
Research article

Theory on a new bivariate trigonometric Gaussian distribution

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ABSTRACT

The main contribution of this article is the introduction of a new bivariate distribution, referred to as the bivariate cosine Gaussian distribution. We develop its theoretical framework, deriving the marginal and conditional distributions, and analyzing the independence properties of the components of the associated random vector. A graphical study is provided to illustrate the behavior of various types of probability density functions, and a simulation procedure for generating samples from the distribution is also described. Together, these results establish the theoretical foundations for potential applications of the bivariate cosine Gaussian distribution in two-dimensional modeling.

1. Introduction

The study of distributions in higher dimensions plays a central role in statistics, probability theory, and their numerous applications. In particular, bivariate distributions provide a natural framework for modeling dependence structures and interactions between random variables. Classical approaches often rely on Gaussian-type distributions due to their analytical tractability and wide applicability. However, the search for new families of distributions that retain Gaussian-like distribution properties while offering additional flexibility remains an active area of research. See, for example, [5, 2, 4, 6].

The purpose of this article is to contribute to this line of research by introducing the bivariate cosine Gaussian (BCG) distribution. This distribution is constructed by incorporating a trigonometric component

into the standard bivariate independent Gaussian distribution framework, thereby enriching its structure while preserving fundamental features such as symmetry and integrability. Particular attention is given to the behavior of the associated probability density function. Theoretical results on its marginal and conditional distributions are established, together with an analysis of independence properties. Numerical illustrations and a simulation procedure are also provided, highlighting the flexibility and potential applications of the BCG distribution in two-dimensional modeling problems.

The remainder of the article is organized as follows: Section 2 introduces the BCG distribution, with a focus on the associated probability density function. The corresponding marginal and conditional distributions are studied in Section 3. Section 4 is devoted to the analysis of independence. A simulation procedure is presented in Section 5. Finally, concluding remarks are given in Section 6.

2. A new bivariate trigonometric Gaussian distribution

We begin by presenting a specific probability density function that extends the standard bivariate independent Gaussian framework through the inclusion of a trigonometric component.

Proposition 2.1. *Let $\theta \in \mathbb{R}$. Then the following function is a valid bivariate probability density function:*

$$f(x, y) = C^{-1} e^{-x^2/2 - y^2/2} \cos^2(\theta xy), \quad (x, y) \in \mathbb{R}^2,$$

where

$$C = \pi \left(1 + \frac{1}{\sqrt{1 + 4\theta^2}} \right).$$

Proof. First of all, it is clear that, for any $(x, y) \in \mathbb{R}^2$, $f(x, y) \geq 0$. We now complete the proof by establishing the following normalization condition: $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$. We can write

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = C^{-1} I, \quad (2.1)$$

where

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2/2 - y^2/2} \cos^2(\theta xy) dx dy.$$

Using the following standard trigonometric formula: $\cos^2(a) = [1 + \cos(2a)]/2$ for $a \in \mathbb{R}$, we have

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2/2 - y^2/2} \cos^2(\theta xy) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2/2 - y^2/2} \frac{1 + \cos(2\theta xy)}{2} dx dy \\ &= \frac{1}{2} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2/2 - y^2/2} dx dy + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2/2 - y^2/2} \cos(2\theta xy) dx dy \right) \\ &= \frac{1}{2} (I_1 + I_2), \end{aligned} \quad (2.2)$$

where

$$I_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2/2 - y^2/2} dx dy$$

and

$$I_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2/2-y^2/2} \cos(2\theta xy) dx dy.$$

Let us first determine I_1 . Using the following classical integral formula: $\int_{-\infty}^{\infty} e^{-px^2} dx = \sqrt{\pi/p}$ (see [3, Formula 3.4613]) with $p = 1/2$, we get

$$I_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2/2-y^2/2} dx dy = \left(\int_{-\infty}^{\infty} e^{-x^2/2} dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2/2} dy \right) = (\sqrt{2\pi})^2 = 2\pi. \quad (2.3)$$

Let us now focus on I_2 . By the Fubini-Tonelli integral theorem, we can write

$$\begin{aligned} I_2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2/2-y^2/2} \cos(2\theta xy) dx dy \\ &= \int_{-\infty}^{\infty} e^{-y^2/2} \left(\int_{-\infty}^{\infty} e^{-x^2/2} \cos(2\theta xy) dx \right) dy. \end{aligned}$$

Using the following integral formula: $\int_{-\infty}^{\infty} e^{-px^2} \cos(qx) dx = \sqrt{\pi/p} e^{-q^2/(4p)}$ (see [3, Formula 3.8964]) with $p = 1/2$ and $q = 2\theta y$, we obtain

$$\int_{-\infty}^{\infty} e^{-x^2/2} \cos(2\theta xy) dx = \sqrt{\frac{\pi}{1/2}} e^{-(2\theta y)^2/(4 \times 1/2)} = \sqrt{2\pi} e^{-2\theta^2 y^2}.$$

This and again the following integral formula: $\int_{-\infty}^{\infty} e^{-px^2} dx = \sqrt{\pi/p}$ with a suitable p yield

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{-y^2/2} \left(\int_{-\infty}^{\infty} e^{-x^2/2} \cos(2\theta xy) dx \right) dy \\ &= \int_{-\infty}^{\infty} e^{-y^2/2} \sqrt{2\pi} e^{-2\theta^2 y^2} dy = \sqrt{2\pi} \int_{-\infty}^{\infty} e^{-(1/2+2\theta^2)y^2} dy \\ &= \sqrt{2\pi} \sqrt{\frac{\pi}{1/2+2\theta^2}} = \frac{2\pi}{\sqrt{1+4\theta^2}}. \end{aligned}$$

