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Research article

An Updated Lindley Distribution: Properties, Estimation, Acceptance Sampling, Actuarial Risk Assessment and Applications

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ABSTRACT

This paper introduces the updated Lindley (UL) distribution, an innovative mixture of exponential and gamma distributions designed to model lifetime data with various failure rate patterns. The statistical properties of the UL distribution, including moments, skewness, kurtosis, and tail behavior, are thoroughly examined to highlight its flexibility and applicability. Parameter estimation methods using non-Bayesian are evaluated for their accuracy using bias and mean squared error metrics across various scenarios. The paper further explores applications of the UL distribution in acceptance sampling plans, actuarial risk assessment, and reliability analysis. Comprehensive simulation studies and graphical visualizations underscore the robustness and versatility of the distribution in addressing real-world problems, making it a valuable tool in statistical modeling and decision-making processes.

1. Introduction

In this paper, the Updated Lindley (UL) distribution, a novel extension of the Lindley distribution, designed to address the limitations of existing models in describing lifetime data with complex failure rate patterns is introduced. It is not new to distribution theorists and those in the field of modeling that lifetime data, influenced by inherent and external factors, often defy accurate modeling with conventional distributions, leading to suboptimal decision-making in reliability, risk assessment, and statistical inference. This paper seeks to address this gap by formulating the UL distribution, following the method introduced by [17] and which has been modified and extended by many authors like [14], [5], [1], [23], [2], and others. Several authors such as [18], [20], [22], [21], [10], [13], [12], [11], and others have also applied the same method in developing new probability models. The method combines exponential and gamma distributions with shape and scale parameters to provide greater flexibility and applicability. The purpose of this paper is to thoroughly explore the statistical properties, estimation methods, and practical applications of the UL distribution, demonstrating its effectiveness in fields such as reliability analysis and actuarial science.

The rest of the sections of this study are arranged in the sequence in Figure 1.

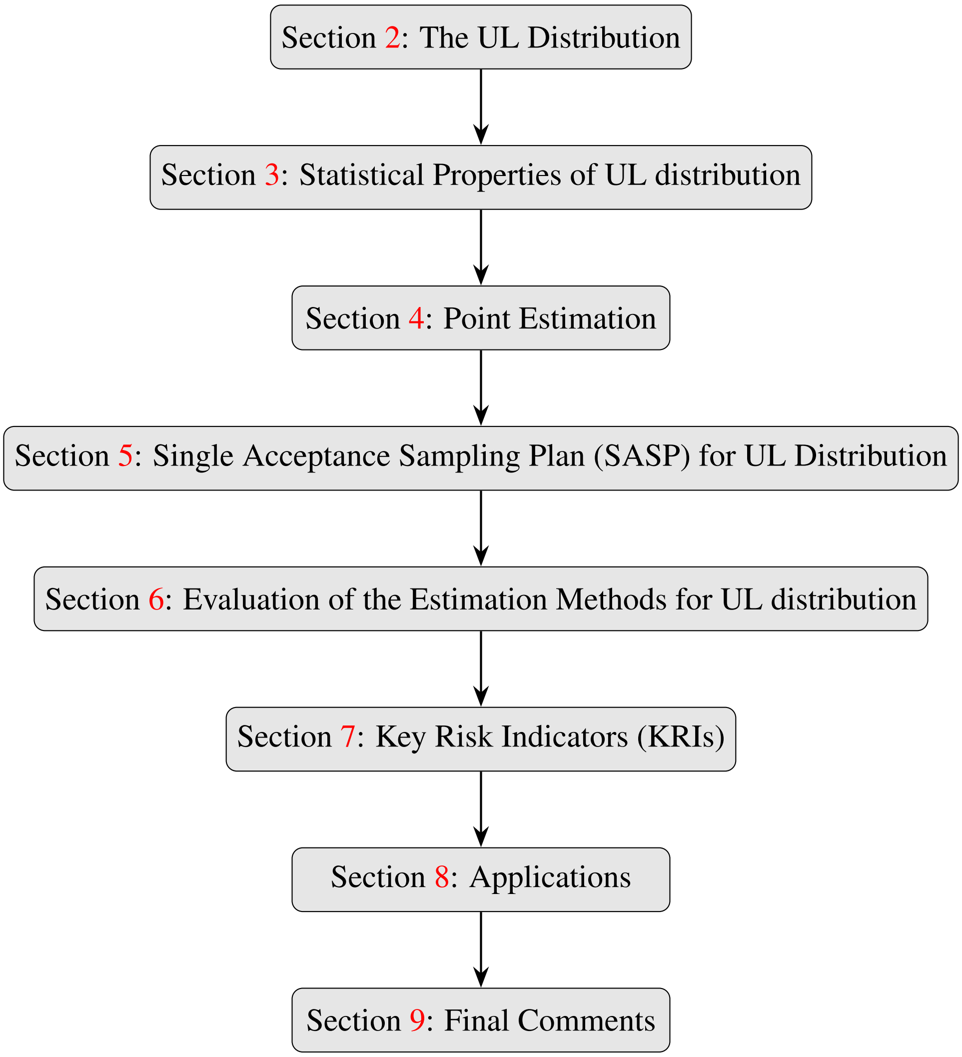


Figure 1. Sequence of the article

2. The UL Distribution

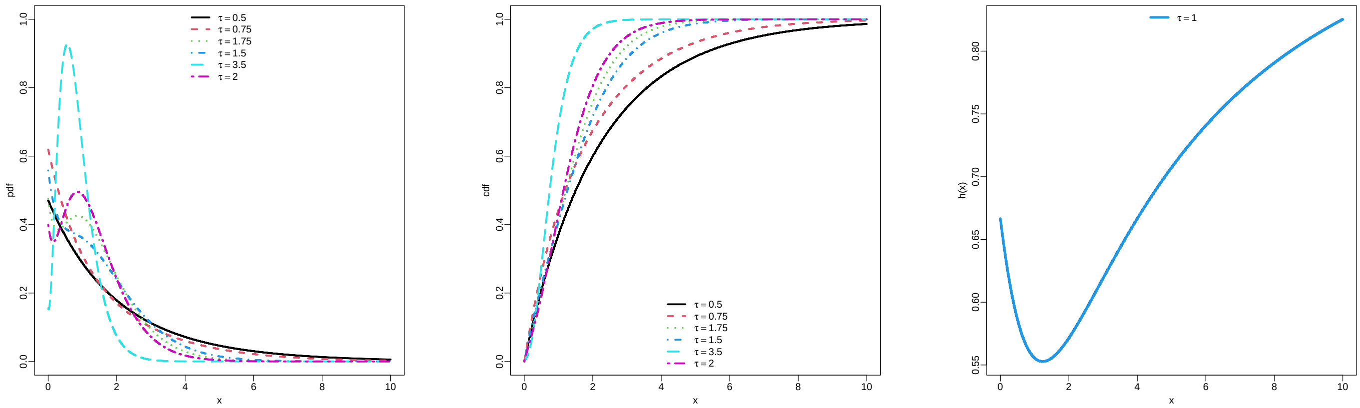
A random variable X with parameter τ is said to have UL distribution, if the probability density function (pdf) is given by

$$f(x,\tau) = \frac{\tau}{2(\tau^3+2)}(4+\tau^5 x^2)\mathrm{e}^{-\tau x}; \quad x>0, \tau>0. \tag{2.1}$$

The pdf in equation (2.1) is a mixture of two component distributions. That is, exponential distribution with scale parameter τ and gamma distribution with shape and scale parameters 3 and τ, respectively. The mixture is of the form $f(x,\tau) = pf_1(x,\tau) + (1-p)f_2(x,3,\tau)$ where $p = \frac{2}{\tau^3+2}$ is the mixing proportion. The cumulative distribution function (cdf) is given in equation (2.2).

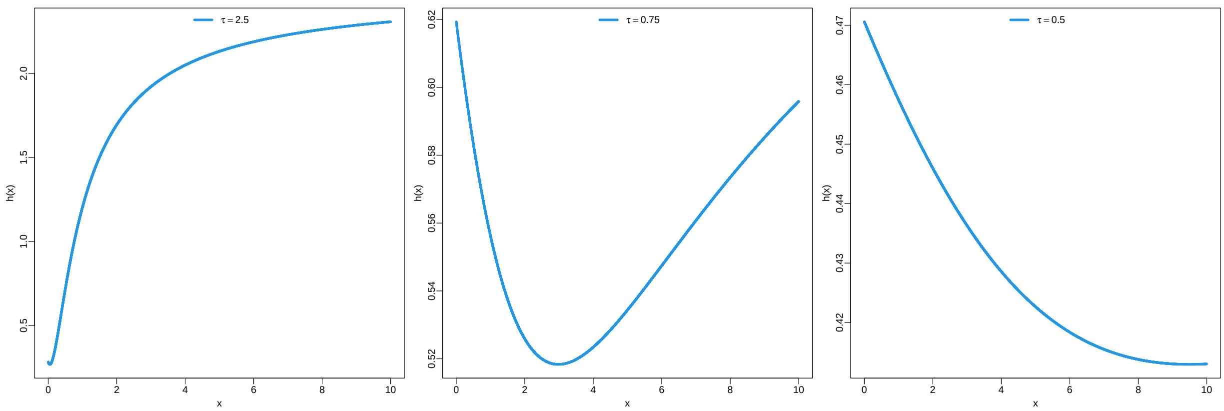
$$F(x,\tau) = 1 - \left[1 + \frac{\tau^4 x(\tau x+2)}{2(\tau^3+2)}\right]\mathrm{e}^{-\tau x}; \quad x>0, \quad \tau>0. \tag{2.2}$$

Figures 2 and 3 contain the plots of the pdf and hazard functions at different parameter values. The images with varying shapes illustrate the potential of the UL distribution to capture various life scenarios.



