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Research article

On the Versatility of the Unit Logistic Exponential Distribution: Capturing Bathtub, Upside-Down Bathtub, and Monotonic Hazard Rates

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ABSTRACT

In this paper, a unit logistic exponential (ULE) distribution is proposed. The ULE distribution appears to be more flexible than some of the existing well-known univariate distributions defined on the unit interval due to its five failure rate shapes, i.e. bathtub and upside-down bathtub, decreasing, increasing and constant shapes. Some statistical properties of the ULE distribution have been discussed. The quantile regression analysis is conducted. A simulation study has been conducted to check the consistency of the adopted MLE procedure. Three real-life data sets have been used to compare the ULE distribution with some existing distributions and the ULE distribution performs best fit among all.

1. Introduction

The exponential distribution is the powerful and appropriate lifespan distribution. It is repeatedly used to estimate the queue time until an event: success or failure. For example, the exponential distribution may be appealed to estimate how long it takes hold of a customer to purchase anything in a store (success). Exponential distributions are usually used in the real world where events happen randomly and independently gradually. Modeling how long it takes for a call to be picked up at a call center, how long it takes for a vehicle to arrive at the call desk, and how long it takes for an electronic device to fail are some of the most common applications.

Probability distribution functions help to provide answers to many mathematical problems. Recently, many researchers discussed different extensions of exponential distribution. For instance, weighted exponential (WE), generalized exponential, Weibull and gamma distributions and their different generalized forms.

The logistic distribution is popular as a strong competitor to the normal distribution because it has a clear formula for the cumulative distribution function (cdf).

Recently, different extensions and generalizations of exponential and logistic distributions have been discussed. For instance, some of the are a double exponential [13], a Weibull exponential [29], a unit gamma [18], a Kumaraswamy exponential Weibull [11], a gamma exponentiated exponential-Weibull [21], an odd log-logistic modified Weibull [24], a single general exponential log-logic [26], an inverse logistic exponential [4], two parameter logistic exponential [5] , a unit Weibull [19], a logistic exponential [17], a top-order single logistic exponential [2], a unit exponential [7], an exponential odd logistic Weibull [23], a two parameter unit Kumarswami type [14] distributions. For more information, see [3, 1, 10, 6].

The cdf and probability density function (pdf) of the ULE distribution are defined as:

$$F_{ULE}(x,\alpha,\vartheta) = \left((\mathrm{e}^{\vartheta\left(1/\mathrm{x}-1\right)} - 1)^{\alpha} + 1\right)^{-1}, \quad x \in (0,1)\,, \alpha > 0, \vartheta > 0\,. \tag{1.1}$$

A pdf plays a vital role in finding most of the results of any distribution, like r^{th} moments, moment generating function etc. The pdf of ULE distribution can be obtained by differentiating the cdf $F_{ULE}(x,\alpha,\lambda)$. The pdf may be obtained as:

$$f_{ULE}(x,\alpha,\vartheta) = \frac{\alpha\vartheta \mathrm{e}^{\vartheta\left(1/\mathrm{x}-1\right)}(\mathrm{e}^{\vartheta\left(1/\mathrm{x}-1\right)} - 1)^{\alpha-1}}{x^2((\mathrm{e}^{\vartheta\left(1/\mathrm{x}-1\right)} - 1)^{\alpha} + 1)^2}\,. \tag{1.2}$$

The remainder of this manuscript is organized in the following way. In Section 2, we discuss several useful structural properties of the ULE distribution. Section 3 deals with the quantile regression of the ULE distribution. A simulation study for the ULE distribution is discussed in Section 4. The maximum likelihood estimation (MLE) is discussed in Section 5. For illustrative purposes, three actual standard datasets are e-examined to exhibit the efficacy of the ULE distribution in Section 5. Finally, Section 6 contains concluding remarks.

2. Structural Properties

Some basic statistical functions of the ULE distribution have been discussed in this section.