We thus have

$$I_2 = \frac{2\pi}{\sqrt{1+4\theta^2}}. \quad (2.4)$$

Finally, based on Equations (2.2), (2.3) and (2.4), we derive

$$I = \frac{1}{2} \left(2\pi + \frac{2\pi}{\sqrt{1+4\theta^2}} \right) = \pi \left(1 + \frac{1}{\sqrt{1+4\theta^2}} \right) = C.$$

It follows from this and Equation (2.1) that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = C^{-1} C = 1.$$

The function $f(x, y)$ is thus a valid bivariate probability density function. The proof is concluded. $\square$

We emphasize that $\theta \in \mathbb{R}$, without restriction. Based on this proposition, we define the BCG distribution with parameter θ as the distribution whose probability density function is given by

$$f(x, y) = C^{-1} e^{-x^2/2 - y^2/2} \cos^2(\theta xy), \quad (x, y) \in \mathbb{R}^2,$$

where

$$C = \pi \left(1 + \frac{1}{\sqrt{1 + 4\theta^2}} \right).$$

We begin the study of this distribution by examining some properties of $f(x, y)$. First, since both the square and cosine functions are even, it is even in each coordinate, i.e., for any $(x, y) \in \mathbb{R}^2$,

$$f(-x, y) = f(x, -y) = f(x, y).$$

Moreover, it is clearly symmetric, i.e., for any $(x, y) \in \mathbb{R}^2$,

$$f(x, y) = f(y, x).$$

When $\theta = 0$, since $\cos(0) = 1$ and $C = 2\pi$, we have

$$f(x, y) = \frac{1}{2\pi} e^{-x^2/2 - y^2/2}.$$

The BCG distribution thus reduces to the standard bivariate independent Gaussian distribution.

Since, for any $(x, y) \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$, $0 \leq \cos^2(\theta xy) \leq 1$, we have the following inequality:

$$f(x, y) \leq C^{-1} e^{-x^2/2 - y^2/2},$$

ensuring diverse integral convergence properties. This is crucial to the existence of various moments associated with the BCG distribution.

To provide a visual representation, Figures 1, 2, 3 and 4 display perspective plots of $f(x, y)$ for $\theta = 1$, $\theta = 3$, $\theta = 5$ and $\theta = 11$, respectively.

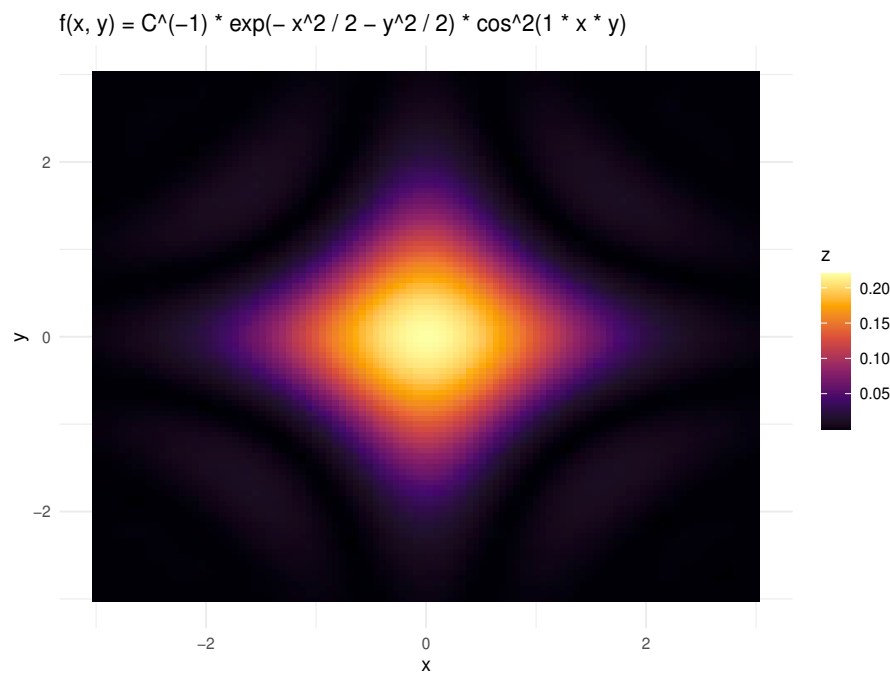


Figure 1. Perspective plot of the probability density function of the BCG distribution for $\theta = 1$.

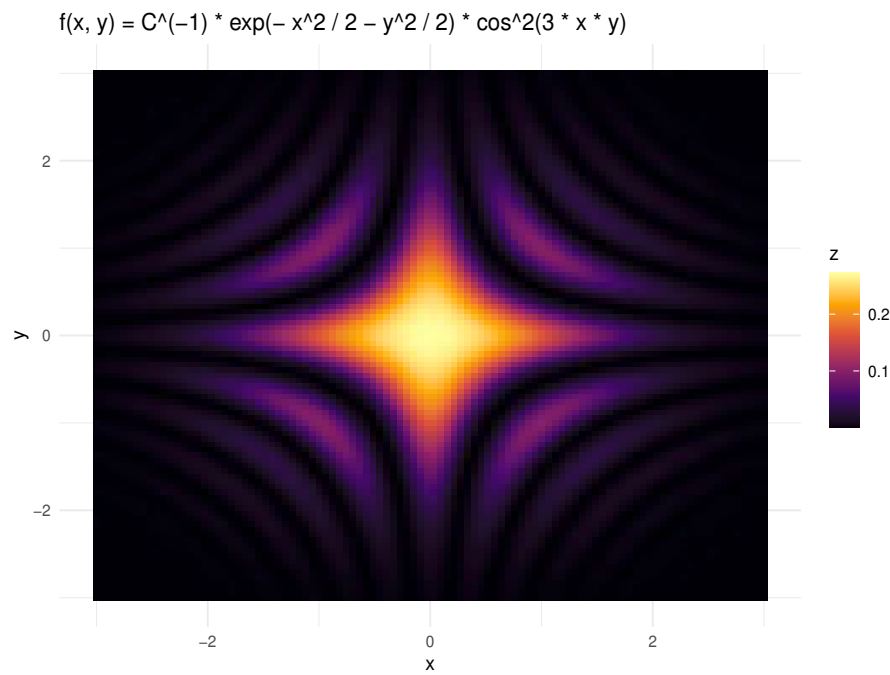


Figure 2. Perspective plot of the probability density function of the BCG distribution for $\theta = 3$.