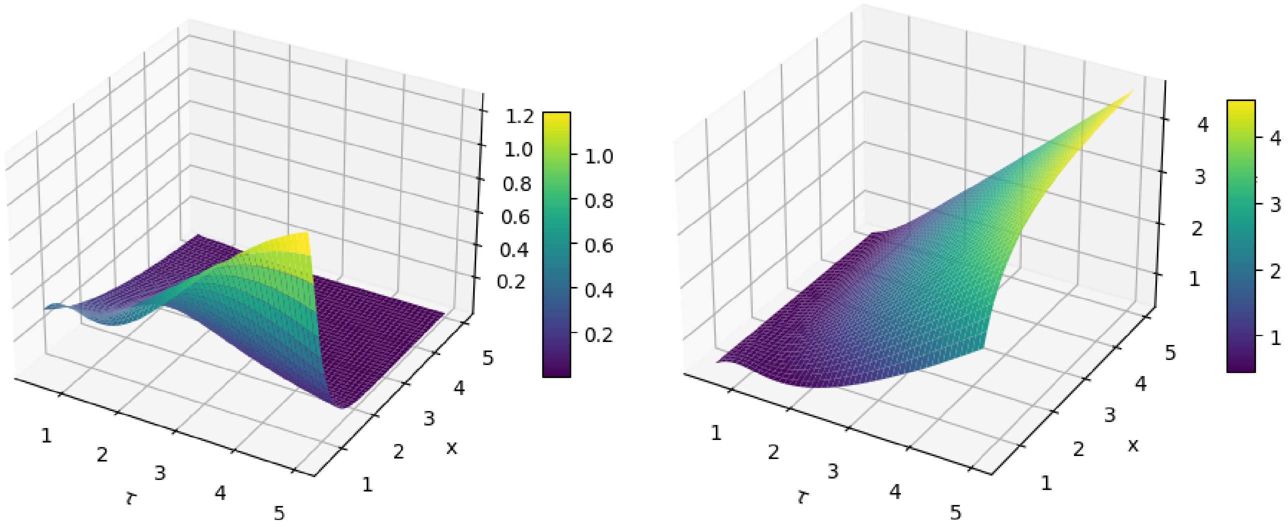
(a) pdf of UL (τ) **(b)** cdf of UL (τ) **(c)** hrf of UL (τ)

Figure 2. Visualization of the UL distribution



(a) hrf of UL (τ) **(b)** hrf of UL (τ) **(c)** hrf of UL (τ)

Figure 3. visualization of the UL distribution



(a) 3D pdf of UL (τ) **(b)** 3D hazard function of UL (τ)

Figure 4. 3D pdf and hazard function

The survival and hazard function of the UL distribution $S(x)$ and $h(x)$ are respectively expressed as

$$S(x) = 1 - F(x) = \left(1 + \frac{\tau^4 x(\tau x + 2)}{2(\tau^3 + 2)}\right) e^{-\tau x}, \tag{2.3}$$

and

$$h(x) = \frac{\tau(4 + \tau^5 x)}{z\tau^3 + 4 + \tau^5 x^2 + 2\tau^4 x}. \tag{2.4}$$

It is important to note the following behaviors about the UL distribution. For the UL distribution, $h(0) = f(0) = 2\tau/(\tau^3 + 2)$. This follows the same pattern as the Lindley distribution, see [14, 4] for details. In addition, $h(0)$ has an increasing function in x and in τ, implying that $2\tau/(\tau^3 + 2) < h(x) < \tau$. When comparing the UL distribution with the exponential distribution, one will notice that $h(x) = \tau$ for the exponential distribution, thus giving the UL distribution more flexibility than the exponential distribution, see Figure 3 for the hazard rate plots.

3. Statistical Properties of the UL Distribution

In this section, a number of properties of the proposed UL distribution are derived and extensively studied.

3.1. Moments

Define X_j to be a random variable that follows the UL distribution with parameter τ, then the crude moment r^{th} can be derived as follows

$$\mu_r' = \int_0^\infty x^r f(x;\tau)dx.$$

By substituting for $f(x,\tau)dx$, given in equation (2.1), the r^{th} crude moment of the UL distribution becomes

$$\mu_r^{'} = E(X^r) = \frac{r!(r+1)(4+\tau^3(r+2)(r+3))}{2(\tau^3+2)\tau^r}. \tag{3.1}$$

For $r = 1, 2, 3$, and 4, the crude moments are given by

$$\mu_1^{'} = \frac{4+12\tau^3}{\tau(\tau^3+2)}, \quad \mu_2^{'} = \frac{2(4+20\tau^3)}{\tau^2(\tau^3+2)}, \quad \mu_3^{'} = \frac{12(4+30\tau^3)}{\tau^3(\tau^3+2)} \quad \text{and} \quad \mu_4^{'} = \frac{12(4+42\tau^3)}{\tau^4(\tau^3+2)}.$$

The 2^{nd}, 3^{rd} and 4^{th} non-central moments are defined below

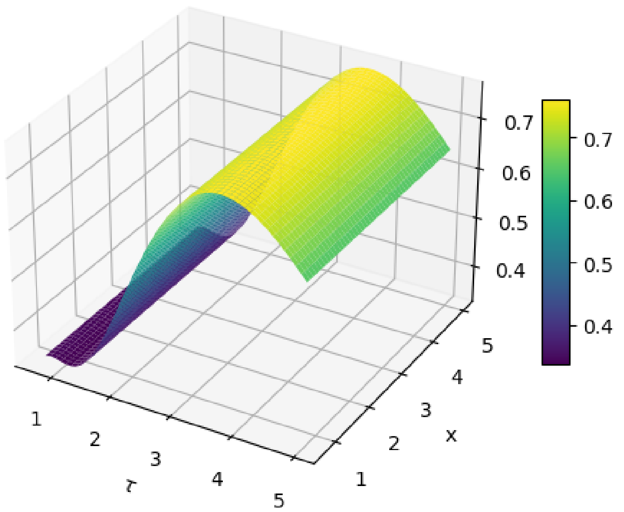
$$\mu_2 = \mu_2^{'} - (\mu_1^{'})^2 = \frac{8+40\tau^3-208\tau^5+384\tau^8+144\tau^{11}+32\tau^2}{\tau^2(\tau^3+2)(\tau(\tau^3+2))^2}, \tag{3.2}$$

$$\mu_3 = \mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2(\mu_1^{'})^3 = \frac{80+744\tau^3+2016\tau^6+3456\tau^9}{(\tau^3(\tau^3+2))^2}, \tag{3.3}$$

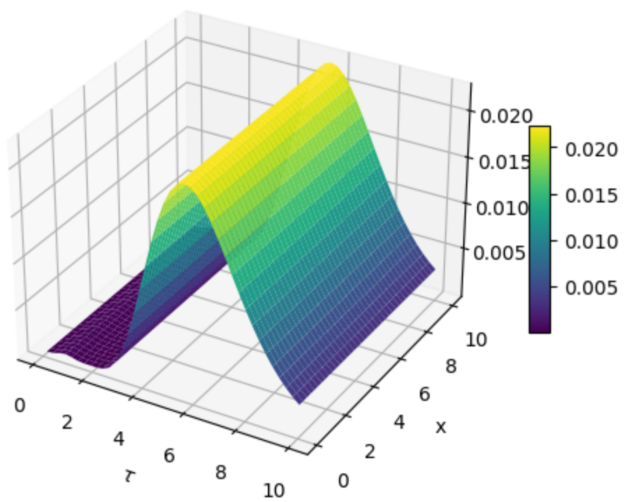
and

$$\begin{aligned}\mu_4 &= \mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_2^{'}(\mu_1^{'})^2 - 3(\mu_1^{'})^4 \\ &= \frac{14712\tau^3+6912\tau^5+34560\tau^8-100224\tau^6-82944\tau^9-3024}{\tau^4(\tau^3+2)^4}.\end{aligned} \tag{3.4}$$

$\mu_1^{'}$ and μ_2 respectively represent the mean and variance of the UL distribution. The standard deviation can be obtained by taking the square root of μ_2. Figure 5 represents the mean and variance.



(a) Mean of UL (τ)



(b) Variance of UL (τ)

Figure 5. Measure of central tendency and dispersion plots

3.2. The Coefficient of Skewness and Kurtosis

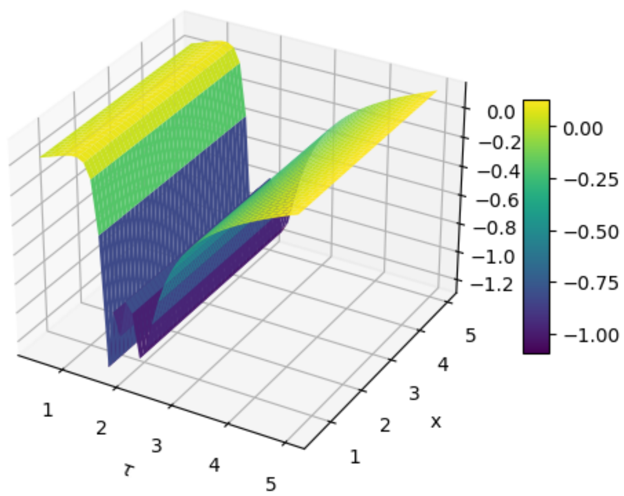
The degree of peakedness and the tail of any distribution can be examined using the Kurtosis and skewness of the distribution. If we let ∇_1 and ∇_2 denote the coefficients of skewness and kurtosis of the UL distribution, then their statistics are obtained as follows

$$\nabla_1 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} = \frac{(80 + 744\tau^3 + 2016\tau^6 + 3456\tau^9)(\tau^4(\tau^3 + 2)^3)^{\frac{3}{2}}}{\tau^6(\tau^3 + 2)^2(8 + 40\tau^3 - 208\tau^5 + 384\tau^8 + 144\tau^{11} + 32\tau^2)^{\frac{3}{2}}}. \tag{3.5}$$

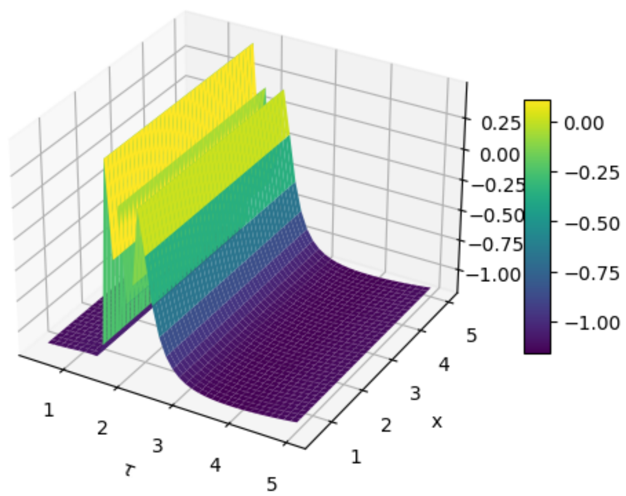
In addition, the coefficient of kurtosis is given by

$$\nabla_2 = \frac{\mu_4}{\mu_2^2} = \frac{14712\tau^3 + 6912\tau^5 + 34560\tau^8 - 100224\tau^6 - 82944\tau^9 - 3024\tau^4(\tau^3 + 2)^2}{(8 + 40\tau^3 - 208\tau^5 + 384\tau^8 + 144\tau^{11} + 32\tau^2)^2}. \tag{3.6}$$

Figure 6 illustrates the skewness and kurtosis of the UL distribution



(a) Skewness of UL (τ)



(b) Kurtosis of UL (τ)

Figure 6. Measure of asymmetry and peakedness plots

To study the shape of the UL distribution, which aids in determining if a data set can be modeled by the formulated model, it is significant that we derive the mode. Note that:

$$\lim_{x\to 0} f(x;\tau) = \frac{2\tau}{\tau^3 + 2} \textit{and} \lim_{x\to\infty} f(x;\tau) = 0.$$

Theorem 3.1. *The pdf of X, $f(x,\tau)$ is a decreasing value for $\tau > 0$.*

Proof.
We first obtain the first derivative of $\ln f(x;\tau)$ from equation (2.1),

$$\ln f(x;\tau) = \ln\tau = \ln\tau - \ln(2(\tau^3 + 2)) + \ln(4 + \tau^5 x^2) - \tau x, \tag{3.7}$$

$$\frac{\delta \ln f(x;\tau)}{\delta x} = \frac{2\tau^5 x}{4 + \tau^5 x} - \tau.$$

It is easy to see that for $\tau > 0$, the function $(\ln f(x;\tau))'$ will yield a negative value. Thus, implying that $f(x;\tau)$ is a decreasing function with

$$\lim_{x\to 0} f(x;\tau) = \frac{2\tau}{\tau^3+2} \textit{and} \lim_{x\to\infty} f(x;\tau) = 0.$$

□

3.3. Mean Past Lifetime of the UL Distribution

The mean past lifetime of a probability distribution is a concept used in reliability theory and survival analysis. It is a measure that quantifies the expected time a system or entity has existed, given that it has survived until now.