2.1. Shapes

In this subsection, we are interested to discuss some of the graphical behaviour of density and hazard rate function of the ULE distribution. In Figure 1, left side graph is showing that λ is a scale parameter and the right side graph is showing that α is a shape parameter. In Figure 2, for different values of the parameters, hr function is representing bathtub, upside-down bathtub, increasing, decreasing and constant shapes.

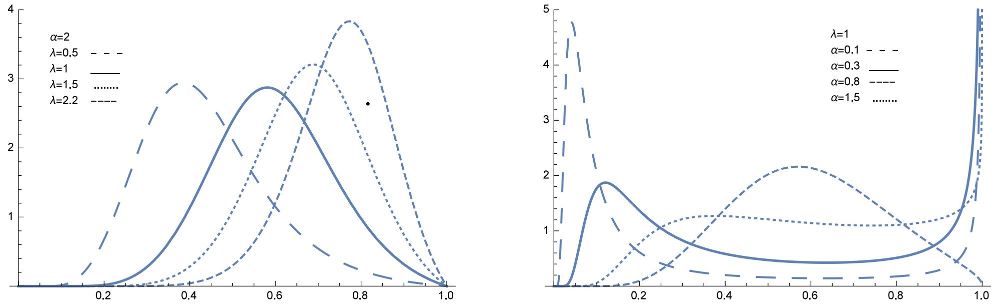

Figure 1. The graphs of ULE pdf

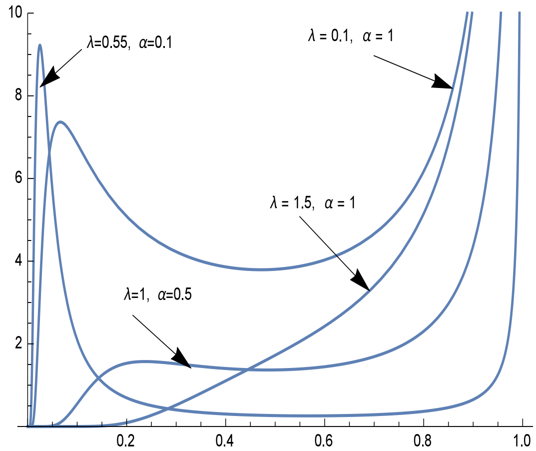

Figure 2. Plots of the ULE hr function

2.2. Survival and Failure Rate Functions

The survival function (sf) and failure rate (fr) function of the ULE distribution, respectively derived as:

$$S(x) = \frac{(\mathrm{e}^{\vartheta(1/\mathrm{x}-1)} - 1)^{\alpha}}{(\mathrm{e}^{\vartheta(1/\mathrm{x}-1)} - 1)^{\alpha} + 1}, \tag{2.1}$$

and

$$fr(x) = \frac{\alpha\vartheta \mathrm{e}^{\vartheta(1/\mathrm{x}-1)}}{x^2((\mathrm{e}^{\vartheta(1/\mathrm{x}-1)} - 1)^{\alpha} + 1)}. \tag{2.2}$$

2.3. Cumulative and Reverse Failure Rate Function

The corresponding cumulative fr (cfr) and reverse fr functions are obtained as:

$$\eta(x) = -\log(S(x)) = \log\left(\frac{(\mathrm{e}^{\vartheta(1/\mathrm{x}-1)})^{\alpha} + 1}{(\mathrm{e}^{\vartheta(1/\mathrm{x}-1)})^{\alpha}}\right), \tag{2.3}$$

and

$$Rfr(x) = x^{-2}(1 - \mathrm{e}^{\vartheta(1/\mathrm{x}-1)})^{\alpha-1}\, \alpha\vartheta \mathrm{e}^{\vartheta(1/\mathrm{x}-1)}\left((1 - \mathrm{e}^{\vartheta(1/\mathrm{x}-1)})^{\alpha} + 1\right)^{-1}. \tag{2.4}$$