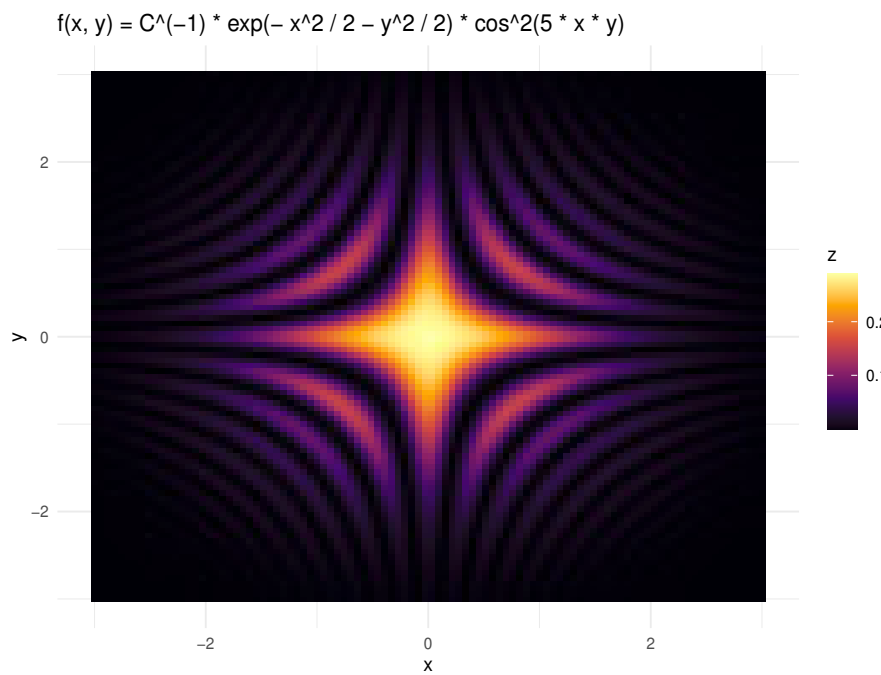


Figure 3. Perspective plot of the probability density function of the BCG distribution for $\theta = 5$.

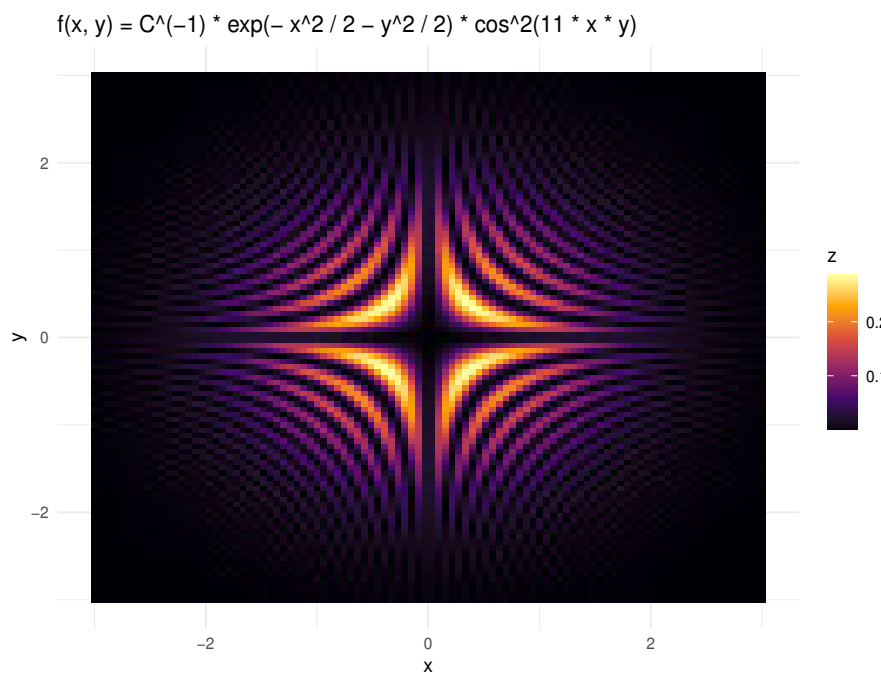


Figure 4. Perspective plot of the probability density function of the BCG distribution for $\theta = 11$.

From these figures, we observe that the probability density function of the BCG distribution may be centered around the origin, or may exhibit more complex oscillatory patterns depending on the value of θ . In particular, star-like and circular shapes can be distinguished. These are very different from the usual elliptical contours associated with a classical bivariate Gaussian distribution.

This probability density function could be useful for modeling the interactions of two variables with periodic effects, such as coupled oscillators with Gaussian noise in physics, cyclic dependencies in financial returns or features that are nonlinearly dependent with a hidden periodic structure in machine learning.

In the next section, we further develop the theory by examining the corresponding marginal and conditional distributions.

3. Marginal and conditional distributions

3.1. Marginal distributions

The proposition below establishes the marginal distributions associated with the BCG distribution.