To understand this better, let's consider a probability distribution that represents the lifetime of a system or object, where the lifetime is a non-negative random variable. This random variable's pdf describes the likelihood of different lifetimes occurring.

Mathematically, let's denote the lifetime random variable from UL distribution as T and its pdf as $f(t)$. The mean past lifetime, denoted as $\epsilon(t)$, is defined as follows

$$\epsilon(t) = \mathbb{E}[t - T/T \leq t] = t - \frac{\int_0^\infty xf(x)dx}{F(t)} = x - \frac{\frac{4\gamma(2,x)+\tau^3\gamma(4,x)}{2\tau(\tau^3+2)}}{1-\left(1+\frac{\tau^4 x(\tau x+2)}{2(\tau^3+2)}\right)e^{-\tau x}}. \tag{3.8}$$

The values of mean past lifetime range from negative to positive infinity. A product with an infinite mean past lifetime may indicate that it has not experienced any failures or reached the end of its useful life within the observed time frame.

3.4. The Tail Behaviour of the UL Distribution

We seek to examine the behavior of the tail of the UL distribution. First, the ratio of $S(x)$ and $e^{-\tau x}$ for the exponential distribution is investigated to verify the weight of the tail. Knowledge of this aids in understanding the speed of decay of the survival function of the UL distribution. Comparing the tail weights, we have:

$$\begin{aligned}\frac{\text{Survival function of UL distribution}}{\text{Survival function of exponential distribution}} &= \frac{\left(1+\frac{\tau^4 x(\tau x+2)}{2(\tau^3+2)}\right)e^{-\tau x}}{e-\tau x}\\ &= \frac{\left(\frac{2\tau^3}{x^2}+\frac{4}{x^2}+\tau^5+\frac{2\tau^4}{x}\right)}{\left(\frac{2\tau^3}{x^2}+\frac{4}{x^2}\right)}.\end{aligned} \tag{3.9}$$

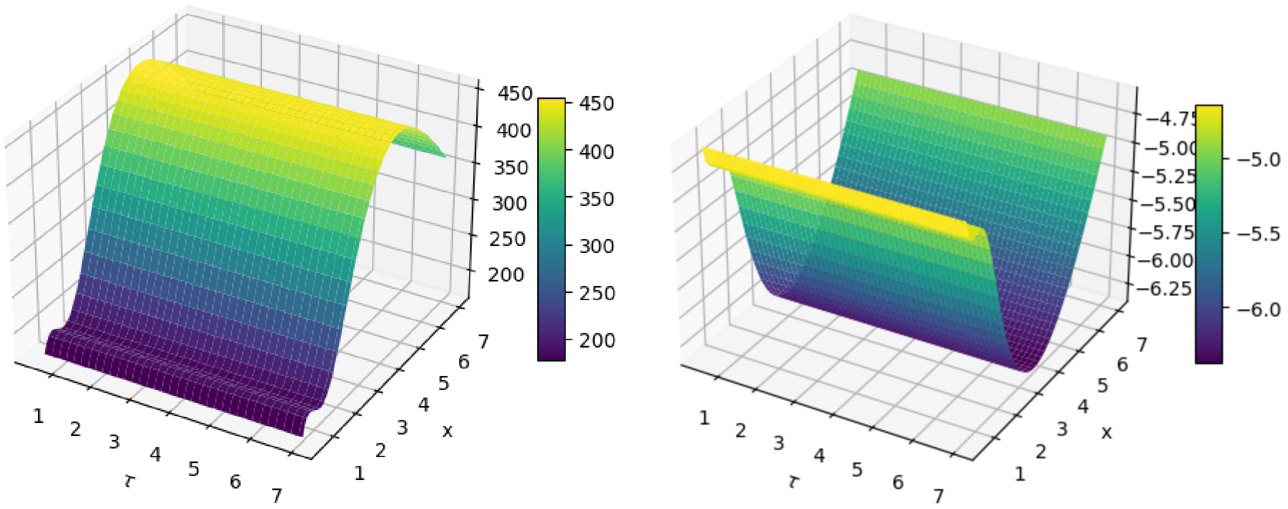
Taking the limit of equation (3.9) as $x \to \infty$ gives ∞. These results give an indication that the UL distribution has a heavier tail than the exponential distribution. Let us also consider the Mean Excess Loss Function of the UL distribution, which will enable the provision of expected excess loss over a specified threshold, given that the threshold has already been exceeded. The excess loss function or mean residual life function $m(x)$ can be expressed as

$$m(x) = \frac{1}{1-F(x)}\int_x^\infty (1-F(t))dt. \tag{3.10}$$

Putting the value of $F(x)$ in Equation (3.10) as can be seen in equation (2.2) and evaluating, we obtain the mean residual life function of the UL distribution as

$$m(x) = \frac{\tau^2(\tau^2x^2 + 4\tau x + 6)}{2\tau^3 + \tau^5x^2 + 2\tau^4x + 4}. \tag{3.11}$$

Figure 7(a) shows the plot of the mean residual life of the UL distribution.



(a) $m(x)$ of UL (τ) **(b)** Rény entropy of UL (τ)

Figure 7. Measure of additional life and Information Loss plots

The mean residual life of UL distribution is a decreasing function that tends to become stationary as x increases. Also, for $m(0)$, we have the constant $(3\tau^2(\tau^3 + 2))$.

The tail of UL distribution can equally be examined by investigating the maximum domain of attraction. Given that the extreme value theorem provides three extremal types, the Gumbel, Fréchet and Weibull distributions, the goal is to ascertain which one the UL distribution tail falls into. Let us look at three conditions (Shown in Theorem (3.2), (3.3), and (3.4)) that will help us attain our goals.

Theorem 3.2. *Let the right extremity of F be given by $\omega(F)$. $F \in D(Gumbel)$ if and only if $\exists\ \kappa(t) > 0$ such that:*

$$\lim_{t\to\omega(F)} \frac{1 - F(t + \kappa(t)x)}{1 - F(t)} = \exp(-x) \quad \textit{for } x \geq 0, \quad \omega(F) < \infty.$$

Theorem 3.3. *$F \in D(Frechet)$ if and only if*

$$\omega(F) = \infty \quad \textit{and} \quad \lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \quad x > 0.$$

and

Theorem 3.4. *$F \in D(Weibull)$ if and only if*

$$\omega(F) < \infty \quad \textit{and} \quad \lim_{t\to 0} \frac{1 - F(\omega(F) - tx)}{1 - F(\omega(F) - t)} = x^{\alpha}, \quad x > 0.$$

To find the maximum domain of attraction for the UL distribution, making use of the necessary and sufficient conditions for a distribution to belong to the Gumbel class as shown in Theorem (3.3), we set $\kappa = \frac{1}{\tau}$ and $\omega(F) = \infty$, where $\omega(F)$ is the upper end point. thus, we have

$$\lim_{t\to\infty} \frac{1 - F(t + \frac{x}{\tau})}{1 - F(t)} = \lim_{t\to\infty} \left(\frac{1 - \left(1 - \left(1 + \frac{\tau^4(t+\frac{x}{\tau})(\tau(t+\frac{x}{\tau})+2)}{2(\tau^2+2)}\right) e^{-\tau(t+\frac{x}{\tau})}\right)}{1 - \left(1 - \left(1 + \frac{\tau^4 t(\tau t+2)}{2(\tau^2+2)e^{-\tau t}}\right)\right)} \right).$$

When we resolve the expression above, we obtain

$$\frac{\tau^2 e^{-\tau\left(t+\frac{x}{\tau}\right)}}{\tau^5 e^{-\tau t}} = \exp(-x).$$

Since $\exp(-x)$ has been obtained from the function, it shows that UL belongs to the Gumbel domain of attraction. The salient features of distributions falling in the Gumbel domain include exhibiting a considerable range of tail heaviness and possessing infinite or finite upper-end points.

3.5. Stochastic Ordering

The stochastic ordering of a non-negative continuous random variable is a vital tool for comparing the behavior of system components. These stochastic orders characterize different aspects of the shapes of the distributions, like skewness, location, and spread. They are useful for comparing distributions in reliability theory, economics, and other fields. A random variable X is said to be smaller than another random variable Y in the

(i) Stochastic order ($X \leq_{st} Y$) if $F_X(x) \geq F_Y(x)$ $\forall x$.

(ii) Hazard rate order ($X \leq_{hr} Y$) if $h_X(x) \geq h_Y(x)$ $\forall x$.

(iii) Mean residual life order ($X \leq_{mrl} Y$) if $m_X(x) \geq m_Y(x)$ $\forall x$.

(iv) Likelihood ratio order ($X \leq_{lr} Y$) if $\frac{f_X(x)}{F_Y(x)}$ decreases in x .

