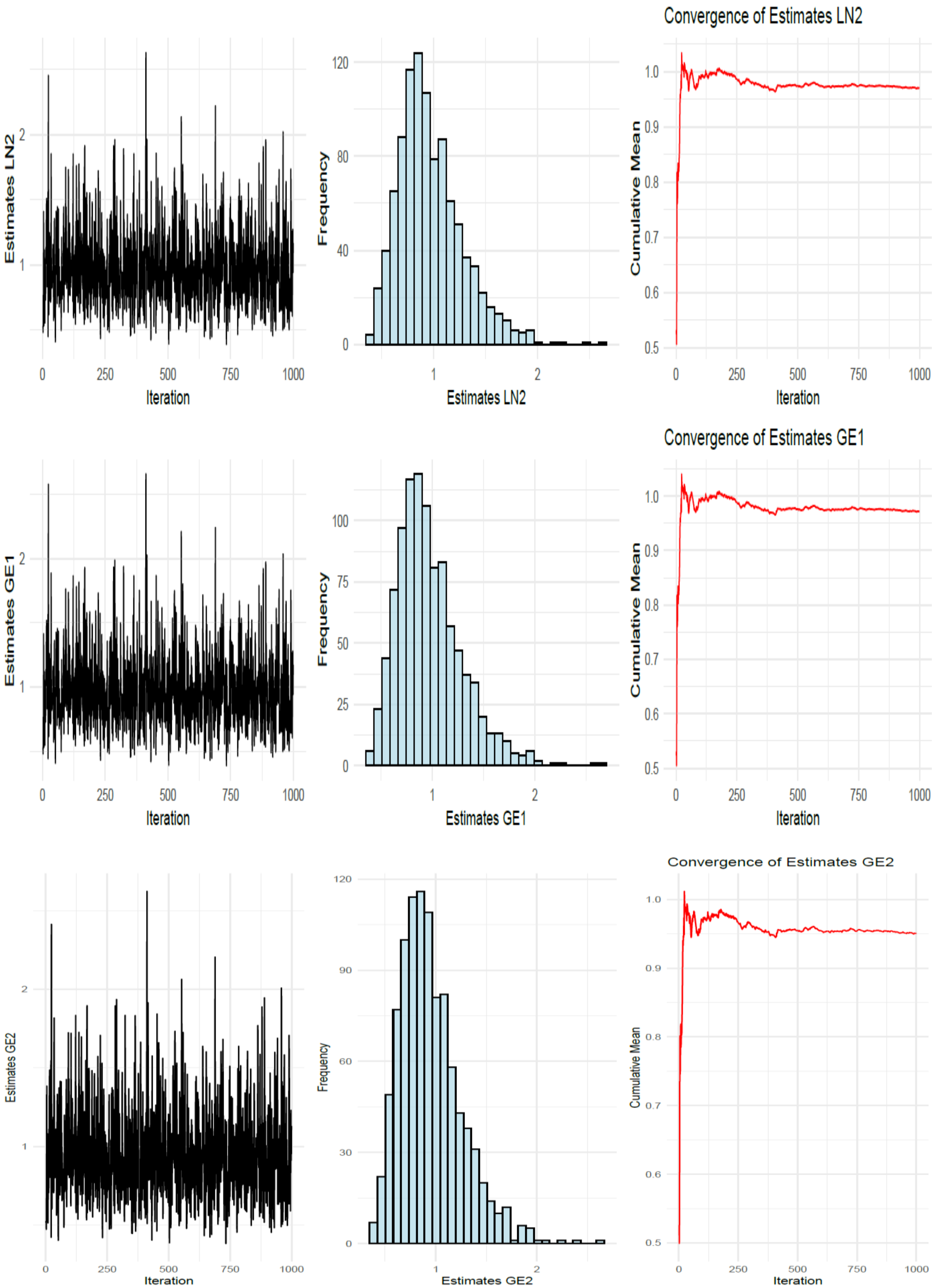

Figure8: MCMC plots of the OBXII-E distribution with parameters λ= 0.7, ρ=2.3, δ=1 for LN2, GE1, and GE2 loss functions.

6. Applications

This part shows how well the OBXII-E distribution fits two real datasets, showing that it fits better than other distributions. We utilized R Statistical Software to figure out all the results. The maximum probability technique will be used to guess the unknown parameters. To compare candidate distributions like the Generalized Rayleigh Inverse Weibull distribution (GRIW) [22], the Truncated Exponentiated Exponential Inverse Weibull distribution (TEEIW) [23], the Odd Generalized Exponential Kumaraswamy Inverse Exponential distribution (OEKIE) [24], the Novel Generalized Exponent Power Weibull distribution (NGEPWe) [25], the Beta Exponential distribution (BeE) [26], the Odd Lomax Exponential distribution (OLOE) [27], and the Truncated Inverse Weibull Exponential distribution (TIWE) [28], we use the negative log-likelihood (-2l), the Akaike information criterion (AIC), the consistent Akaike information criterion (CAIC), the Bayesian information criterion (BIC), the Hannan-Quinn information criterion (HQIC), the Cramer-von Mises (W), the Anderson-Darling (A), the Kolmogorov-Smirnov (KS), and its p-value statistics.
Tables (12) and (14) show how to utilize MLEs to find model parameters and statistic measures for the two sets of data. We also get the values for the log-likelihood and goodness-of-fit measures in Tables (13) and (15).
The suggested OBXII-E distribution is better than its rivals in both datasets. Its p values are 0.87775 and 0.98660, which explains why. The new model's performance metrics, such as -2l, W, A, KS, AIC, CAIC, HQIC, and BIC, are the lowest, which means that the OBXII-E distribution is superior at estimating parameters. Figures 11–15 show the graphical optimization of parameter estimates using the MLE approach. The reading matches the data shown in Tables (12–15).