Proposition 3.1. *Let $\theta \in \mathbb{R}$. Then the marginal distributions associated with the BCG distribution with parameter θ are defined by the two following probability density functions:*

- with respect to x :

$$g(x) = D^{-1} e^{-x^2/2} \left(1 + e^{-2\theta^2 x^2}\right), \quad x \in \mathbb{R},$$

where

$$D = \sqrt{2\pi} \left(1 + \frac{1}{\sqrt{1 + 4\theta^2}}\right),$$

- with respect to y :

$$h(y) = D^{-1} e^{-y^2/2} \left(1 + e^{-2\theta^2 y^2}\right) = g(y), \quad y \in \mathbb{R}.$$

Proof.

- By definition, the probability density function associated with the marginal distribution with respect to x is given by

$$g(x) = \int_{-\infty}^{\infty} f(x, y) dy, \quad x \in \mathbb{R}.$$

Using the following standard trigonometric formula: $\cos^2(a) = [1 + \cos(2a)]/2$ for $a \in \mathbb{R}$, we have

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} C^{-1} e^{-x^2/2 - y^2/2} \cos^2(\theta xy) dy \\ &= C^{-1} e^{-x^2/2} \int_{-\infty}^{\infty} e^{-y^2/2} \frac{1 + \cos(2\theta xy)}{2} dy \\ &= C^{-1} \frac{1}{2} \left(e^{-x^2/2} \int_{-\infty}^{\infty} e^{-y^2/2} dy + e^{-x^2/2} \int_{-\infty}^{\infty} e^{-y^2/2} \cos(2\theta xy) dy \right) \\ &= C^{-1} \frac{1}{2} (g_1(x) + g_2(x)), \end{aligned} \tag{3.1}$$

where

$$g_1(x) = e^{-x^2/2} \int_{-\infty}^{\infty} e^{-y^2/2} dy$$

and

$$g_2(x) = e^{-x^2/2} \int_{-\infty}^{\infty} e^{-y^2/2} \cos(2\theta xy) dy.$$

Let us first determine $g_1(x)$. Using the following integral formula: $\int_{-\infty}^{\infty} e^{-py^2} dy = \sqrt{\pi/p}$ with $p = 1/2$, we get

$$g_1(x) = e^{-x^2/2} \int_{-\infty}^{\infty} e^{-y^2/2} dy = \sqrt{2\pi} e^{-x^2/2}. \quad (3.2)$$

Let us now focus on $g_2(x)$. Using the following integral formula: $\int_{-\infty}^{\infty} e^{-py^2} \cos(qy) dy = \sqrt{\pi/p} e^{-q^2/(4p)}$ with $p = 1/2$ and $q = 2\theta x$, we obtain

$$\int_{-\infty}^{\infty} e^{-y^2/2} \cos(2\theta xy) dy = \sqrt{\frac{\pi}{1/2}} e^{-(2\theta x)^2/(4 \times 1/2)} = \sqrt{2\pi} e^{-2\theta^2 x^2}.$$

We thus get

$$g_2(x) = e^{-x^2/2} \int_{-\infty}^{\infty} e^{-y^2/2} \cos(2\theta xy) dy = \sqrt{2\pi} e^{-2\theta^2 x^2} e^{-x^2/2}. \quad (3.3)$$

Finally, based on Equations (3.1), (3.2) and (3.3), we have

$$\begin{aligned} g(x) &= C^{-1} \frac{1}{2} \left(\sqrt{2\pi} e^{-x^2/2} + \sqrt{2\pi} e^{-2\theta^2 x^2} e^{-x^2/2} \right) \\ &= C^{-1} \sqrt{\frac{\pi}{2}} e^{-x^2/2} (1 + e^{-2\theta^2 x^2}) \\ &= D^{-1} e^{-x^2/2} (1 + e^{-2\theta^2 x^2}). \end{aligned}$$

- By definition, the probability density function associated with the marginal distribution with respect to y is given by

$$h(y) = \int_{-\infty}^{\infty} f(x, y) dx, \quad y \in \mathbb{R}.$$

Since, for any $(x, y) \in \mathbb{R}^2$, $f(x, y) = f(y, x)$, we have $h(y) = g(y)$, i.e.,

$$h(y) = D^{-1} e^{-y^2/2} (1 + e^{-2\theta^2 y^2}), \quad y \in \mathbb{R}.$$

This completes the proof. □

Before presenting further analytical results, Figure 5 displays $g(x)$ for $\theta = 0.5$, $\theta = 1$, and $\theta = 2$.

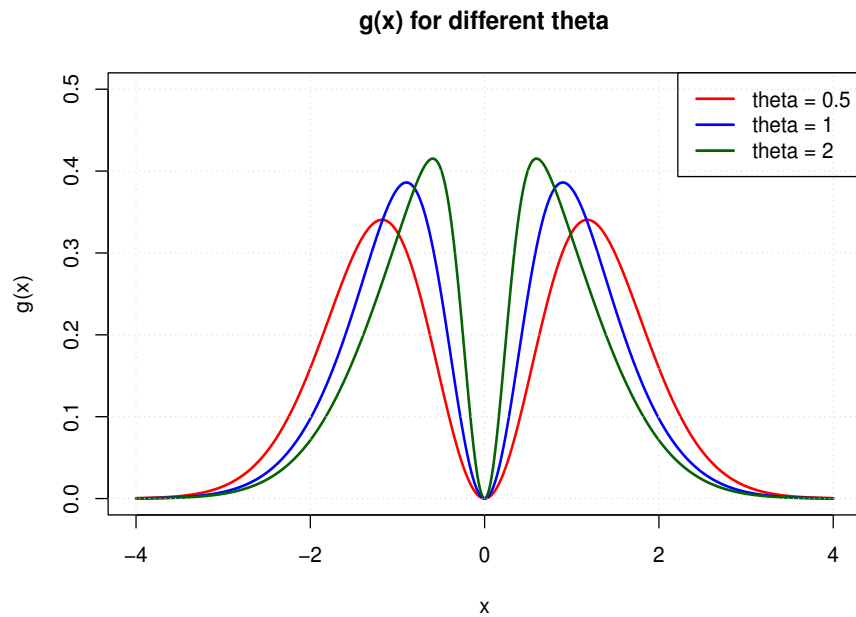


Figure 5. Standard plots of the marginal probability density function of the BCG distribution with respect to x for $\theta = 0.5$, $\theta = 1$ and $\theta = 2$.