See [16, 19] for more details. From the theorem, it can be inferred that

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y \Rightarrow X \leq_{mrl} Y$$

In this section, it can be proven that the UL distribution is ordered for the strongest "likelihood ratio", as shown in the theorem below

Theorem 3.5. *Let $X \sim UL(\tau_1)$ and $Y \sim UL(\tau_2)$. If $\tau_1 > \tau_2$ then $X \leq_{lr} Y$ hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.*

Proof.

$$\frac{f_X(x)}{f_Y(x)} = \frac{\frac{\tau_1}{2(\tau_1^3+2)}\left(4 + \tau_1^5 x\right) e^{-\tau_1 x}}{\frac{\tau_2}{(\tau_2^3+2)}\left(4 + \tau_2^5 x^2\right) e^{-\theta_2 x}} = \left(\frac{\tau_1(\tau_2^3 + 2)}{\theta_2(\tau_1^3 + 2)}\right)\left(\frac{4 + \tau_1^5 x^2}{4 + \tau_2^5 x^2}\right) e^{-(\tau_1-\tau_2)x}.$$

Taking the natural logarithm of the expression above and differentiating partially for x, we obtained the following

$$\frac{\partial}{\partial x}\ln\left(\frac{F_X(x)}{F_Y(x)}\right) = \frac{8x\left(\tau_1^5 - \tau_2^5\right)}{\left(4+\tau_2^5 x^2\right)^2} - (\tau_1 - \tau_2).$$

From the above expression one can deduce that for $\tau_1 < \theta_2$, $\frac{\partial}{\partial x}\ln\left(\frac{F_X(x)}{F_Y(x)}\right) < 0$. This means $X \leq_{lr} Y$ and hence, $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$. Then, we can say that the UL distribution follows the strongest likelihood ratio ordering. □

3.6. *Quantile Function and Distribution of Order Statistics*

Quantile function and order statistics are related concepts for continuous probability distributions. The quantile function $Q(p)$ is the inverse of the cumulative distribution function (CDF). For a probability p, it returns the value x such that $P(X \leq x) = p$, where X is the random variable following that distribution. In addition, order statistics refer to the sorted values of a random sample drawn from a distribution. For a sample of size n, the order statistics are denoted as $X_1, X_2, \cdots, X_n$, where $X_1 \leq X_2 \leq \cdots \leq X_n$. X_i represents the i^{th} order statistic, or the i^{th} smallest value in the sample. There is a direct relationship between the two for continuous distributions: $X(i) = Q(i/(n+1))$. The i^{th} order statistic corresponds to the quantile value at percentile $i/(n+1)$. So, order statistics allow you to empirically estimate quantiles and the entire quantile function for a distribution based on a sample. The larger the sample size n, the better the estimation. Let X with pdf given in equation (2.1), the quantile function is defined by

$$2(\tau^3+2) + \tau^4 x(\tau x+2)\mathrm{e}^{-\tau x} = 2(\tau^3+2)(1-p). \tag{3.12}$$

Resolving equation (3.12) gives the quantile function of UL distribution. Let $X_1, X_1, \cdots, X_n$ be a random sample of size n, then the probability density function of the ω^{th} order statistics, say $X_\omega : n$ can be expressed as

$$f_X(x) = \frac{n!}{(\omega-1)!(n-\omega)!}F^{\omega-1}(x)(1-F(x))^{n-\omega}f(x). \tag{3.13}$$

Based on equations (2.1) and (2.2) and the application of binomial expansion theory, we have pdf and cdf of UL distribution, respectively, given as

$$\begin{aligned} f_X(x) = {} & \frac{\tau^4 l+m+1n!2^l-m}{(\omega-1)!(n-\omega)!(2(\tau^3+2))^l+1}\sum_{j=0}^{n-\omega}\binom{n-\omega}{j}(-1)^j\sum_{k=0}^{\infty}\binom{\omega+j-1}{k}(-1)^k \\ & \times\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}(4+\tau^5x^2)x^{l+m}\mathrm{e}^{-\tau(1+k)x}, \end{aligned} \tag{3.14}$$

$$F_X(x) = \sum_{j=\omega}^{n}\binom{n}{j}\sum_{k=0}^{n-j}(-1)^k\sum_{l=0}^{\infty}\binom{j+k}{l}(-1)^l\sum_{m=0}^{l}\binom{l}{m}\sum_{n=0}^{m}\binom{m}{n}\frac{2^{m-n}\theta^{4m+n}}{(2(\tau^3+2))^m}x^n\mathrm{e}^{-\tau \mathrm{lx}}. \tag{3.15}$$

Figures (8) and (9) demonstrate the behavior of the pdf of the order statistics of UL distribution.

Figure 8. Plot of order statistics

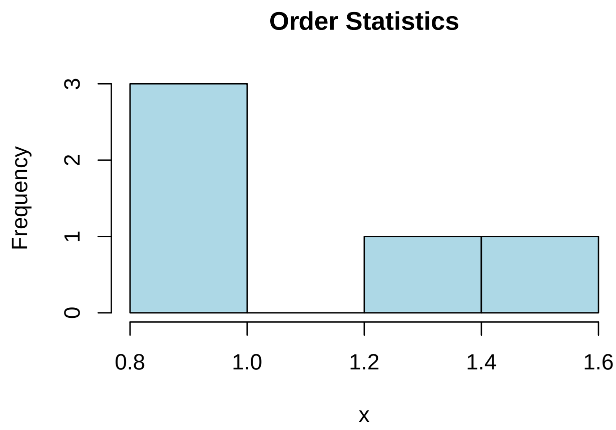


Figure 9. Histogram of the order statistics

4. Point Estimation

In this section, we study the estimation of the parameter of UL distribution using non-Bayesian methods. The methods studied include MLE, MPSE, LSE, WSLE, CVM, ADE and RTADE and Bayesian estimation under various loss functions

4.1. Maximum Likelihood Estimation

Following the method described in [7], the maximum likelihood of UL distribution is derived as

$$\varphi(\tau) = n \ln \tau - n \ln\left(2\left(\tau^2 + 2\right)\right) + \sum_{i=1}^{n} \ln\left(4 + \tau^5 x_i^5\right) - \tau \sum_{i=1}^{n} x_i. \tag{4.1}$$

The maximum likelihood estimation of $\hat{\tau}_{MLE}$ of the parameter τ can be estimated by maximizing the log-likelihood function $\varphi(\tau)$. The value of τ is the solution to the partial derivation of the non-linear equation below

$$\frac{\partial \varphi(\tau)}{\partial \tau} = \frac{n}{\tau} - \frac{2n\tau}{\tau^3 + 2} + \sum_{i=1}^{n} \frac{5\tau^4 x_i^2}{4 + \tau^5 x_i^2} - \sum_{i=1}^{n} x_i = 0.$$

4.2. Maximum Product Spacing (MPS)

Employing the method introduced by [8], the derivative of the MPS of the UL distribution is obtained by maximizing the function

$$\beth(\tau) = \frac{1}{n+1} \sum_{i=1}^{n+1} \ln f(\tau). \tag{4.2}$$

The estimator $\hat{\tau}$ for τ is derived by finding the solution to the non-linear equation below

$$\frac{\partial}{\partial \tau}\beth(\tau) = \frac{1}{n+2} \sum_{i=1}^{n+1} \frac{1}{f(\tau)} \left(\epsilon\left(t_{i:n}|\tau\right) - \epsilon\left(t_{i:n}|\tau\right)\right) = 0,$$

where

$$
\begin{aligned}
(\epsilon(t_{i:n}|\tau)) = \left(2\left(\tau^3+2\right)\right)^2 \Big(&3\tau^7x^2+4\tau^6x+20\tau^4x^2+32\tau^3x-4\tau^7-16\tau^4-16\tau \\
&-2\tau^9x^2-4\tau^8x-4\tau^6x^2-8\tau^5x\Big)e^{-\tau x}.
\end{aligned} \tag{4.3}
$$

MPS aids in Bayesian inference and numerical analysis to reduce the correlation between parameters and enhance the accuracy of parameter estimates and convergence of numerical algorithms.

4.3. Ordinary Least Squares and Weighted Least Squares

Given that

$$
E(F_{t:n}|\tau) = \frac{i}{n+1} \quad \text{and} \quad V(F_{t:n}|\tau) = \frac{i(n-i+1)}{(n+1)^2(n+2)}. \tag{4.4}
$$

Ref. [24] proposed that the least squares estimate of $\hat{\tau}_{OLS}$ of τ is obtained by minimizing the function

$$
T(\tau) = \sum_{i=1}^{n}\left(F(t_{i:n}|\tau) - \frac{i}{n+1}\right)^2.
$$

Differentiating partially yields

$$
\sum_{i=1}^{n}\left(F(t_{i:n}|\tau) - \frac{i}{n+1}\right)\epsilon(t_{i:n}|\tau) = 0. \tag{4.5}
$$

Similarly, the weighted least squares estimate $\hat{\tau}_{WLSE}$ for the UL distribution parameter τ is achieved by minimizing the function $\omega(\tau)$ with respect to τ

$$
\omega(\tau) = \underset{(\tau)}{\arg\min} \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(t_{i:n}|\tau) - \frac{i}{n+1}\right]^2. \tag{4.6}
$$

Resolving partially, we obtain the following non-linear equation

$$
\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(t_{i:n}|\tau) - \frac{i}{n+1}\right]\epsilon(t_{i:n}|\tau) = 0, \tag{4.7}
$$

where $\epsilon(t_{i:n}|\tau)$ is as defined in equation (4.3).

4.4. Cramér-von-Mises Estimation Method (CVME)

The method for Cramér-von-Mises estimates of $\hat{\tau}_{CVME}$, of the UL distribution parameter τ, is defined as

$$
C(\tau) = \underset{(\tau)}{\arg\min}\left\{\frac{1}{12n} + \sum_{i=1}^{n}\left[F(t_{i:n}|\tau) - \frac{2i-1}{2n}\right]^2\right\}. \tag{4.8}
$$

The estimates are obtained by solving the following non-linear equations

$$
\sum_{i=1}^{n}\left(F(t_{i:n}|\tau) - \frac{2i-1}{2n}\right)\epsilon(t_{i:n}|\tau) = 0.
$$

4.5. Anderson-Darling Estimation

The Anderson-Darling estimate $\hat{\tau}$ for the UL distribution with parameter τ is obtained by minimizing the function $\vartheta(\tau)$ with respect to τ

$$\vartheta(\tau) = \underset{(\tau)}{\arg\min} \sum_{i=1}^{n} (2i-1)\Big\{ \ln F(x_{i:n}|\tau) + \ln\left[1 - F(x_{n+1-i:n}|\tau)\right]\Big\}. \tag{4.9}$$

The estimates are obtained by solving the following sets of non-linear equations

$$\sum_{i=1}^{n} (2i-1)\left[\frac{\varepsilon(x_{i:n}|\tau)}{F(x_{i:n}|\tau)} - \frac{\varepsilon(x_{n+1-i:n}|\tau)}{1 - F(x_{n+1-i:n}|\tau)}\right] = 0, \tag{4.10}$$

where $\varepsilon(x.|\tau)$ is as defined in equation (4.3).