5.3.2.1 The first data (DI)

The first data set comprises 36 observations of daily COVID-19-related fatalities documented in Canada from April 10 to May 15, 2020.[29] : (3.1091, 3.3825, 3.1444, 3.2135, 2.4946, 3.5146, 4.9274, 3.3769, 6.8686, 3.0914, 4.9378, 3.1091 ,3.2823, 3.8594, 4.0480, 4.1685, 3.6426, 3.2110, 2.8636, 3.2218, 2.9078, 3.6346, 2.7957, 4.2781, 4.2202, 1.5157, 2.6029, 3.3592, 2.8349, 3.1348, 2.5261, 1.5806, 2.7704, 2.1901, 2.4141, 1.9048).

5.3.2.2 The first dataset (DII)

The first dataset presents the tensile strength in GPa of 74 carbon fibers assessed at 20mm gauge lengths. [30]: (1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958,1.966,1.997, 2.006, 2.021, 2.027, 2.055, 2.063,2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770,2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585).

Table 11: Descriptive analysis of the two data

Dataset	N	Mean	SD	Median	mad	Min	Max	Rang	CS	CK	SE
Data I	36	3.28	1	3.18	0.64	1.52	6.87	5.35	1.16	2.81	0.17
Data II	74	2.48	0.49	2.51	0.46	1.31	3.58	2.27	-0.15	-0.13	0.06

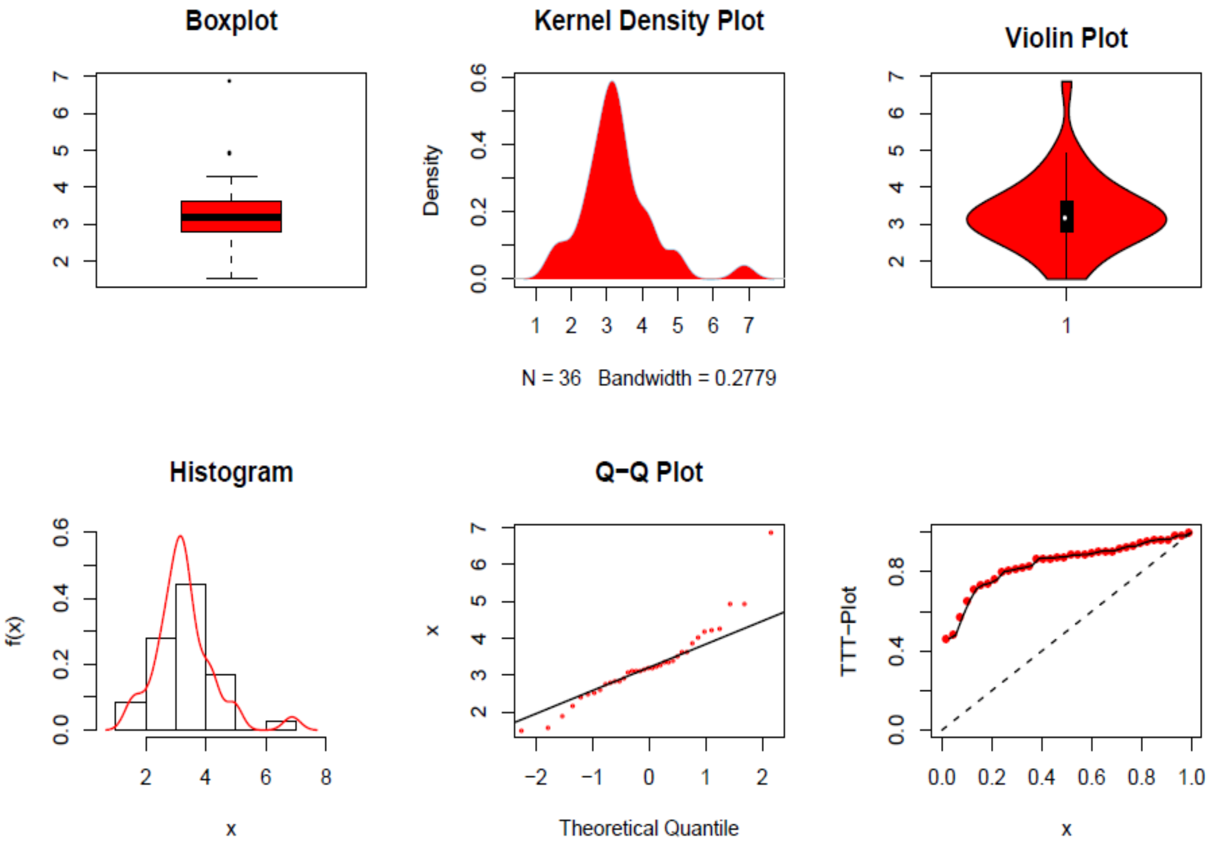

Figure 9: Display DI and its representation.