We observe that $g(x)$ is symmetric and bimodal, except in the special case $\theta = 0$, where it is reduced to a unimodal distribution, i.e., the standard Gaussian distribution. This behavior can be explained analytically, since $g(x)$ has a mixture Gaussian probability density function representation, as detailed below. For convenience, let $s = 1 + 4\theta^2$ and $\sigma_2^2 = 1/s$. Then, for any $x \in \mathbb{R}$, we can write

$$g(x) = \alpha \varphi_1(x) + (1 - \alpha) \varphi_{\sigma_2}(x),$$

where

$$\alpha = \frac{\sqrt{s}}{\sqrt{s} + 1}$$

and $\varphi_{\sigma}(x)$ is the probability density function associated with the Gaussian distribution with mean 0 and variance σ^2 , i.e.,

$$\varphi_{\sigma}(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-x^2/(2\sigma^2)}, \quad x \in \mathbb{R}.$$

From this representation, the shapes observed in Figure 5 can be naturally explained. Moreover, let (X, Y) denote a random vector following the BCG distribution with parameter θ . Using the mixture Gaussian probability density function representation, we can easily derive the mean and variance of X as

$$\mathbb{E}(X) = 0$$

and

$$\mathbb{V}(X) = \alpha \times 1 + (1 - \alpha) \times \frac{1}{s} = \frac{1 + (1 + 4\theta^2)^{-3/2}}{1 + (1 + 4\theta^2)^{-1/2}},$$

with $\mathbb{V}(X) = 1$ only when $\theta = 0$. The same expressions hold for $\mathbb{E}(Y)$ and $\mathbb{V}(Y)$, respectively.

These results can be used to better understand the impact of the parameter θ on the BCG distribution. In particular, the variances of X and Y decrease as $|\theta|$ increases, reflecting the concentration of mass around the origin induced by the cosine term. This behavior highlights how the BCG distribution departs from the classical bivariate Gaussian distribution. In particular, it illustrates its ability to capture stronger localization and oscillatory effects. Such insights are valuable both for theoretical analysis and for guiding potential applications in multivariate modeling where additional flexibility beyond the Gaussian distribution framework is required.

3.2. Conditional distributions

The proposition below determines the conditional distributions associated with the BCG distribution.

Proposition 3.2. *Let $\theta \in \mathbb{R}$. Then the conditional distributions associated with the BCG distribution with parameter θ are defined by the two following conditional probability density functions:*

- with respect to y given $x \in \mathbb{R}$:

$$k(y | x) = \sqrt{\frac{2}{\pi}} \frac{e^{-y^2/2} \cos^2(\theta xy)}{1 + e^{-2\theta^2 x^2}}, \quad y \in \mathbb{R},$$

- with respect to x given $y \in \mathbb{R}$:

$$\ell(x | y) = \sqrt{\frac{2}{\pi}} \frac{e^{-x^2/2} \cos^2(\theta xy)}{1 + e^{-2\theta^2 y^2}}, \quad x \in \mathbb{R}.$$

Proof.

- By the definition of a conditional distribution with respect to y given $x \in \mathbb{R}$, we have

$$k(y | x) = \frac{f(x, y)}{g(x)}, \quad y \in \mathbb{R}.$$

We therefore derive

$$k(y | x) = \frac{C^{-1} e^{-x^2/2 - y^2/2} \cos^2(\theta xy)}{D^{-1} e^{-x^2/2} (1 + e^{-2\theta^2 x^2})} = \sqrt{\frac{2}{\pi}} \frac{e^{-y^2/2} \cos^2(\theta xy)}{1 + e^{-2\theta^2 x^2}}.$$

- Similarly, by the definition of a conditional distribution with respect to x given $y \in \mathbb{R}$, we have

$$\ell(x | y) = \frac{f(x, y)}{h(y)}, \quad x \in \mathbb{R}.$$

We therefore derive

$$\ell(x | y) = \frac{C^{-1} e^{-x^2/2 - y^2/2} \cos^2(\theta xy)}{D^{-1} e^{-y^2/2} (1 + e^{-2\theta^2 y^2})} = \sqrt{\frac{2}{\pi}} \frac{e^{-x^2/2} \cos^2(\theta xy)}{1 + e^{-2\theta^2 y^2}}.$$

This completes the proof. $\square$

For a graphical illustration, Figure 6 displays $k(y | x)$ for $\theta = 0.5$, $\theta = 1$ and $\theta = 2$, and $x = 1$.