4.6. Right-Tailed Anderson-Darling Estimation

The Right-Tailed Anderson-Darling estimates $\hat{\tau}_{RTAD}$ of the UL distribution parameter τ obtained by minimizing the function $Q(\tau)$ with respect to τ

$$Q(\tau) = \underset{(\tau)}{\arg\min}\left\{\frac{n}{2} - 2\sum_{i=1}^{n} F(x_{i:n}|\tau) - \frac{1}{n}\sum_{i=1}^{n} (2i-1)\ln\left[1 - F(x_{n+1-i:n}|\tau)\right]\right\}. \tag{4.11}$$

The estimates can be obtained by solving the following set of non-linear equations

$$-2\sum_{i=1}^{n} \frac{\varepsilon(x_{i:n}|\tau)}{F(x_{i:n}|\tau)} + \frac{1}{n}\sum_{i=1}^{n} (2i-1)\left[\frac{\varepsilon(x_{n+1-i:n}|\tau)}{1 - F(x_{n+1-i:n}|\tau)}\right] = 0, \tag{4.12}$$

where $\varepsilon(x.|\tau)$ is as defined in equation (4.3). Anderson-Darling right-tailed method is best when the dataset is censored or truncated.

5. Single Acceptance Sampling Plan (SASP) for the UL Distribution

Assume that the lifetime of a product follows the UL distribution with CDF defined in equation (2.2). Here, τ is the parameter of the UL distribution. The manufacturer fixed the standard median life cycle of the industry product as ϕ_0. The goal is to design a sampling plan to decide whether to accept the proposed lot based on the actual median life cycle ϕ exceeding ϕ_0. The test must be completed by time t_0, recording the number of failures.

5.1. Steps for designing the SASP for UL distribution

1. Specify the parameters
   - Consumers's risk α, the probability of accepting a lot that doesn't meet the required standards.
   - Test duration $t_0 = a\phi_0$, where a is a constant greater than 1.
2. Sampling Procedure

- Draw a random sample of n units from the proposed lot.
- Conduct the life test for ϕ_0 time units
- Accept the lot if c or fewer units (accepting number) fail; otherwise reject the lot.

Using the binomial distribution for large lots, the probability of accepting a lot is given by:

$$\zeta(p) = \sum_{i=0}^{c} \binom{n}{i} p^i (1-p)^{n-i}, \tag{5.1}$$

where p is the probability of a unit failing by time t_0. For the UL distribution, p can be calculated as:

$$p = F(t_0;\tau) = 1 - \left(1 + \frac{\tau^4 t_0 (\tau t_0 + 2)}{2(\tau^3+2)}\right) e^{-\tau t_0}. \tag{5.2}$$

Given that $t_0 = a\phi_0$, we denote p_0 as:

$$p_0 = 1 - \left(1 + \frac{\tau^4 (a\phi_0)(\tau(a\phi_0) + 2)}{2(\tau^3+2)}\right) e^{-\tau(a\phi_0)}. \tag{5.3}$$

To determine the sample size, we aim at finding the least positive integer n such that the probability of accepting the lot meets the consumer's risk threshold:

$$\zeta(p_0) = \sum_{i=0}^{c} \binom{n}{i} p_0^i (1-p_0)^{n-i} \leq 1 - \alpha,$$

where p_0 is derived from the above equation.

5.2. Assumptions and Inference

1. Consumer risk α: Set the value at 0.95
2. Acceptance number c: Possible values include 0, 2, 4, 8 and 10.
3. Constant a: Values might include 0.10, 0.25, 0.50, and 0.75. For $a = 1$, t_0 equals the median lifetime ϕ_0.
4. Parameter τ: Various values can be used to see the effect on sample size and acceptance probability.

From the results, we infer:

- As α, c and n increase, $\zeta(p_0)$ decreases.
- The required sample size n decreases as a increases, while $\zeta(p_0)$ increases.
- The required sample size n increases and $\zeta(p_0)$ decreases as τ increases.

Table 1. The SASP for the UL distribution with parameter $\tau = 0.35$ and $\tau = 0.40$

α^*	c	$a = 0.1$		$a = 0.2$		$a = 0.4$		$a = 0.8$		$a = 1$	
		n	$\zeta(p_0)$	n	$\zeta(p_0)$	n	$\zeta(p_0)$	n	$\zeta(p_0)$	n	$\zeta(p_0)$
						$\tau = 0.35$					
0.25	0	5	0.75646	3	0.75662	2	0.75693	1	1.00000	1	1.00000
	2	26	0.76427	14	0.76568	8	0.77067	5	0.78942	4	0.87500
	4	51	0.75399	27	0.75569	15	0.76159	9	0.78330	8	0.77344
	8	103	0.75046	54	0.75262	29	0.77727	17	0.80307	15	0.78803
	10	129	0.75474	67	0.76482	37	0.75722	22	0.75461	19	0.75966
	0	20	0.26561	10	0.28507	5	0.32827	3	0.32934	3	0.25000
	2	58	0.25211	30	0.25293	16	0.25534	9	0.26327	7	0.34375
0.75	4	92	0.25812	48	0.25068	25	0.27176	14	0.28420	12	0.27441
	8	159	0.25567	82	0.25732	43	0.27620	24	0.29481	21	0.25172
	10	192	0.25388	99	0.25573	52	0.27399	29	0.29496	25	0.27063
	0	43	0.05337	22	0.05348	11	0.06174	6	0.06225	5	0.06250
	2	92	0.05061	47	0.05086	24	0.05699	13	0.05908	11	0.05469
0.95	4	134	0.05044	68	0.05301	35	0.05880	19	0.06213	16	0.05923
	8	211	0.05162	108	0.05211	56	0.05739	31	0.05445	26	0.05388
	10	249	0.05024	127	0.05245	67	0.05070	36	0.06363	30	0.06802
						$\tau = 0.40$					
	0	5	0.75578	3	0.75601	2	0.75648	1	1.00000	1	1.00000
	2	26	0.76285	14	0.76443	8	0.76975	5	0.78914	4	0.87500
0.25	4	51	0.75199	27	0.75394	15	0.76031	9	0.78292	8	0.77344
	8	102	0.75652	54	0.75015	29	0.77556	17	0.80256	15	0.78802
	10	129	0.75163	67	0.76214	37	0.75519	22	0.75395	19	0.75966
	0	20	0.26447	10	0.28404	5	0.32749	3	0.32909	3	0.25000
	2	58	0.25017	30	0.25122	16	0.25406	9	0.26285	7	0.34375
0.75	4	92	0.25563	47	0.26588	25	0.27009	14	0.28364	12	0.27441
	8	159	0.25238	82	0.25441	43	0.27397	24	0.29406	21	0.25172
	10	192	0.25028	99	0.25253	52	0.27154	29	0.29414	25	0.27063
	0	43	0.05286	22	0.05304	11	0.06138	6	0.06213	5	0.06250
	2	91	0.05248	47	0.05020	24	0.05646	13	0.05890	11	0.05469
0.95	4	133	0.05172	68	0.05217	35	0.05812	19	0.06190	16	0.05923
	8	211	0.05043	108	0.05105	56	0.05654	31	0.05418	26	0.05388
	10	248	0.05059	127	0.05129	66	0.05676	36	0.06330	30	0.06802

Table 2. The SASP for the UL distribution with parameter $\tau = 0.15$ and $\tau = 0.20$

α^*	c	$a = 0.1$ n	$a = 0.1$ $\zeta(p_0)$	$a = 0.2$ n	$a = 0.2$ $\zeta(p_0)$	$a = 0.4$ n	$a = 0.4$ $\zeta(p_0)$	$a = 0.8$ n	$a = 0.8$ $\zeta(p_0)$	$a = 1$ n	$a = 1$ $\zeta(p_0)$
$\tau = 0.15$											
0.25	0	5	0.75775	3	0.75776	2	0.75778	1	1.00000	1	1.00000
	2	26	0.76694	14	0.76804	8	0.77240	5	0.78996	4	0.87500
	4	51	0.75773	27	0.75900	15	0.76402	9	0.78405	8	0.77344
	8	103	0.75572	54	0.75727	29	0.78052	17	0.80405	15	0.78802
	10	130	0.75282	68	0.75449	37	0.76104	22	0.75588	19	0.75966
0.75	0	20	0.26776	10	0.28701	5	0.32975	3	0.32983	3	0.25000
	2	58	0.25578	30	0.25619	16	0.25778	9	0.26408	7	0.34375
	4	93	0.25421	48	0.25484	25	0.27496	14	0.28527	12	0.27441
	8	160	0.25529	82	0.26289	44	0.25329	24	0.29624	21	0.25172
	10	193	0.25472	99	0.26186	53	0.25391	29	0.29655	25	0.27063
0.95	0	44	0.05069	22	0.05434	11	0.06244	6	0.06248	5	0.06250
	2	92	0.05204	47	0.05213	24	0.05802	13	0.05943	11	0.05469
	4	135	0.05002	69	0.05016	36	0.05069	19	0.06257	16	0.05923
	8	213	0.05031	109	0.05054	57	0.05142	31	0.05497	26	0.05388
	10	250	0.05104	128	0.05130	67	0.05234	36	0.06426	30	0.06802
$\tau = 0.20$											
0.25	0	5	0.75760	3	0.75763	2	0.75768	1	1.00000	1	1.00000
	2	26	0.76663	14	0.76776	8	0.77219	5	0.78989	4	0.87500
	4	51	0.75729	27	0.75861	15	0.76373	9	0.78396	8	0.77344
	8	103	0.75510	54	0.75673	29	0.78013	17	0.80393	15	0.78802
	10	130	0.75212	68	0.75387	37	0.76059	22	0.75573	19	0.75966
0.75	0	20	0.26750	10	0.28678	5	0.32957	3	0.32978	2	0.50000
	2	58	0.25534	30	0.25580	16	0.25749	9	0.26398	7	0.34375
	4	93	0.25366	48	0.25435	25	0.27458	14	0.28514	12	0.27441
	8	160	0.25455	82	0.26223	44	0.25280	24	0.29607	21	0.25172
	10	193	0.25391	99	0.26113	53	0.25338	29	0.29636	25	0.27063
0.95	0	44	0.05058	22	0.05423	11	0.06236	6	0.06245	5	0.06250
	2	92	0.05187	47	0.05198	24	0.05790	13	0.05939	11	0.05469
	4	134	0.05200	68	0.05444	36	0.05055	19	0.06252	16	0.05923
	8	213	0.05005	109	0.05030	57	0.05124	31	0.05491	26	0.05388
	10	250	0.05075	128	0.05105	67	0.05215	36	0.06418	30	0.06802

Tables 1 and 2 present the values of p_0, which govern failure probability by time t_0 under different environmental settings characterized by τ (the shape parameter), α (consumer risk), c (the acceptance number), and a (scaling factor tied to test duration). Thus, the exercise here is to check the behavior of p_0 under specific parameter values. With increments in τ, p_0 decreases at a constant α, c, a value setting. This goes for higher tau values, where the cdf significantly drops in probability over a given time period. Thus, the probability of failure at $t = to$ is higher when τ is lower, since under $t = t_0$ high values imply that the-U-function is dropping steeply; hence, the failure probability lessens. At higher α values, e.g., $\alpha = 95$, p_0

tends to be low in value relative to lower ones under α, e.g., under $\alpha = 25$. Plus, when α is higher, there is a stricter requirement for rejecting lots; thus, with less tolerance for failures, p_0 goes further down. In this context, c is expected to positively correlate with p_0 at relatively lower values of τ, α, and a. Larger acceptance numbers c mean more failures are allowed in the sample, which increases acceptance probability associated with larger p_0. The implication is that results of the parameters presented reflect the dynamic balance maintained among UL distribution shape, acceptance parameters, and risk thresholds for defining p_0.