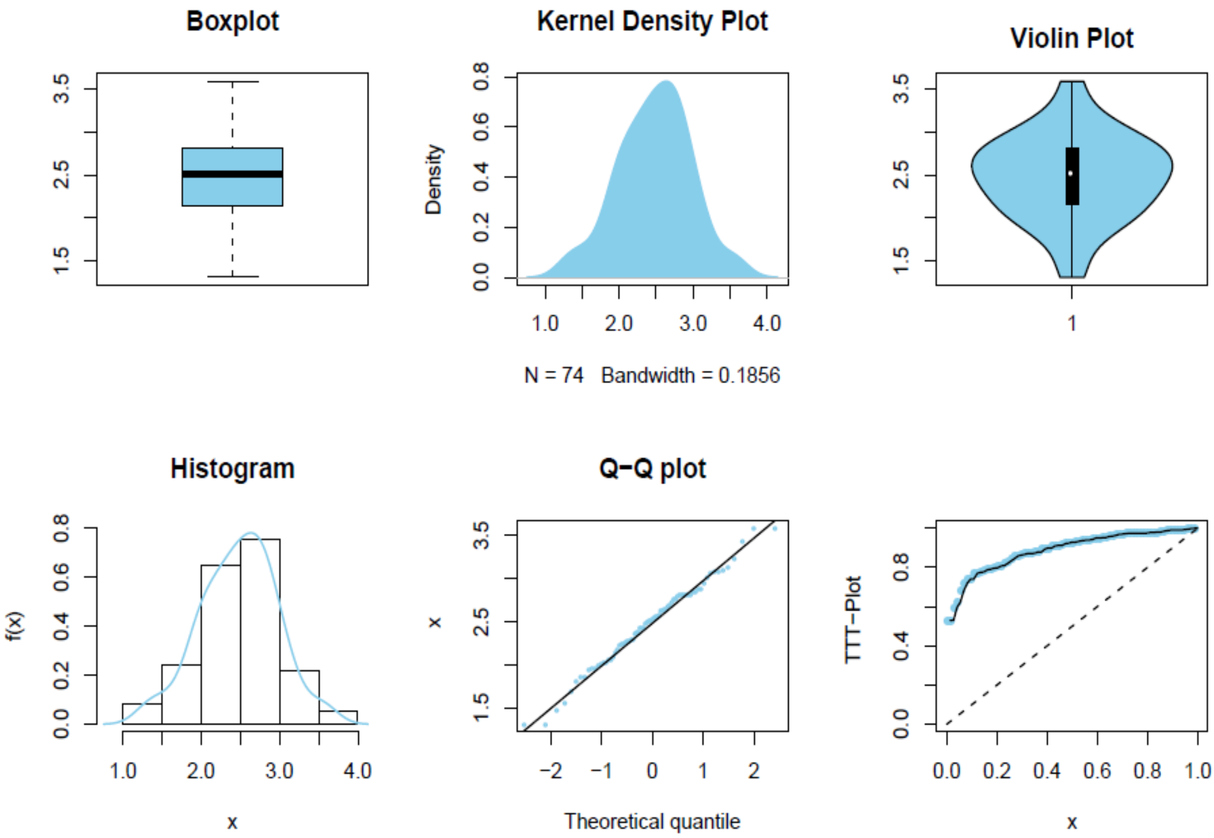

Figure 10: Display DII and its representation.

Table 12: MLEs, and statistic measures for different models for DI.

Model	MLEs	W	A	KS	P-value
OBXII-E	$\boldsymbol{\hat{\lambda}}$**:2.98542** $\boldsymbol{\hat{\rho}}$**:0.96982** $\boldsymbol{\hat{\delta}}$**:0.30547**	**0.05956**	**0.34673**	**0.09826**	**0.87775**
GRIW	$\hat{\lambda}$:0.11865 $\hat{\rho}$:5.57278 $\hat{\delta}$:0.39227 $\hat{c}$:0.67881	0.09706	0.56787	0.10654	0.80846
TEEIW	$\hat{\lambda}$:2.06406 $\hat{\rho}$:0.89474 $\hat{\delta}$:4.20866 $\hat{c}$:1.20294	0.09414	0.55261	0.11344	0.74315
OEKIE	$\hat{\lambda}$:4.04273 $\hat{\rho}$:1.04451 $\hat{\delta}$:0.22600 $\hat{c}$:3.47103 $\hat{a}$:3.47103	0.09698	0.56337	0.10654	0.80842
NGEPWe	$\hat{\lambda}$:0.88573 $\hat{\rho}$:0.04583 $\hat{\delta}$:2.57202	0.10280	0.57907	0.12225	0.65494
BeE	$\hat{\lambda}$:6.71893 $\hat{\rho}$:2.61208 $\hat{\delta}$:0.66399	0.09652	0.56442	0.10802	0.79483
OLOE	$\hat{\lambda}$:1.30998 $\hat{\rho}$:4.75594 $\hat{\delta}$:0.91685	0.09435	0.52975	0.16765	0.26368
TIWE	$\hat{\lambda}$:0.75922 $\hat{\rho}$:3.30881 $\hat{\delta}$:1.16947	0.11838	0.70809	0.12665	0.61050

Table 13: The log-likelihood and goodness-of-fit measures for different models for DI.

Model	-2ℓ	AIC	CAIC	BIC	HQIC
OBXII-E	**47.0694**	**100.139**	**100.889**	**104.889**	**101.797**
GRIW	48.1160	104.232	105.522	110.566	106.442
TEEIW	47.9615	103.923	105.213	110.257	106.133
OEKIE	48.6246	107.249	109.249	115.166	110.012
NGEPWe	48.4609	102.921	103.671	107.672	104.580
BeE	48.0968	102.193	102.943	106.944	103.851
OLOE	50.5778	107.155	107.905	111.906	108.813
TIWE	48.6702	103.340	104.090	108.091	104.998