2.4. The Quantile Function

The quantile function ($Q(p)$) is derived as:

$$Q(p) = (ln((p^{-1} - 1)^{\frac{1}{\alpha}} + 1)^{\frac{1}{\vartheta}} + 1)^{-1}. \tag{2.5}$$

2.5. Odd Function

The odd function of ULE distribution is obtained as:

$$O(x) = (\mathrm{e}^{\vartheta(1/\mathrm{x}-1)})^{\alpha}. \tag{2.6}$$

2.6. Moments

The r^{th} order moments of ULE distribution are derived as:

$$m(x) = \alpha\vartheta(-1)^{\alpha-1}\int_0^1 x^{r-2}\mathrm{e}^{\vartheta(\vartheta-1)}(1-\mathrm{e}^{\vartheta(1/\mathrm{x}-1)})^{\alpha-1}((\mathrm{e}^{\vartheta(1/\mathrm{x}-1)}-1)^{\alpha}+1)^{-2}\mathrm{dx}.$$

The following series expansions will be used to derive moments:

$$(1-z)^{\varsigma-1} = \sum_{n=0}^{\infty}\binom{\varsigma-1}{n}(-1)^n z^n, \tag{2.7}$$

and

$$(1+z)^{-m} = \sum_{l=0}^{\infty}\binom{m+l-1}{l}(-1)^l z^l, \tag{2.8}$$

By making use of the series expansions (2.7) and (2.8), one obtains

$$\begin{aligned} m(x) &= \alpha\vartheta(-1)^{\alpha-1}\int_0^1 x^{r-2}\mathrm{e}^{\vartheta(1/\mathrm{x}-1)}\sum_{\mathrm{n}=0}^{\infty}\binom{\alpha-1}{\mathrm{n}}(-1)^{\mathrm{n}}\mathrm{e}^{\mathrm{n}\,\vartheta(1/\mathrm{x}-1)}\sum_{\mathrm{j}=0}^{\infty}\binom{1+\mathrm{j}}{\mathrm{j}}(-1)^{\mathrm{j}}(\mathrm{e}^{\vartheta(1/\mathrm{x}-1)}-1)^{\alpha\mathrm{j}}\mathrm{dx} \\ &= \alpha\,\vartheta\sum_{n=0}^{\infty}\sum_{\mathrm{j}=0}^{\infty}\binom{\alpha-1}{n}\binom{1+j}{j}(-1)^{n+\alpha+\mathrm{j}-1}\int_0^1 x^{r-2}\mathrm{e}^{\vartheta(\mathrm{n}+1)(1/\mathrm{x}-1)}(\mathrm{e}^{\vartheta(1/\mathrm{x}-1)}-1)^{\alpha\mathrm{j}}\mathrm{dx}. \end{aligned} \tag{2.9}$$

One can numerically verify the integral in the (2.9) in computational packages Mathematica, MAPLE, MATLAB or R. The following relationship is used to obtain the central or actual moments, the first moment about mean is always equal to zero and second moment about mean is equal to variance $\mu_2 = \mu_2' - \left(\mu_1'\right)^2, \mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2\left(\mu_1'\right)^3$ and $\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\left(\mu_1'\right)^2 - 3\left(\mu_1'\right)^4$. The moment based measure of skewness and Kurtosis are obtained by using $\beta_1 = \frac{\mu_3^2}{\mu_2^3}$ and $\beta_2 = \frac{\mu_4}{\mu_2^2}$ respectively. The values of the moments, variance, covariance, skewness and kurtosis are given in Table 1.

Comment on Table 1: From Table 1, we can make the following comments:

- For fixed α, as the value of λ increases, the amount of kurtosis increases while the skewness values decrease.
- For fixed λ, as the value of α increases, the amount of kurtosis decreases while the skewness values increase.

Table 1. Possible values for the first four moments, variance, standard deviation, covariance, skewness, and kurtosis of the ULE distribution.