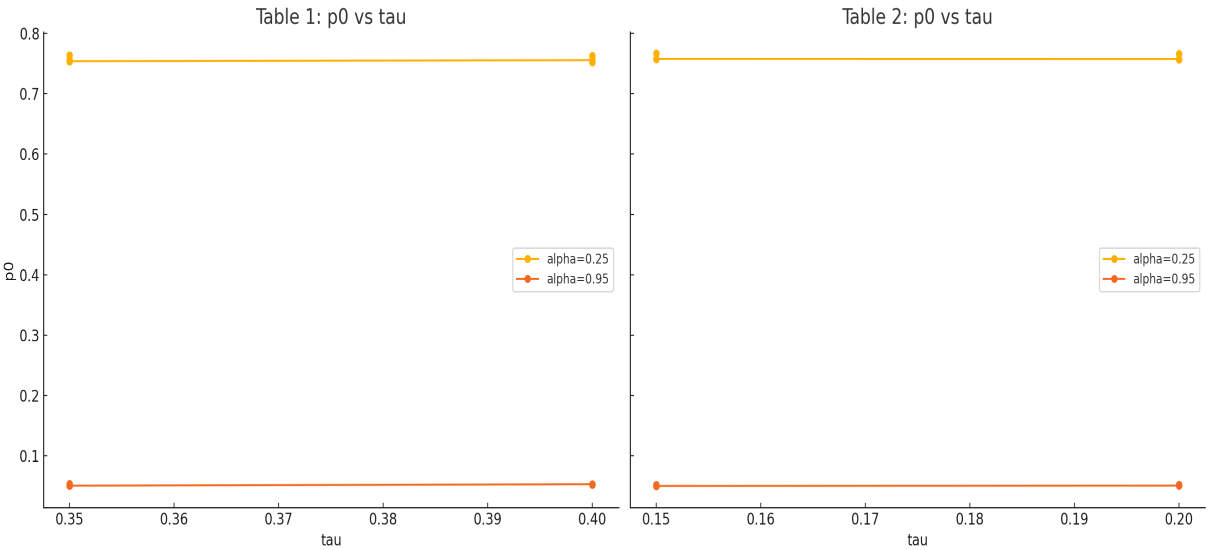


Figure 10. Behaviour of p_0 for values of τ, α, and c.

The graphs visually represent the changing probability of failure p_0 based on varying parameters τ, α, and c from the datasets of Tables 1 and 2. For Table 1, we can see that as the consumer becomes stricter (higher α), the system has less and less tolerance for failures, and consequently, p_0 gets smaller. In fact, p_0 decreases with increasing τ. With higher values of τ, the failure probability of the updated Lindley distribution rapidly decreases, causing the associated failure probability during the test time to decrease even further. Somewhat similar trends can be seen in the Table 2 graph, probably with slightly higher values of p_0 in the corresponding values of τ, α, and c due to the smaller shape parameter τ.

6. Evaluation of the Estimation Methods for the UL Distribution: Bias, MSE, and Performance Analysis

Examining Tables 3, 4 and 5 below exhibit distinct patterns in the behavior of different estimation methods for the UL distribution with varying values of τ and sample size n. As τ increases, most of the methods show a stable decrease in bias and mean square error (MSE). This is because a larger τ reduces variability in the data, making parameter estimation more accurate.

For lower values of τ, the maximum likelihood estimate (MLE), the maximum product space (MPS), and the right-tailed Anderson-Darling estimate (RTADE) consistently outperform others in terms of lower bias and MSE. Among these, RTADE achieved the smallest bias and MSE, demonstrating its robustness and reliability for small sample sizes.

Table 3. Bias and MSE For various sample sizes and non-Bayesian methods

Method	$n = 50$		$n = 100$		$n = 150$		$n = 200$	
	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
				$\tau = 0.10$				
ML	0.000350	0.000010	0.000290	0.000000	0.000133	0.000003	0.000118	0.000002
MPS	0.000430	0.000010	0.000180	0.000000	0.000203	0.000003	0.000149	0.000002
LS	0.000140	0.000010	0.000260	0.000010	0.000083	0.000004	0.000058	0.000003
WLS	0.000130	0.000010	0.000260	0.000010	0.000095	0.000003	0.000072	0.000002
CVM	0.000210	0.000010	0.000300	0.000010	0.000108	0.000004	0.000077	0.000003
AD	0.000120	0.000010	0.000240	0.000000	0.000081	0.000003	0.000065	0.000002
RTAD	0.000090	0.000010	0.000210	0.000000	0.000056	0.000003	0.000045	0.000002
				$\tau = 0.11$				
ML	0.002470	0.000270	0.000910	0.000120	0.001037	0.000082	0.000197	0.000054
MPS	0.001880	0.000240	0.001620	0.000110	0.000822	0.000079	0.001254	0.000054
LS	0.001860	0.000370	0.000470	0.000160	0.001144	0.000112	0.000069	0.000071
WLS	0.001760	0.000330	0.000470	0.000140	0.001036	0.000099	0.000023	0.000063
CVM	0.002270	0.000370	0.000670	0.000160	0.001280	0.000112	0.000034	0.000071
AD	0.001710	0.000320	0.000410	0.000140	0.000975	0.000097	0.000081	0.000063
RTAD	0.001160	0.000280	0.000250	0.000130	0.000785	0.000089	0.000199	0.000058

Table 4. Bias and MSE For various sample sizes and Non-Bayesian Methods

Method	MLE	LS	WLSE	MPSE	CVME	ADE	RTADE	PERCE
				n = 25	$\tau = 0.01$			
Mean	0.0104000	0.0102819	0.0102528	0.0109448	0.0075858	0.0102428	0.0101652	0.0176710
MSE	0.0000050	0.0000065	0.0000059	0.0000063	0.0000137	0.0000054	0.0000049	0.0000831
RMSE	0.0022274	0.0025402	0.0024254	0.0025180	0.0036960	0.0023334	0.0022066	0.0091161
Bias	0.0004014	0.0002819	0.0002528	0.0009448	0.0024142	0.0002428	0.0001652	0.0076710
				n =100	$\tau = 0.01$			
Mean	0.0100992	0.0100746	0.0100737	0.0102915	0.0071778	0.0100665	0.0100388	0.0188545
MSE	0.0000010	0.0000014	0.0000012	0.0000012	0.0000091	0.0000012	0.0000011	0.0000852
RMSE	0.0010192	0.0011719	0.0011102	0.0010781	0.0030225	0.0010995	0.0010457	0.0092324
Bias	0.0000992	0.0000746	0.0000737	0.0002915	0.0028222	0.0000665	0.0000388	0.0088545
				n = 150	$\tau = 0.01$			
Mean	0.0100619	0.0100412	0.0100401	0.0102016	0.0071520	0.0100355	0.0100242	0.0190746
MSE	0.0000007	0.0000009	0.0000008	0.0000008	0.0000089	0.0000008	0.0000007	0.0000872
RMSE	0.0008358	0.0009568	0.0009091	0.0008722	0.0029774	0.0009028	0.0008615	0.0093370
Bias	0.0000619	0.0000412	0.0000401	0.0002016	0.0028480	0.0000355	0.0000242	0.0090746
				n= 200	$\tau = 0.01$			
Mean	0.0100470	0.0100325	0.0100338	0.0101606	0.0071253	0.0100291	0.0100190	0.0192215
MSE	0.0000005	0.0000007	0.0000006	0.0000006	0.0000088	0.0000006	0.0000006	0.0000887
RMSE	0.0007208	0.0008245	0.0007790	0.0007469	0.0029707	0.0007752	0.0007424	0.0094154
Bias	0.0000470	0.0000325	0.0000338	0.0001606	0.0028747	0.0000291	0.0000190	0.0092215

Table 5. Bias and MSE For various sample sizes and non-Bayesian methods

				n = 25	$\tau = 0.02$			
Method	**MLE**	**LS**	**WLSE**	**MPSE**	**CVME**	**ADE**	**RTADE**	**PERCE**
Mean	0.0207751	0.0205666	0.0205039	0.0218927	0.0152287	0.0204834	0.0203229	0.0354072
MSE	0.0000193	0.0000252	0.0000230	0.0000252	0.0000526	0.0000215	0.0000193	0.0003315
RMSE	0.0043950	0.0050234	0.0047984	0.0050215	0.0072544	0.0046397	0.0043945	0.0182072
Bias	0.0007751	0.0005666	0.0005039	0.0018927	0.0047713	0.0004834	0.0003229	0.0154072
				n = 100	$\tau = 0.02$			
Mean	0.0201166	0.0200344	0.0200381	0.0205006	0.0143044	0.0200266	0.0199974	0.0375323
MSE	0.0000040	0.0000052	0.0000047	0.0000044	0.0000371	0.0000046	0.0000042	0.0003336
RMSE	0.0020002	0.0022876	0.0021687	0.0021063	0.0060935	0.0021498	0.0020557	0.0182649
Bias	0.0001166	0.0000344	0.0000381	0.0005006	0.0056956	0.0000266	0.0000026	0.0175323
				n = 150	$\tau = 0.02$			
Mean	0.0201308	0.0201103	0.0201075	0.0204154	0.0142933	0.0200954	0.0200622	0.0381513
MSE	0.0000027	0.0000036	0.0000032	0.0000029	0.0000356	0.0000032	0.0000029	0.0003473
RMSE	0.0016375	0.0018978	0.0017912	0.0017109	0.0059645	0.0017778	0.0016926	0.0186370
Bias	0.0001308	0.0001103	0.0001075	0.0004154	0.0057067	0.0000954	0.0000622	0.0181513
				n = 200	$\tau = 0.02$			
Mean	0.0201073	0.0200681	0.0200688	0.0203306	0.0142506	0.0200605	0.0200465	0.0384543
MSE	0.0000021	0.0000027	0.0000024	0.0000023	0.0000353	0.0000024	0.0000022	0.0003548
RMSE	0.0014517	0.0016489	0.0015626	0.0015029	0.0059445	0.0015561	0.0014924	0.0188365
Bias	0.0001073	0.0000681	0.0000688	0.0003306	0.0057494	0.0000605	0.0000465	0.0184543

However, increasing n generally reduces bias and MSE across all estimation methods, as larger sample sizes provide more data for stable parameter estimation. Note, the rate of improvement varies. MLE and RTADE show the most consistent performance, with bias and MSE approaching zero faster than Least Squares (LS) and Cramér-von-Mises Estimation (CVME). Weighted Least Squares (WLSE) performs comparably to LS, but both methods tend to exhibit slightly higher bias and MSE for smaller sample sizes, making them less reliable under such conditions. In general, RTADE emerges the best method for estimating parameters of the UL distribution. It demonstrates the lowest bias and MSE across varying τ and n, maintaining accuracy and precision under less favorable conditions. MLE is a close second, showing strong performance and scalability with increasing sample sizes.