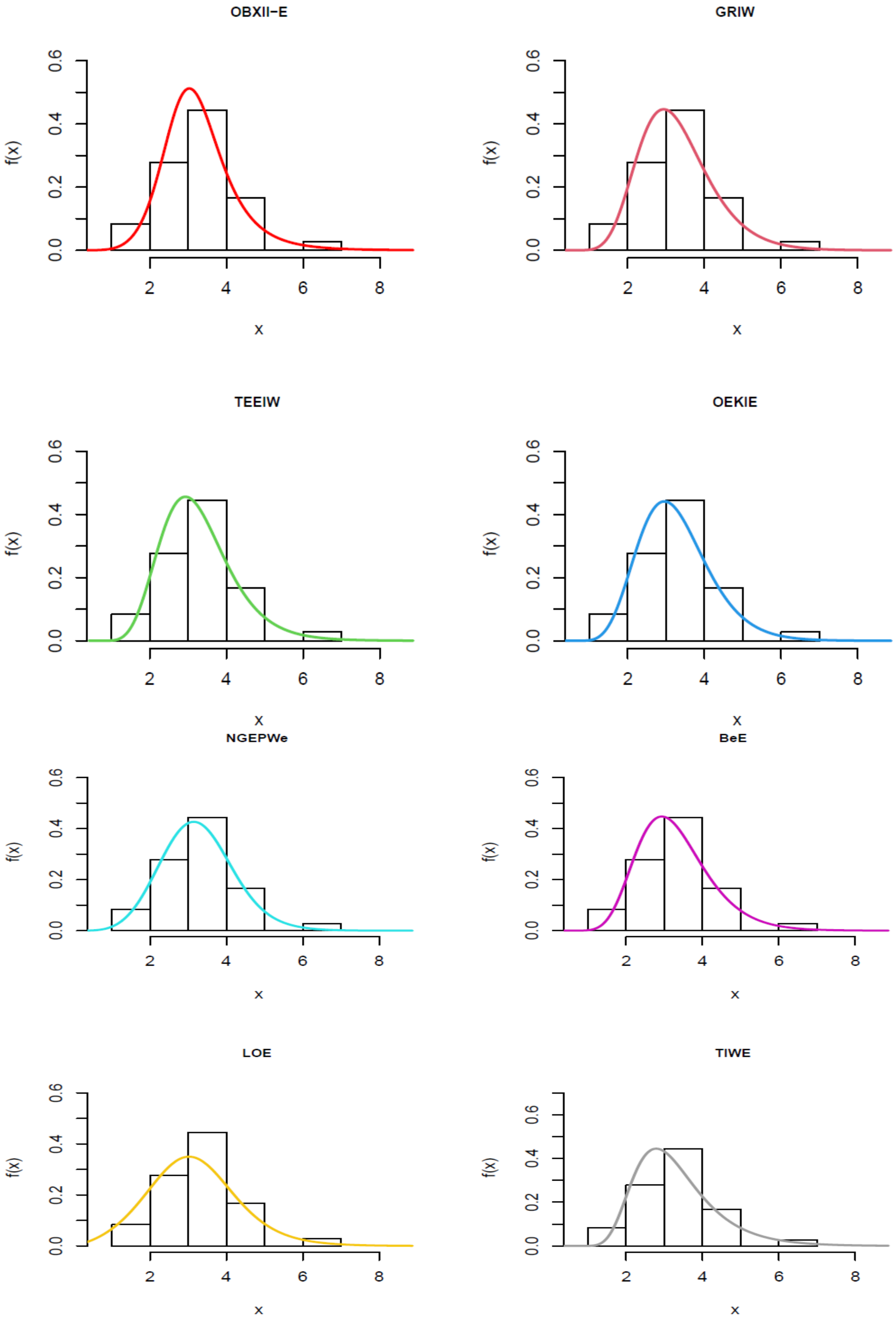

Figure 11: The Plot illustrates the empirical pdf for the OBXII-E model and comparative models for DI.