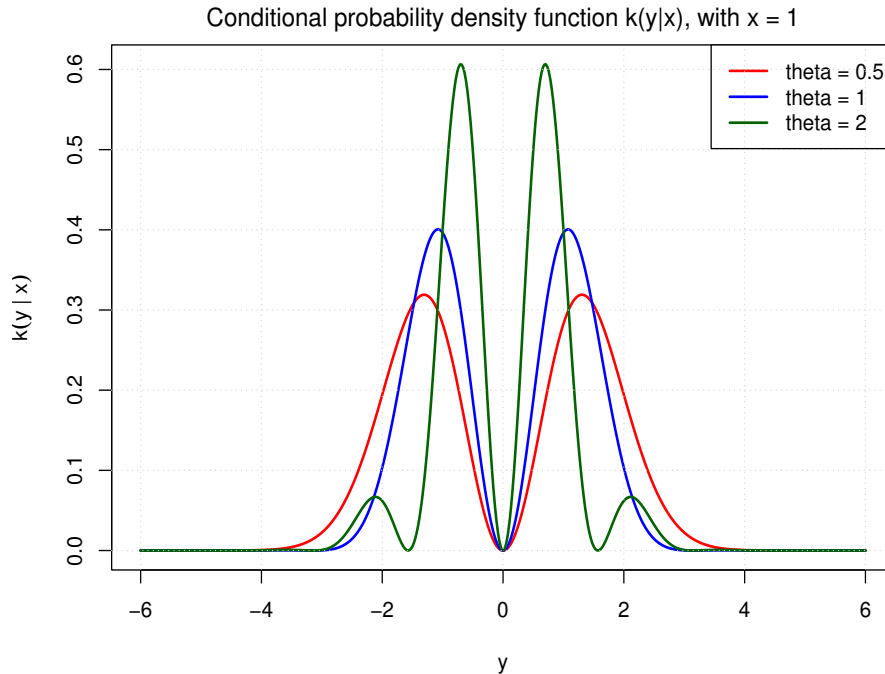


Figure 6. Standard plots of the conditional probability density function of the BCG distribution for $\theta = 0.5$, $\theta = 1$ and $\theta = 2$, and $x = 1$.

From this figure, we observe both the symmetry and the multimodal patterns induced by the cosine component.

Let (X, Y) be a random vector following the BCG distribution with parameter θ . Based on Proposition 3.2 and after some integral calculations (omitted here for brevity), we obtain

$$\mathbb{E}(Y | X = x) = 0, \quad \mathbb{E}(X | Y = y) = 0,$$

$$\mathbb{V}(Y | X = x) = \frac{1 + (1 - 4\theta^2 x^2)e^{-2\theta^2 x^2}}{1 + e^{-2\theta^2 x^2}}$$

and

$$\mathbb{V}(X | Y = y) = \frac{1 + (1 - 4\theta^2 y^2)e^{-2\theta^2 y^2}}{1 + e^{-2\theta^2 y^2}}.$$

These conditional means and variances can be used to show the non-Gaussian distribution nature of the BCG distribution. In particular, the conditional variances exhibit a nontrivial dependence on the conditioning variable. This dependence is modulated by both the parameter θ and the magnitude of x or y , leading to a reduction in conditional variability as $|x|$ or $|y|$ increases. These results emphasize the flexibility of the BCG distribution in capturing interaction effects between variables. They also provide a theoretical foundation for potential applications in areas such as dependence modeling, risk analysis, and spatial statistics.

The next section presents a classical independence analysis, which had only been outlined briefly in the preceding sections.

4. Independence analysis

The proposition below focuses on the basic stochastic dependence associated with the BCG distribution.

Proposition 4.1. *Let $\theta \in \mathbb{R}$, and (X, Y) be a random vector following the BCG distribution with parameter θ . Then X and Y are stochastically independent if and only if $\theta = 0$.*

Proof. By the definition of the stochastic independence, using the expressions of the corresponding probability density functions, X and Y are independent if and only if, for any $(x, y) \in \mathbb{R}^2$,

$$f(x, y) = g(x)h(y),$$

i.e.,

$$C^{-1}e^{-x^2/2-y^2/2}\cos^2(\theta xy) = D^{-1}e^{-x^2/2}\left(1 + e^{-2\theta^2 x^2}\right)D^{-1}e^{-y^2/2}\left(1 + e^{-2\theta^2 y^2}\right).$$

This is possible if and only if $\theta = 0$. The proof is completed. $\square$

The proposition below concerns the linear dependence associated with the BCG distribution.

Proposition 4.2. *Let $\theta \in \mathbb{R}$, and (X, Y) be a random vector following the BCG distribution with parameter θ . Then X and Y are linearly independent.*

Proof. By considering the covariance of X and Y defined by $\mathbb{C}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$, we aim to prove that $\mathbb{C}(X, Y) = 0$. Since, for any $(x, y) \in \mathbb{R}^2$,

$$f(-x, y) = f(x, y),$$

we have

$$\mathbb{E}(XY) = \mathbb{E}((-X)Y) = -\mathbb{E}(XY).$$

which implies that

$$\mathbb{E}(XY) = 0.$$

Moreover, since, for any $x \in \mathbb{R}$,

$$g(-x) = D^{-1}e^{-(-x)^2/2}\left(1 + e^{-2\theta^2(-x)^2}\right) = D^{-1}e^{-x^2/2}\left(1 + e^{-2\theta^2 x^2}\right) = g(x),$$

we have $\mathbb{E}(X) = 0$. We also have $\mathbb{E}(Y) = 0$. This implies that $\mathbb{C}(X, Y) = 0$. The proof is concluded. $\square$

For a random vector (X, Y) following the BCG distribution with parameter θ , the dependence between X and Y can be characterized as oscillatory and circular. To the best of our knowledge, no suitable dependence measure has yet been established for this type of relationship, and clarifying the exact role of θ in this dependence remains an open mathematical challenge.

A simple simulation procedure to generate samples of values from the BCG distribution is developed in the next section.

5. Simulation procedure

Since the BCG distribution does not admit a closed-form sampling method, we rely on the accept-reject algorithm with the standard bivariate independent Gaussian distribution as proposal. For more details on this method, see [1].