(α,λ)	μ_1	μ_2	μ_3	μ_4	σ^2	σ	Cov	$Skewness$	$Kurtosis$
(1.8,1.0)	0.595	0.376	0.250	0.174	0.022	0.148	0.249	0.089	2.594
(1.9,1.0)	0.595	0.374	0.247	0.170	0.020	0.142	0.239	0.095	2.658
(1.9,1.1)	0.615	0.398	0.269	0.189	0.019	0.139	0.226	0.028	2.654
(1.9,1.2)	0.634	0.421	0.290	0.207	0.019	0.136	0.215	-0.032	2.662
(2.1,1.2)	0.634	0.418	0.285	0.201	0.016	0.126	0.199	-0.022	2.778
(2.2,1.2)	0.634	0.417	0.283	0.198	0.015	0.122	0.192	-0.017	2.832
(2.2,1.5)	0.681	0.477	0.342	0.251	0.013	0.114	0.167	-0.163	2.895
(2.2,1.7)	0.706	0.511	0.377	0.284	0.012	0.108	0.154	-0.241	2.960
(2.3,1.7)	0.707	0.510	0.376	0.282	0.011	0.105	0.148	-0.236	3.007
(3.0,2.0)	0.740	0.553	0.418	0.319	0.006	0.078	0.105	-0.288	3.375

2.7. *Order Statistics*

Let $\xi_1, \xi_2, \ldots, \xi_n$ be a random sample of the ULE distribution and the order statistics (OS) are denoted by $\xi_{1:n} \leq \xi_{2:n} \leq \ldots \leq \xi_{n:n}$. The cdf and pdf of the i^{th} order statistic $\xi_{i:n}$ of ULE distribution are

$$F_{i:n}(\xi) = \frac{m!}{(q-1)!\,(m-q)!}\sum_{l=0}^{m-q}\sum_{\mathrm{j}=0}^{\infty}\binom{m-q}{l}\frac{(-1)^{(\mathrm{j}+1)}}{q+l}\binom{2q+2l+\mathrm{j}-1}{\mathrm{j}}\mathrm{e}^{\alpha\vartheta\mathrm{j}(\xi^{-1}-1)}, \tag{2.10}$$

and

$$\begin{aligned} f_{i:n}(\xi) &= \frac{m!\alpha\vartheta\mathrm{e}^{\vartheta(1/\xi-1)}(\mathrm{e}^{\vartheta(1/\xi-1)}-1)^{\alpha-1}}{(q-1)!(m-i)!\xi^2((\mathrm{e}^{\vartheta(1/\xi-1)}-1)^{\alpha}+1)^2}\sum_{l=0}^{m-q}\binom{m-q}{l}(-1)^l\left((\mathrm{e}^{\vartheta(1/\xi-1)}-1)^{\alpha}+1\right)^{-2(q-1+l)} \\ &= \frac{\alpha\vartheta m!(-1)^{\alpha-1}\mathrm{e}^{\vartheta(1/\xi-1)}}{(q-1)!\,(m-q)!\xi^2}\sum_{l=0}^{r-i}\sum_{q,k,j=0}^{\infty}(-1)^{q+j+k+l}\,\mathrm{e}^{\mathrm{q}\vartheta(1/\xi-1)}\,(\mathrm{e}^{\vartheta(1/\xi-1)}-1)^{\alpha(\mathrm{j+k})} \\ &\quad\times\binom{r-q}{l}\binom{\alpha-1}{q}\binom{k+1}{k}\binom{2q+2l+j-3}{j}. \end{aligned} \tag{2.11}$$