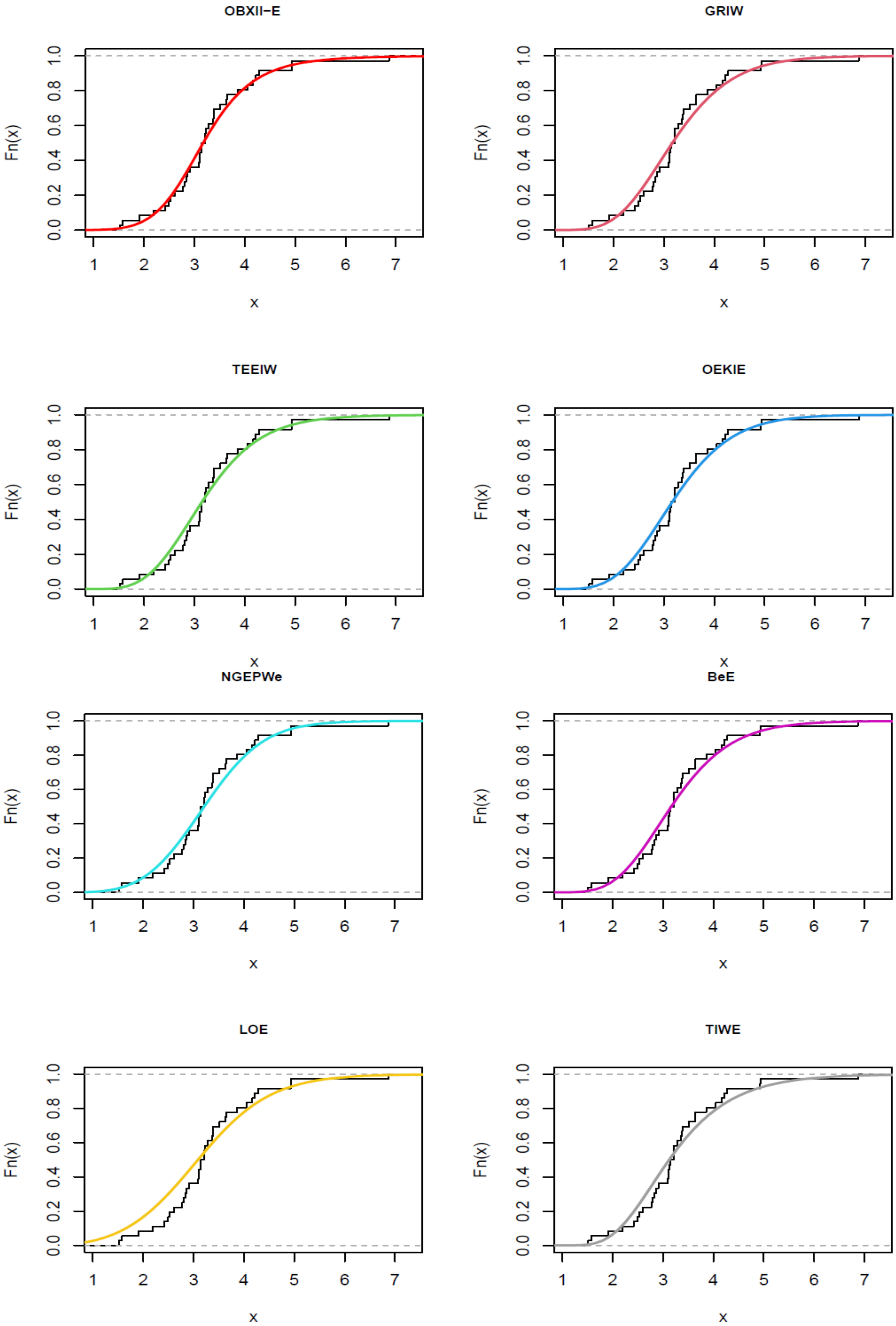

Figure 12: The plot exhibits the empirical cdf for the OBXII-E model and comparative models for DI.

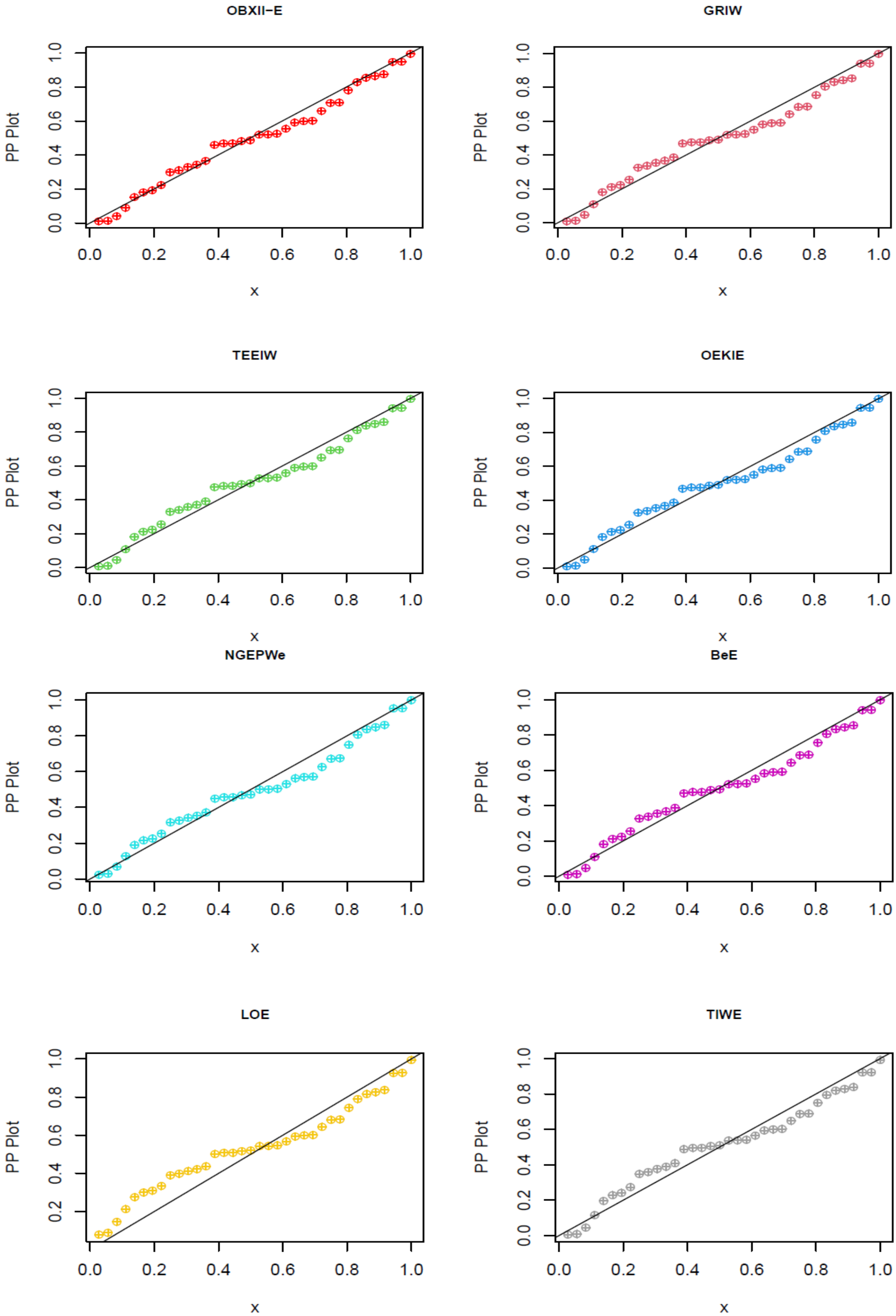

Figure 12: Plots of estimated PP for the OBXII-E model and comparative models for DI.