The procedure can be summarized in the following four steps:

Step 1. Recall that the probability density function of the BCG distribution with parameter θ is given by

$$f(x, y) = C^{-1} e^{-x^2/2 - y^2/2} \cos^2(\theta xy), \quad (x, y) \in \mathbb{R}^2,$$

where

$$C = \pi \left(1 + \frac{1}{\sqrt{1 + 4\theta^2}} \right).$$

Step 2. We choose the standard bivariate independent probability density function defined by

$$\phi(x, y) = \frac{1}{2\pi} e^{-x^2/2 - y^2/2}, \quad (x, y) \in \mathbb{R}^2,$$

as the proposal distribution. Notice that, for any $(x, y) \in \mathbb{R}^2$, we have

$$Cf(x, y) = 2\pi\phi(x, y) \cos^2(\theta xy) \leq 2\pi\phi(x, y).$$

Step 3. To generate a random value (x, y) from the BCG distribution, the process is as follows:

1. Generate a candidate (x^*, y^*) from the standard bivariate independent Gaussian distribution.
2. Generate a value u from the uniform distribution over $(0, 1)$.
3. Accept (x^*, y^*) if

$$u < \frac{Cf(x^*, y^*)}{2\pi\phi(x^*, y^*)} = \cos^2(\theta x^* y^*).$$

Otherwise, reject and repeat.

Step 4. Repeat Step 3 until the desired sample size n is obtained. The accepted points form a sample derived from the BCG distribution.

To provide a visual illustration of this simulation procedure, Figures 7, 8, 9 and 10 display scatter plots of samples of size $n = 500$ generated from the BCG distribution for $\theta = 0.5$, $\theta = 1$, $\theta = 2$ and $\theta = 5$, respectively.

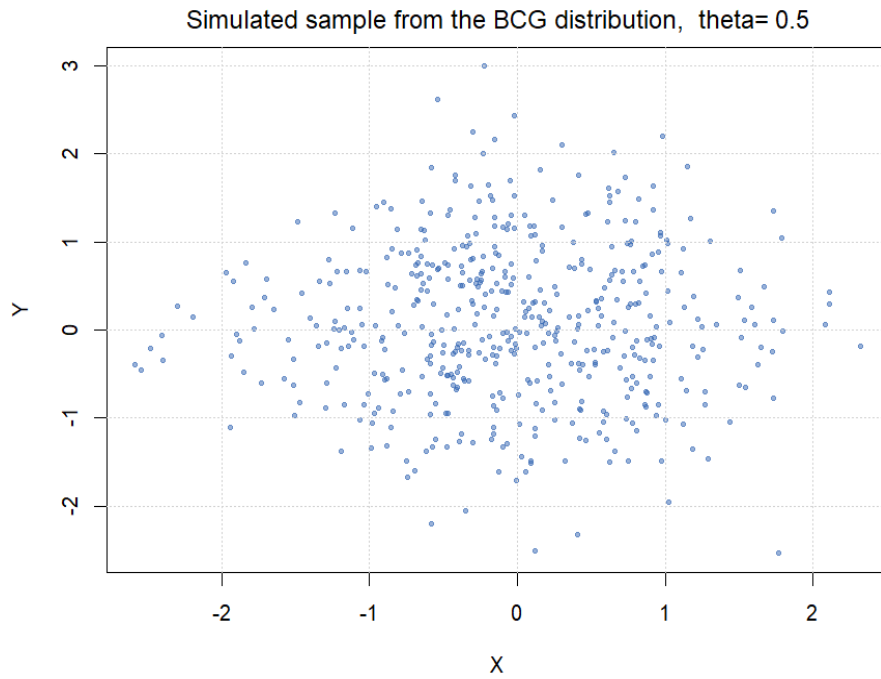


Figure 7. Scatter plot of a sample of size $n = 500$ generated from the BCG distribution for $\theta = 0.5$.

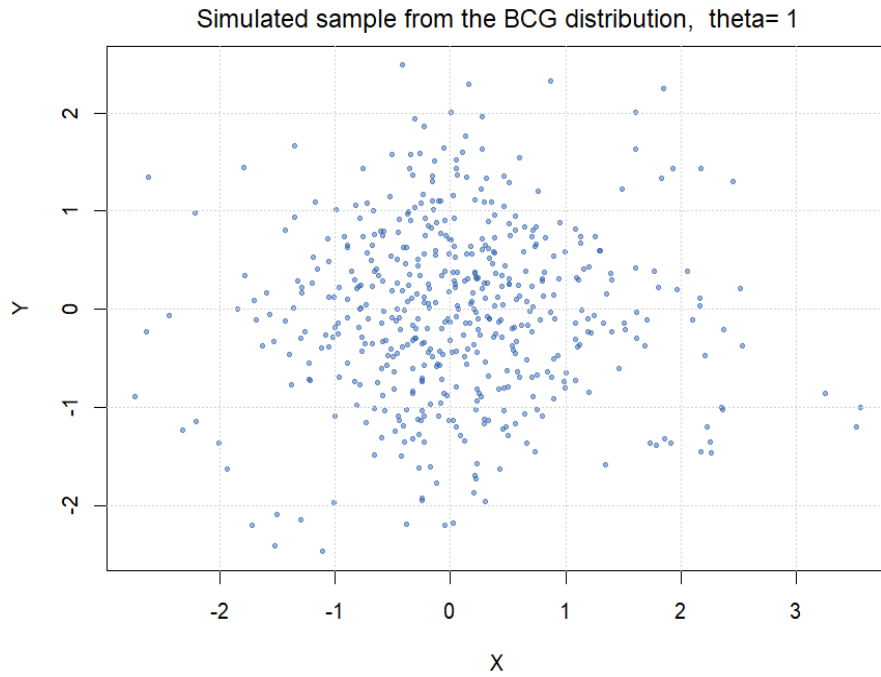


Figure 8. Scatter plot of a sample of size $n = 500$ generated from the BCG distribution for $\theta = 1$.