2.8. *Reversed Order Statistics*

Let $\xi_1, \xi_2, \ldots, \xi_n$ be a chance sample of the ULE distribution and $\xi_{n:n} \leq \xi_{n-1:n} \leq \ldots \leq \xi_{1:n}$ represent the corresponding reversed OS. Let $f_{i:n}(\xi)$ denote, respectively, the pdf of the i^{th} OS $\xi_{i:n}$. After simplification, one can write

$$f_r(re) = c_{s:i} f(\xi)\sum_{l=0}^{s-1}(-1)^l\binom{s-1}{l}(F(\xi))^{i-s+l}, \tag{2.12}$$

where $c_{s:i} = \frac{i!}{(i-s)!(i-1)!}$. Then,

$$f_r(re) = c_{r:n}\alpha\vartheta\xi^{-2}\mathrm{e}^{\vartheta(1/\xi-1)}(\mathrm{e}^{\vartheta(1/\xi-1)}-1)^{\alpha-1}((\mathrm{e}^{\vartheta(1/\xi-1)}-1)^{\alpha}+1)^{-2}$$

$$\times \sum_{l=0}^{u-1}(-1)^l \binom{u-1}{l}((\mathrm{e}^{\vartheta(1/\xi-1)}-1)^{\alpha}+1)^{-2(n-u+l)}$$

$$= c_{u:n}\alpha\lambda\xi^{-2}(-1)^{\alpha-1}\mathrm{e}^{\lambda(\xi^{-1}-1)}\sum_{q,r,k=0}^{\infty}\sum_{l=0}^{n-1}(-1)^{q+r+l+k}\mathrm{e}^{\mathrm{i}\vartheta(1/\xi-1)}(\mathrm{e}^{\vartheta(1/\mathrm{x}-1)}-1)^{\alpha(\mathrm{r+k})}$$

$$\times \binom{\alpha-1}{q}\binom{r+1}{r}\binom{r-1}{l}\binom{n-u+l+k-1}{k}. \tag{2.13}$$

2.9. Upper Record Statistics

The Upper Record Statistics for the ULE distribution is calculated as:

$$f^k_{Y_n}(x) = \frac{k!}{(n-1)!}\left(-\ln\frac{(\mathrm{e}^{\vartheta(1/\mathrm{x}-1)}-1)^{\alpha}}{(\mathrm{e}^{\vartheta(1/\mathrm{x}-1)}-1)^{\alpha}+1}\right)^{(n-1)}\sum_{i=0}^{\infty}(-1)^i\binom{k-i}{i}\sum_{j=0}^{\infty}\binom{i+j-1}{j}(-1)^j(\mathrm{e}^{\vartheta(1/\mathrm{x}-1)}-1)^{\alpha\mathrm{j}}$$
$$\times\frac{\alpha\vartheta\mathrm{e}^{\vartheta\left(1/\mathrm{x}-1\right)}(\mathrm{e}^{\vartheta\left(1/\mathrm{x}-1\right)}-1)^{\alpha-1}}{x^2((\mathrm{e}^{\vartheta\left(1/\mathrm{x}-1\right)}-1)^{\alpha}+1)^2}.$$

2.10. Residual Lifetime (RL) and Reversed Residual Life (RRL) Functions

The sf of the RL for the ULE distribution is obtained as:

$$S_{RL_{(t)}}(x) = \frac{(\mathrm{e}^{\vartheta(\frac{1}{\mathrm{x+t}}-1)}-1)^{\alpha}((\mathrm{e}^{\vartheta(1/\mathrm{t}-1)}-1)^{\alpha}+1)}{((\mathrm{e}^{\vartheta(\frac{1}{\mathrm{x+t}}-1)}-1)^{\alpha}+1)(\mathrm{e}^{\vartheta(1/\mathrm{t}-1)}-1)^{\alpha}}. \tag{2.14}$$

One can easily obtain its pdf of the RL function.

The RRL is defined as $\bar{R}L_{(t)} := t{-}X|X \leq t$ which denotes the time elapsed from the failure of a component given that its life $\leq t$. The survival function of the RRL $\bar{R}_{(t)}$ for the ULE distribution is

$$S_{\bar{RL}_{(t)}}(x) = \frac{(\mathrm{e}^{\vartheta(1/\mathrm{t}-1)})^{\alpha}+1}{(\mathrm{e}^{\vartheta(1/(\mathrm{t-x})-1)})^{\alpha}+1}. \tag{2.15}$$

One can easily obtain pdf and hr of the $\bar{R}_{(t)}$ from the survival function (2.15).