Table 14: MLEs, and statistic measures for different models for DII.

Model	MLEs	W	A	KS	P-value
OBXII-E	$\boldsymbol{\hat{\lambda}}$**:3.13254** $\boldsymbol{\hat{\rho}}$**:4.50472** $\boldsymbol{\hat{\delta}}$**:0.29428**	**0.02455**	**0.18668**	**0.05260**	**0.98660**
GRIW	$\hat{\lambda}$:0.02211 $\hat{\rho}$:1.00076 $\hat{\delta}$:1.21527 $\hat{c}$:2.34258	0.03051	0.24467	0.06060	0.94860
TEEIW	$\hat{\lambda}$:6.47976 $\hat{\rho}$:0.46667 $\hat{\delta}$:4.49976 $\hat{c}$:1.47432	0.03950	0.29886	0.05736	0.96798
OEKIE	$\hat{\lambda}$:4.90338 $\hat{\rho}$:0.50866 $\hat{\delta}$:5.40309 $\hat{c}$:4.78892 $\hat{a}$:4.65819	0.02940	0.24975	0.07183	0.83963
NGEPW	$\hat{\lambda}$:0.41759 $\hat{\rho}$:0.09657 $\hat{\delta}$:2.82767	0.05102	0.34247	0.12950	0.16703
BeE	$\hat{\lambda}$:5.27412 $\hat{\rho}$:4.13705 $\hat{\delta}$:0.29362	0.09003	0.59039	0.06997	0.86173
OLOE	$\hat{\lambda}$:2.53573 $\hat{\rho}$:4.43330 $\hat{\delta}$:1.53516	0.25227	0.18894	0.14998	0.07161
TIW	$\hat{\lambda}$:1.93136 $\hat{\rho}$:2.11007 $\hat{\delta}$:1.98716	0.22909	1.48456	0.09877	0.46584

Table 15: The log-likelihood and goodness-of-fit measures for different models for DII.

Model	-2ℓ	AIC	CAIC	BIC	HQIC
OBXII-E	**51.1226**	**108.245**	**108.588**	**115.157**	**111.002**
GRIW	51.1595	110.319	110.898	119.535	113.995
TEEIW	51.4584	110.916	111.496	120.133	114.593
OEKIE	51.9024	113.804	114.687	125.325	118.400
NGEPWe	58.3450	122.742	123.084	129.654	125.499
BeE	53.2811	112.562	112.905	119.474	115.319
OLOE	58.2145	122.429	122.771	129.341	125.186
TIWE	59.2750	124.550	124.892	131.462	127.307