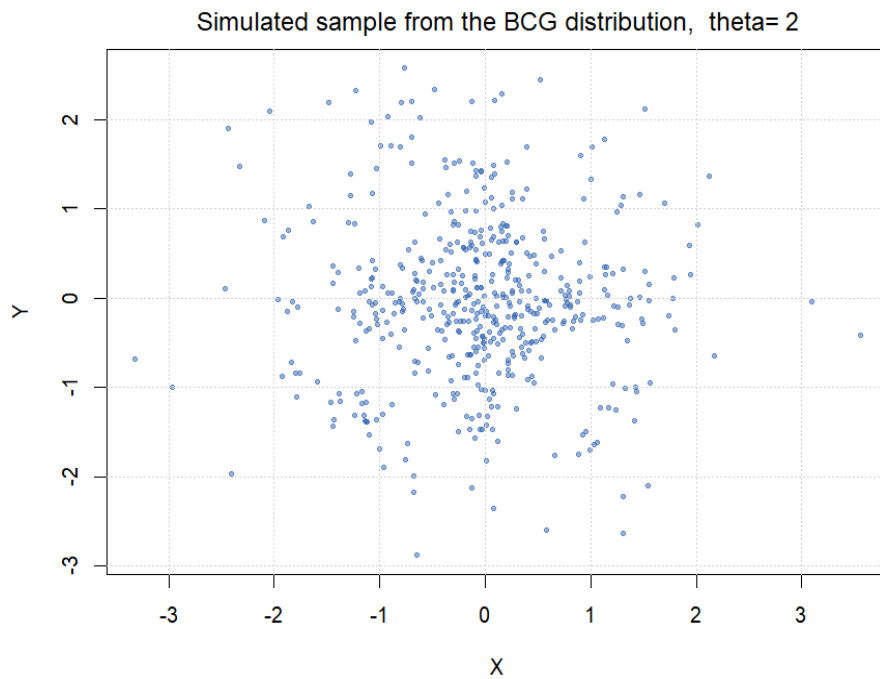


Figure 9. Scatter plot of a sample of size $n = 500$ generated from the BCG distribution for $\theta = 2$.

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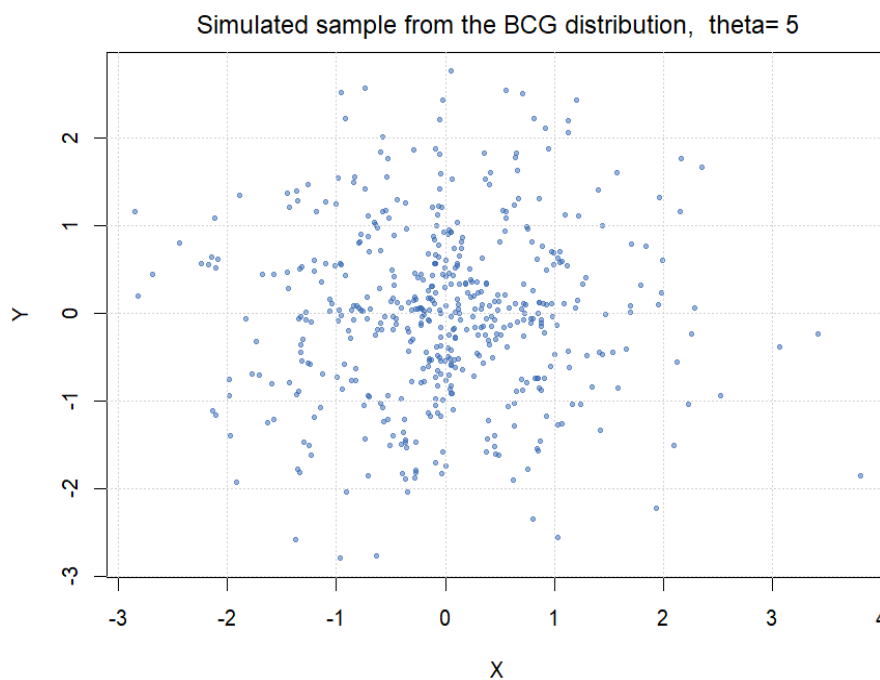


Figure 10. Scatter plot of a sample of size $n = 500$ generated from the BCG distribution for $\theta = 5$.

From these figures, we observe that the points tend to form star-like and circular structures that are visually consistent with the shapes of the probability density function of the BCG distribution. We refer to

Figures 1, 2, 3 and 4.

6. Conclusion

In conclusion, the BCG distribution provides a flexible cosine extension of the classical bivariate independent Gaussian distribution, combining tractable theoretical properties with a richer dependence structure. Future work could consider the complementary bivariate sine Gaussian (BSG) distribution, defined by the following probability density function:

$$f(x, y) = U^{-1} e^{-x^2/2 - y^2/2} \sin^2(\theta xy), \quad (x, y) \in \mathbb{R}^2,$$

where

$$U = \pi \left(1 - \frac{1}{\sqrt{1 + 4\theta^2}} \right).$$

In a more general way, we can think of studying the generalized trigonometric distribution defined by the following probability density function:

$$f(x, y) = W^{-1} e^{-x^2/2 - y^2/2} (1 - \eta \cos(\theta xy)), \quad (x, y) \in \mathbb{R}^2,$$

with $\eta \in [-1, 1]$, and W is such that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$. Focusing on the BCG distribution itself, future research could also explore higher-dimensional generalizations, parameter estimation methods, and practical applications in areas such as finance, environmental modeling, or machine learning, where capturing nontrivial interactions between variables is particularly important.

Conflict of interest

The author declares that there are no conflicts of interest regarding this work.

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