3. Quantile regression

The quantile regression regression of the ULE distribution is elaborated based on the re-parameterizations of the ULE distribution from the quantile function. It is attainable to re-parameterize the parameter ϑ.

$$\mu = Q(x)$$

Now

$$\mu = [ln((\frac{1}{p}-1)^{\frac{1}{\alpha}}+1)^{\frac{1}{\vartheta}}+1]^{-1}$$

$$\frac{1}{\mu} = ln((\frac{1}{p} - 1)^{\frac{1}{\alpha}} + 1)^{\frac{1}{\vartheta}} + 1$$
$$\frac{1}{\mu} - 1 = \frac{1}{\vartheta}\ ln((p^{-1} - 1)^{\frac{1}{\alpha}} + 1)$$
$$\vartheta = (\frac{1}{\mu} - 1)^{-1}\ ln((1/p - 1)^{\frac{1}{\alpha}} + 1)\,. \tag{3.1}$$

In such a manner that ϑ is for the u^{th} quantile of X. The new cdf and pdf of the quantile ULE (QULE) distribution are expressed as:

$$F_{QULE}(x) = \frac{1}{(\mathrm{e}^{(\frac{1}{\mu}-1)^{-1}\ \ln((\frac{1}{\mathrm{p}}-1)^{\frac{1}{\alpha}}+1)\big(1/\mathrm{x}-1\big)} - 1)^{\alpha} + 1}, \tag{3.2}$$

and

$$f_{QULE}(x) = \frac{\alpha(\frac{1}{\mu} - 1)^{-1}\ ln((\frac{1}{p} - 1)^{\frac{1}{\alpha}} + 1)}{x^2((\mathrm{e}^{(\frac{1}{\mu}-1)^{-1}\ \ln((\frac{1}{\mathrm{p}}-1)^{\frac{1}{\alpha}}+1)(1/\mathrm{x}-1-1)} - 1)^{\alpha} + 1)^2}$$
$$\times\ \mathrm{e}^{(\frac{1}{\mu}-1)^{-1}\ \ln((\frac{1}{\mathrm{p}}-1)^{\frac{1}{\alpha}}+1)(1/\mathrm{x}-1)}(\mathrm{e}^{(\frac{1}{\mu}-1)^{-1}\ \ln((\frac{1}{\mathrm{p}}-1)^{\frac{1}{\alpha}}+1)(1/\mathrm{x}-1)} - 1)^{\alpha-1}, \tag{3.3}$$

where $\alpha \geq 0,\ v,\ \mu\epsilon(0, 1)$ are recognized, called the QULE distribution and represented it as QULE(α, u, μ). Assume that i observations from the ULE distribution are $\xi_1, \xi_2, ..., \xi_i$. The QULE regression distribution is given as:

$$h(\mu_i) = z_i\eta^T\ ,$$

where $\eta = (1, z_{i1}, ..., z_{ip})$ is the known regressor vector, $\eta = (\eta_0, \eta_1, ..., \eta_p)^T$ is the undetermined regression vector of coefficients and $h\,\varepsilon(0, 1)$ is a characterized and double deviation link function. The majority general link functions are probit, $\ln - \ln$, logit and these are utilized by statistician. The $h(\mu_i) = log\frac{\mu_i}{1-\mu_i},\ i = 1, 2, ..., n$ is the logit link function.