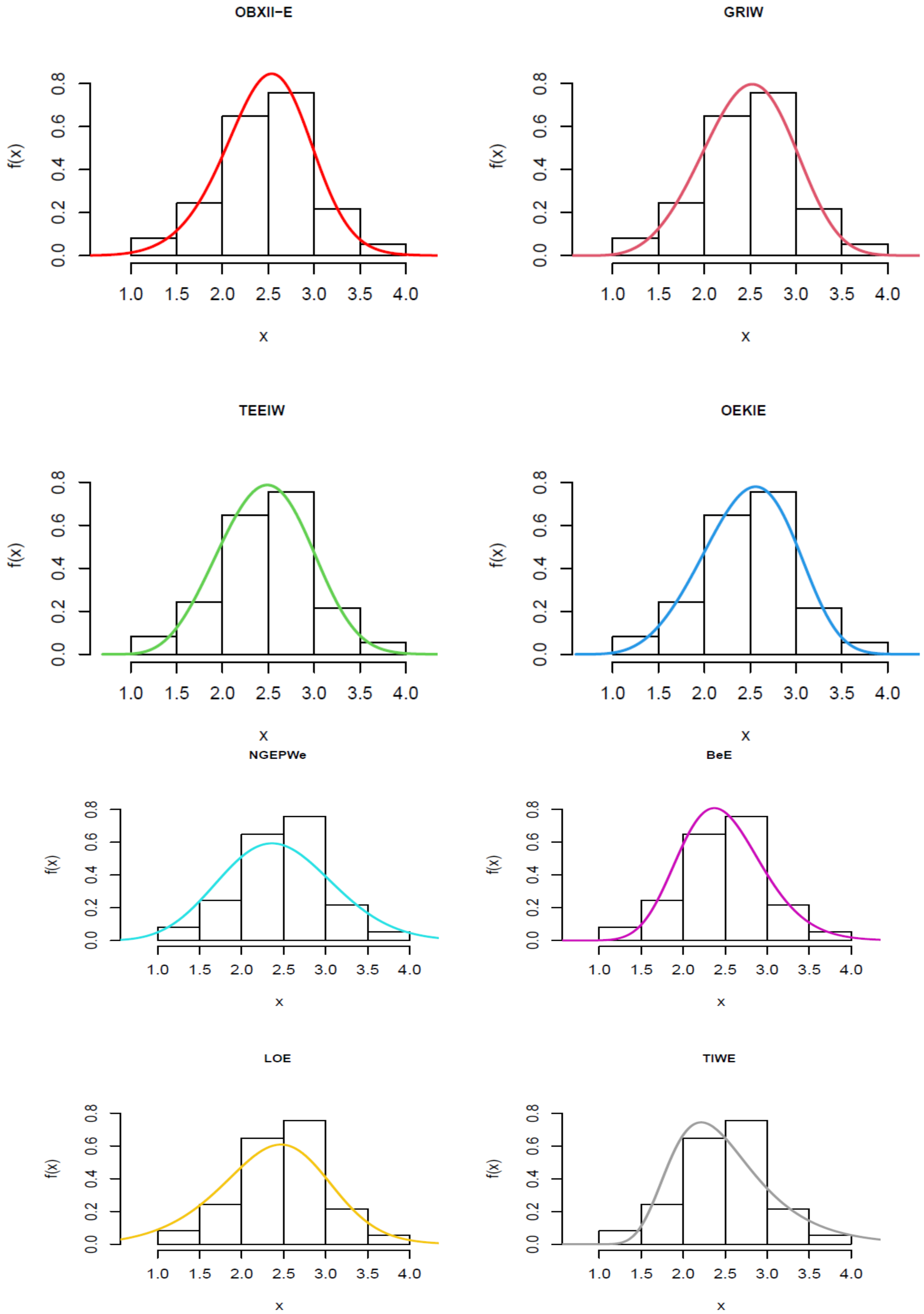

Figure 13: The Plot illustrates the empirical pdf for the OBXII-E model and comparative models for DII.

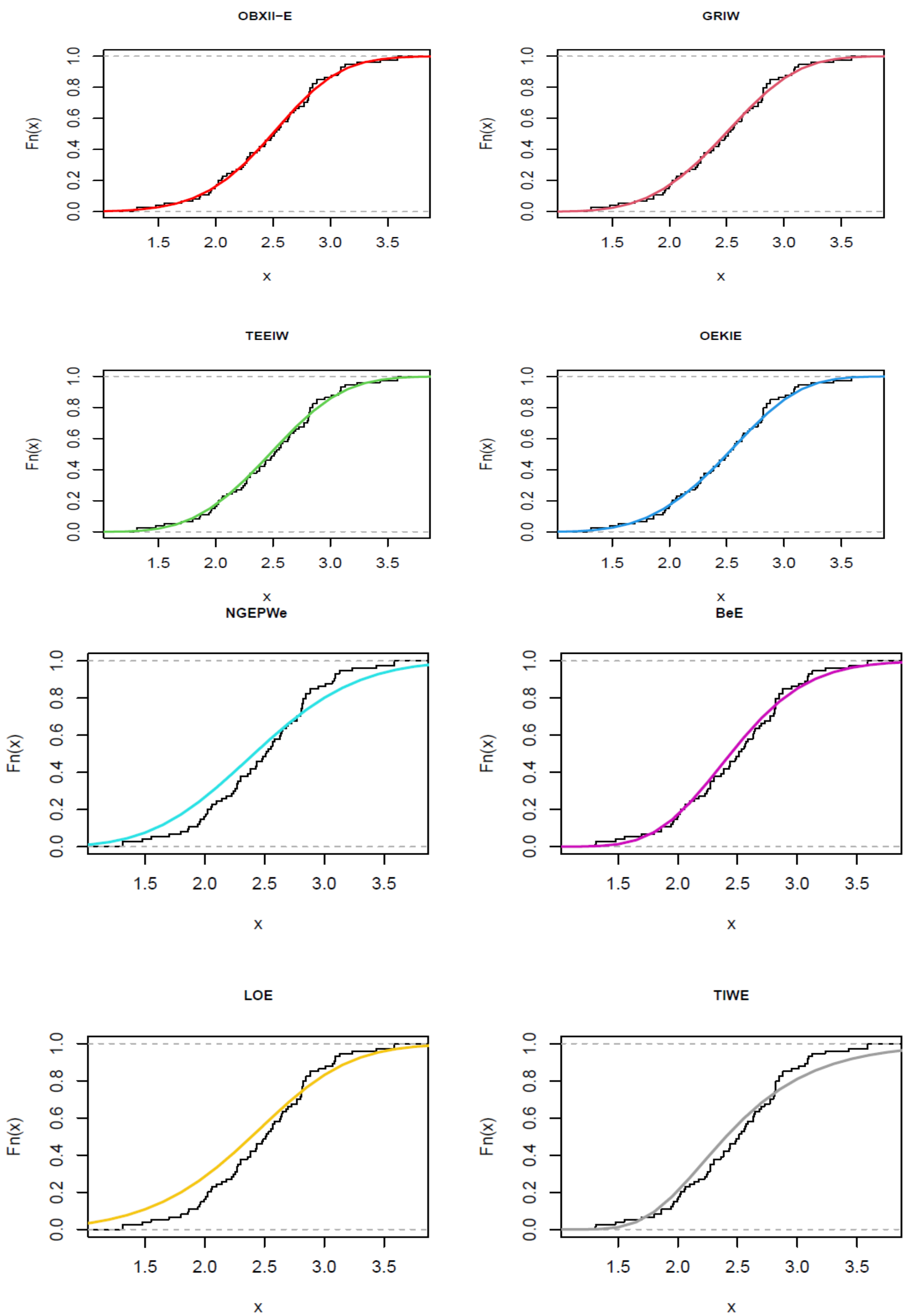

Figure 14: The plot exhibits the empirical cdf for the OBXII-E model and comparative models for DII.