7. Key Risk Indicators

Probability-based distributions effectively summarize risk exposure by providing specific metrics, often referred to as Key Risk Indicators (KRIs). These KRIs are functions of a particular model and offer insights into a company's exposure to various risks. The main KRIs to be considered in the paper are Value-at-Risk (VaR), Tail Value-at-Risk (TVaR) or Conditional VaR (CVaR), Tail Variance (TV), Tail Mean-Variance (TMV),Tail Variance Premium (TVP) and the Mean Excess Loss (MEL) function. For detailed information on the theoretical aspect of these indicators, readers may consider perusing the following articles [3], [15], [6], others. Tables 6 and 7 show the results obtained for the various actuarial measures considered for different values of τ

Table 6. Simulation study of VaR, TVaR, TV, TMV, TVP, and MEL of the UL distribution

τ	q	VaR	CVaR	TV	TMV	TVP	MEL
	0.7	6.0585	11.8911	30.5241	2.5670	14.9435	5.5503
	0.75	6.9785	12.9507	31.0113	2.3946	16.0519	5.6010
0.2	0.8	8.1053	14.2741	31.7165	2.2220	17.4458	5.6664
	0.85	9.5595	16.0270	32.8681	2.0508	19.3138	5.7560
	0.9	11.6119	18.5962	35.1809	1.8918	22.1143	5.8923
	0.95	15.1279	23.2920	42.4210	1.8213	27.5341	6.1512

Table 7. Simulation study of VaR, TVaR, TV, TMV, TVP, and MEL of the UL distribution

τ	q	VaR	CVaR	TV	TMV	TVP	MEL
	0.7	2.1995	4.9902	5.2577	1.0536	5.5159	2.4232
	0.75	2.5561	5.4748	5.5230	1.0088	6.0271	2.4481
0.7	0.8	2.9963	6.0791	5.9366	0.9766	6.6727	2.4715
	0.85	3.5672	6.8711	6.6333	0.9654	7.5344	2.4914
	0.9	4.3720	8.0005	7.9566	0.9945	8.7962	2.5045
	0.95	5.7333	9.9336	11.1686	1.1243	11.0504	2.5013

Table 8. Simulation study of VaR, TVaR, TV, TMV, TVP, and MEL of the UL distribution

τ	Quantile	VaR	CVaR	TV	TMV	TVP	MEL
	0.7	2.0894	3.6518	2.0835	0.5705	3.8601	1.5623
	0.75	2.4026	3.9337	2.0215	0.5139	4.1359	1.5311
1	0.8	2.7754	4.2714	1.9538	0.4574	4.4668	1.4961
	0.85	3.2398	4.6959	1.8783	0.4000	4.8838	1.4561
	0.9	3.8674	5.2764	1.7901	0.3393	5.4555	1.4090
	0.95	4.8831	6.2306	1.6753	0.2689	6.3981	1.3475

The major actuarial risk measures produced by the UL distribution, computed at varying quantiles (q) and parameters (τ) within tables 6, 7 and 8 are evidenced from the tables. In table 6, with $\tau = 0.2$, VaR is greatly amplified with increasing q, indicating greater thresholds for extremely likely losses at higher confidence levels. CVaR grows from q, indicating increasing expected losses even after VaR. This shows a quite heavy tail situation for $\tau = 0.2$ TV has little increase with the change in q, suggesting that greater variability in the tail would obtain. It is however slower than average (CVaR). TMV grows smaller with increasing q, implying that tail variability grows lesser relative to CVaR. This visible normalization is important when analyzing tail stability. Increased premiums for extreme risks are shown by the TVP integrating TV into CVaR. It steepens with q, running from 14.9435 to 27.5341. Consistently, MEL increases steadily and mildly with q, showing a proportional increase in excess losses above the VaR threshold.

Table 7 thus presents the actuarial measures for $\tau = 0.7$. VaR is smaller under $\tau = 0.2$ because the distribution with $\tau = 0.7$ has a thinner tail. CVaR moderately increased from 4.9902 to 9.9336 with q. This also confirms that $\tau = 0.7$ lowers the extremity in loss as compared to smaller τ. The TV increases in a moderate way meaning that the variability in the tail grows less sharply than that of CVaR. In TMV, there is also a trend like in table 6 where proportional reduction in variability is observed in relation to mean. Again, as stated in TVP, the capacity of premiums is increased with respect to q, while integrating the smooth combination of TV and CVaR into it. MEL increases slower as compared to $\tau = 0.2$, which means it has relatively minor excess losses beyond VaR.

Table 8 is all the actuarial measures for $\tau = 1$. With $\tau = 1$, VaR would become lower compared to Tables 6 and 7 because it would not form thicker tails for UL distribution. In the same way, CVaR has an increasing trend from 3.6518 to 6.2306 with q. In fact, TV shows smaller tail variability for larger τ, thus making it smaller with respect to lower values of τ. TMV steadily decreases with q to reach at 0.2689 when $q = 0.95$. That implies greater stability within the tail. TVP Less premiums for extreme risks are applicable because they have thinner tails. MEL has Lowest against Tables 6 and 7, showing reduced average excess loss above VaR. Notice that smaller value of ($\tau = 0.2$) produces a heavy-tailed distribution, leading to higher VaR, CVaR, TV, and MEL. Thus, UL distribution is a good model for extreme loss. Also, higher value of ($\tau = 1$) leads to a thinner tail, thus lessening severity and variability of extreme losses, as exhibited in smaller actuarial measures. Thinner tails are more predictive loss in nature and applicable for general insurance or low-risk portfolios.

8. Applications

In this section, we demonstrate that the UL distribution is more flexible and provide alternative modeling tool compared to some existing distributions in the literature.

8.1. Application to Demographic Data

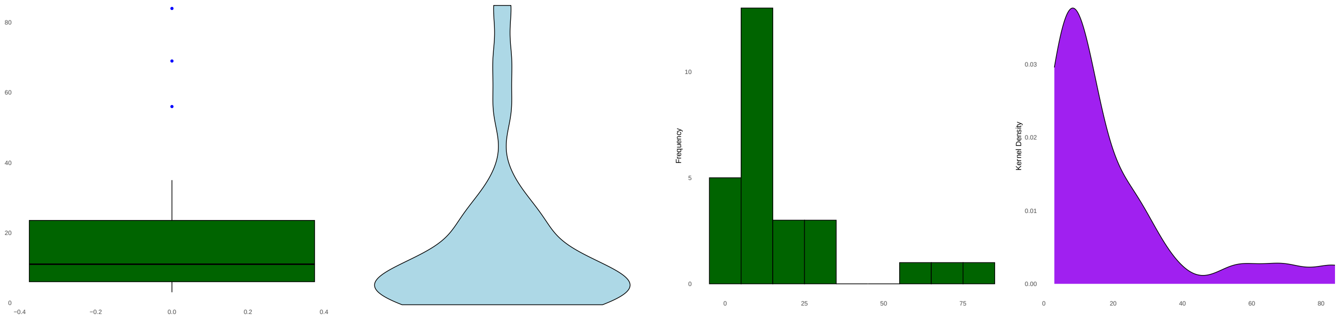
The first set of information is a description of the infant mortality rate per 1,000 live births for a few chosen nations in 2021, as reported by a https://data.world-bank.org/indicator/SP.DYN.IMRT.IN and studied by [9]. This data is as follows

56	10	22	3	69	6	7	11	4	4	19	13	7	27
12	3	4	11	84	27	25	6	35	14	11	12	6	

We study some basic statistics of the Mortality data for insight

Table 9. Summary Statistics for Infant Mortality Rate in 2021

n	$\bar{X}$	σ	Median	trimmed $\bar{X}$	MAD	Min	Max	Range	Skew	Kurtosis	δ_e
27	18.81	20.51	11	15.17	10.38	3	84	81	1.85	2.61	3.95



(a) Box plot **(b)** Violin plot **(c)** Histogram **(d)** Kernel density

Figure 11. Non-parametric graphs for the Mortality Rate Data

Next, we compare the goodness of fit of the UL distribution with the two-parameter Chris-Jerry distribution (TPCJ) by [9], Weibull distribution, gamma distribution, Lomax distribution, and Chris-Jerry (CJ) distribution by [18]. The fitness metrics considered are the log-likelihood (LL), the Akaike information criterion (AIC), the corrected AIC (CAIC), the Bayesian information criterion (BIC), the Hannan–Quinn information criterion (HQIC), Anderson Darling (AD) and Cramér-von-Mises (CVM) statistics. The model with the lowest values of these metrics is chosen as the best performer.

Table 10. Model Performance Indicators using World Infant Mortality Rate per 1000 Live Birth Data

Dist	LL	AIC	CAIC	BIC	HQIC	W^*	A^*	K-S	p-value	scale	shape
UL	-106.24	214.47	214.63	215.77	214.86	0.12	0.80	0.15	0.6006	0.05	-
TPCJ	-106.16	216.31	216.81	218.9	217.08	0.11	0.75	0.16	0.5345	399.51	0.06
Weibull	-106.11	231.36	231.86	233.95	232.13	0.13	0.82	0.32	0.0084	0.90	8.90
Gamma	-105.76	217.90	218.40	220.49	218.67	0.13	0.82	0.18	0.3436	1.80	9.74
LOMAX	-106.17	216.33	216.83	218.92	217.10	0.11	0.71	0.16	0.5158	232.81	13.37
CJ	-112.39	226.77	226.93	228.07	227.16	0.17	1.10	0.28	0.0260	0.15	-

The UL distribution was fitted to the datasets using the Maximum likelihood estimation (MLE) method and compared against five competing distributions in Table 10. The comparative results, including the Akaike Information Criterion (AIC), Corrected AIC (CAIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), W, A, and Kolmogorov-Smirnov (K-S) statistics along with their corresponding p-values, are detailed in these tables. The analysis reveals that the UL distribution consistently achieves the lowest AIC, CAIC, BIC, and HQIC values among the fitted models, establishing it as the best-fitting model for the datasets.

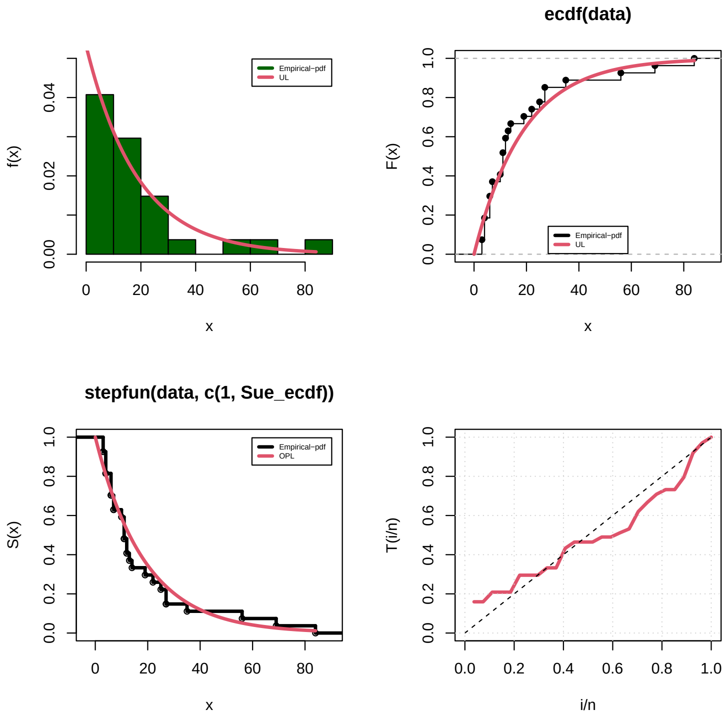


Figure 12. Density, CDF, Survival and TTT plots for the Mortality Rate Data

8.2. Application to Communication Data

To demonstrate the flexibility of the UL distribution, a WhatsApp business conversation time between two brothers, Kelechi Onyekwere and Chika Onyekwere from February 9, 2024 until July 30, 2024 was fitted to the model. The data is as shown below

Business conversation time (seconds)

31	32	40	38	33	183	247	10	4	19	7	14	144	5	25	208	22
50	51	49	32	84	12	34	66	72	56	15	10	37	94	52	30	

We intend to describe the data using RBEAST by providing plots of the data, so as to enable us detect change points and anomalies in the conversation time and show why UL distribution is best for analyzing it. The graphs in Figures 13 and 14 show that the conversation time data, characterized by irregular patterns with abrupt changes and extreme values, reflects variability typical of real-life lifetime or failure time data. The UL distribution, with its ability to model diverse failure rates and accommodate heavy-tailed behavior, is well-suited for the data. Its flexibility in capturing shifts and extreme events aligns with the trends and anomalies observed in the graph, hence, making it an optimal choice for analyzing and interpreting the underlying process.

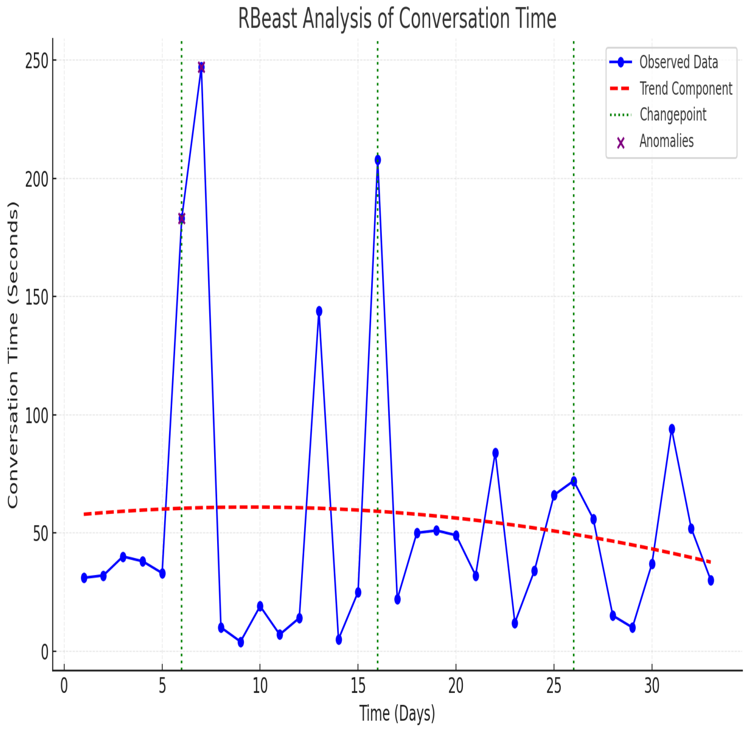


Figure 13. RBeast analysis of the conversation time

Table 11. Goodness-of-Fit Comparison for UL Distribution Using Various Models

Dist.	-LL	AIC	CAIC	BIC	HQIC	W	A	K-S	Pvalue	Scale	Shape
UL	165.080	332.160	332.280	333.650	332.660	0.090	0.570	0.120	0.760	0.020	-
XRama	186.780	375.550	375.680	377.050	376.060	0.090	0.580	0.280	0.010	0.070	-
Weibull	189.430	382.860	383.260	385.860	383.870	0.050	0.310	0.560	0.000	0.420	9.140
Gamma	194.810	393.610	394.010	396.600	394.620	0.090	0.610	0.180	0.230	4.080	9.860
Shanker	167.860	337.710	337.840	339.210	338.210	0.090	0.570	0.180	0.250	0.040	-
Lindley	167.530	337.070	337.200	338.570	337.570	0.090	0.610	0.170	0.270	0.040	-
CJ	174.560	351.120	351.250	352.620	351.620	0.110	0.740	0.240	0.050	0.050	-
Ishita	176.260	354.520	354.650	356.020	355.020	0.090	0.570	0.230	0.060	0.050	-
Akash	176.140	354.290	354.420	355.780	354.790	0.090	0.590	0.230	0.060	0.050	-
Pranav	186.830	375.660	375.790	377.160	376.160	0.090	0.570	0.280	0.010	0.070	-
Rama	186.800	375.610	375.740	377.100	376.110	0.090	0.580	0.280	0.010	0.070	-
Rayleigh	182.640	367.280	367.410	368.770	367.780	0.200	1.240	0.380	0.000	56.430	-

9. Final Comments

The paper explores the updated Lindley (UL) distribution, a mixture of exponential and gamma distributions, which offers enhanced flexibility in modeling lifetime data. It presents the distribution's probability density function, cumulative distribution function, survival and hazard functions, along with key statistical properties such as moments, skewness, kurtosis, and tail behavior. Methods for parameter estimation, including non-Bayesian approaches, are evaluated for bias and mean squared error across varying scenarios.

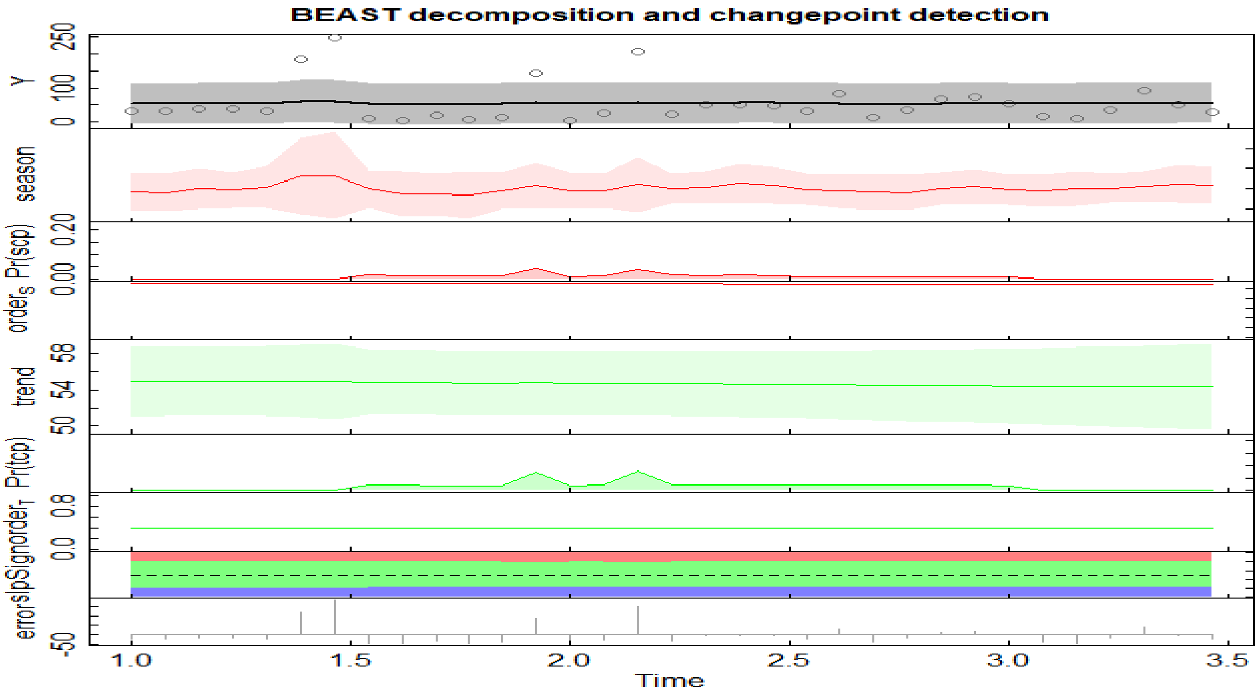


Figure 14. BEAST decomposition and change point detection

In addition, applications of the UL distribution in acceptance sampling, actuarial risk measures, and reliability analysis are discussed, demonstrating its utility in various fields. Graphical analyses and simulation studies further illustrate its behavior, robustness, and practical relevance.

Conflict of Interest

The authors declare there is no existing conflict of interest.

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