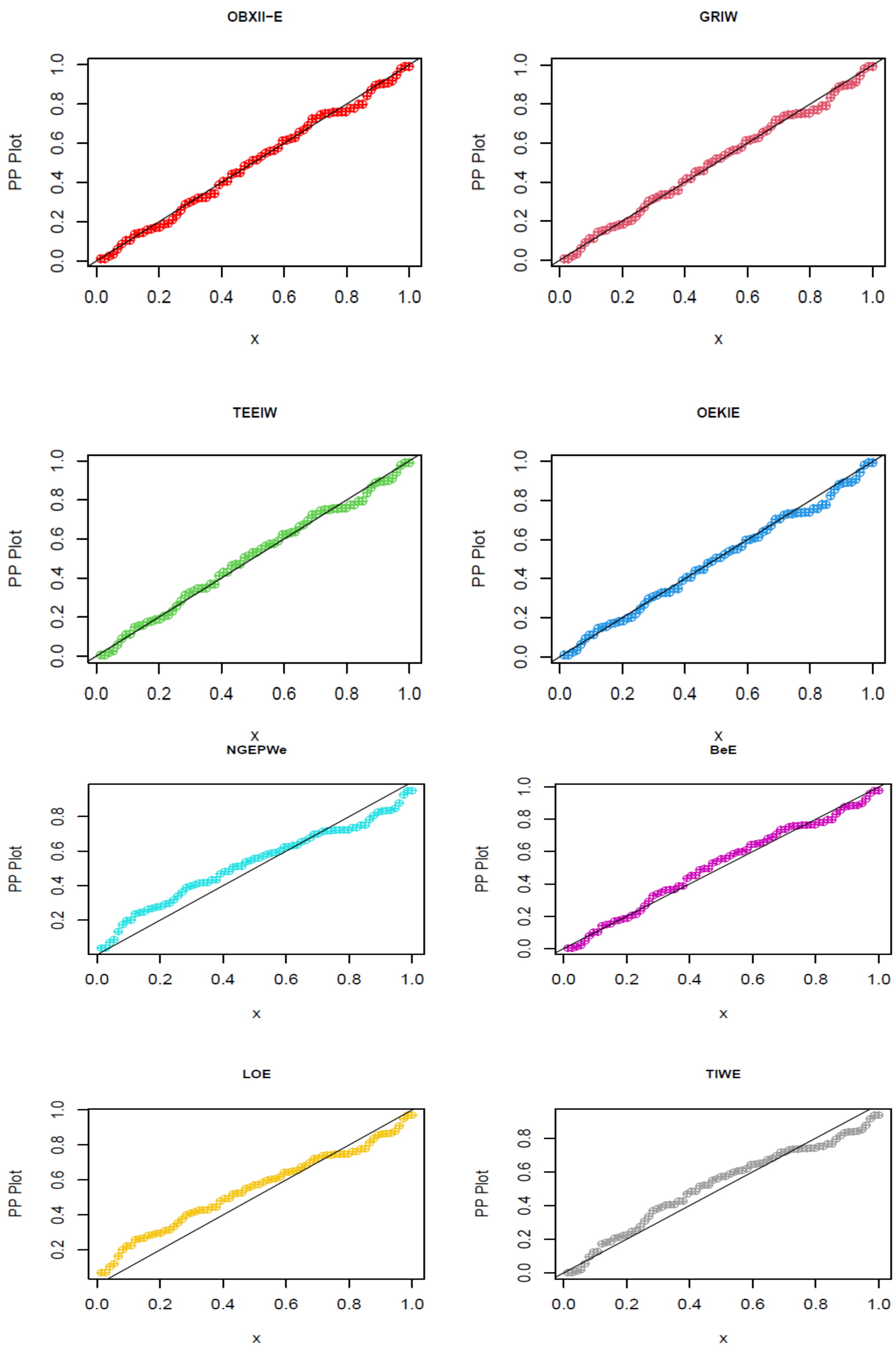

Figure 15: Plots of estimated PP for the OBXII-E model and comparative models for DII.

7. Conclusion

This study systematically evaluated the parameters of the Burr XII-Exponential (BXII-E) distribution using both Bayesian and non-Bayesian methodologies. The findings indicate that both Bayesian and non-Bayesian estimators demonstrate consistency, with estimates converging to true parameter values as sample sizes increase. Among the non-Bayesian methods, the Kernel Estimator (KE) consistently outperformed other techniques across various scenarios, while the Anderson-Darling estimator (AD) followed closely. In the Bayesian framework, the LN2 approach exhibited superior performance in most scenarios, outshining other estimation methods. The analysis revealed that the performance metrics, including mean squared error (MSE) and root mean squared error (RMSE), improved with larger sample sizes, reinforcing the reliability of the estimates. However, coverage probability (CP) showed a different trend, remaining stable across sample sizes. The OBXII-E distribution was successfully fitted to two real datasets, outperforming competing distributions such as the Generalized Rayleigh Inverse Weibull and others. The statistical metrics indicated minimal values for -2 log-likelihood, Akaike information criterion (AIC), and Bayesian information criterion (BIC), affirming the OBXII-E's efficacy in parameter estimation. Graphical representations of parameter estimation via the maximum likelihood method corroborated the numerical results presented in the tables, enhancing the credibility of the findings. The p-values of 0.87775 and 0.98660 for the OBXII-E distribution indicate robust fit, underscoring its superiority in modeling over alternative distributions. In conclusion, the BXII-E distribution, through the application of diverse estimation methods, demonstrates significant potential in accurately modeling complex data structures. Future research can build upon these findings to explore additional distributions and estimation techniques, further enriching the field of statistical modeling.

Authors' Contributions:

All authors have worked equally to write and review the manuscript.

Data Availability Statement:

The data that supports the findings of this study are available within the article.

Conflicts of Interest:

The authors declare no conflict of interest.

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