The MLE approach is used to estimate the QUPE parameters for the real data. Let us consider the unbiased sample $x = \xi_1, \xi_2, ..., \xi_n$ is the perception of the identically independent unbiased unstables $\Xi = (\Xi_1, ..., \Xi_n)$. Therefore, $\Xi_i \sim$ QULE(α, v, μ) as stated by to Eqn. (3.3), in which

$$\mu_i = \frac{\mathrm{e}^{\mathrm{y_i}\eta^{\mathrm{T}}}}{1 + \mathrm{e}^{\mathrm{y_i}\eta^{\mathrm{T}}}}$$

3.1. Log-Likelihood Function of the QULE Distribution

The log-likelihood function can be stated in the following form:

$$\ell_{QULE}(\Phi|X, \mu) = \ \mathrm{n}\log(\alpha) + \mathrm{n}\log\left(\left(\frac{1}{\mu_{\mathrm{i}}} - 1\right)^{-1}\ln\left(\left(\frac{1}{\mathrm{u}} - 1\right)^{1/\alpha} + 1\right)\right)$$
$$+ \left(\frac{1}{\mu_i} - 1\right)^{-1}\ln\left(\left(\frac{1}{u} - 1\right)^{1/\alpha} + 1\right)\sum_{i=1}^{n}\log\left(\frac{1}{x_i} - 1\right)$$

$$- 2 \sum_{i=1}^{n} \log \left(\left(e^{\left(\frac{1}{\mu_i}-1\right)\ln\left(\frac{1}{x_i}-1\right)\left(\left(\frac{1}{u}-1\right)^{1/\alpha}+1\right)} - 1 \right)^{\alpha} + 1 \right)$$

$$- 2 \sum_{i=1}^{n} \log (x_i) + (\alpha - 1) \sum_{i=1}^{n} \log \left(e^{\left(\frac{1}{\mu_i}-1\right)\ln\left(\frac{1}{x_i}-1\right)\left(\left(\frac{1}{u}-1\right)^{1/\alpha}+1\right)} - 1 \right). \quad (3.4)$$

3.2. Score Function of the QULE Distribution

The QULE's score function distribution may be derived as:

$$U(\Phi|x, u) = [U_{\vartheta_r}, U_\alpha],$$

where $U_\eta = \frac{\partial(logL)}{\partial\mu_i}\frac{\partial\mu_i}{\partial\eta_r}$ and $U_\alpha = \frac{\partial(logL)}{\partial\alpha}$ with $i = 1, \cdots, n$ and $r = 1, \cdots, p$. Therefore, the score function with interrelate may be explained as:

$$\frac{\partial(logL)}{\partial\alpha} = \frac{n}{\alpha} - \frac{n\left(\frac{1}{u}-1\right)^{1/\alpha}\log\left(\frac{1}{u}-1\right)}{\alpha^2\left(\left(\frac{1}{u}-1\right)^{1/\alpha}+1\right)} + \sum_{i=1}^{n} \log\left(e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{x_i}-1\right)\left(\left(\frac{1}{u}-1\right)^{1/\alpha}+1\right)\right)} - 1 \right)$$

$$+ (\alpha - 1) \sum_{i=1}^{n} \frac{-\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{x_i}-1\right)\left(\frac{1}{u}-1\right)^{1/\alpha}\log\left(\frac{1}{u}-1\right)e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{x_i}-1\right)\left(\left(\frac{1}{u}-1\right)^{1/\alpha}+1\right)\right)}}{\alpha^2\left(e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{x_i}-1\right)\left(\left(\frac{1}{u}-1\right)^{1/\alpha}+1\right)\right)} - 1\right)}$$

$$- 2\sum_{i=1}^{n} \frac{\left(e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{x_i}-1\right)\left(\left(\frac{1}{u}-1\right)^{1/\alpha}+1\right)\right)} - 1\right)^{\alpha}\log\left(e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{x_i}-1\right)\left(\left(\frac{1}{u}-1\right)^{1/\alpha}+1\right)\right)} - 1\right)}{\left(e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{x_i}-1\right)\left(\left(\frac{1}{u}-1\right)^{1/\alpha}+1\right)\right)} - 1\right)^{\alpha} + 1}$$

$$- \frac{\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{x_i}-1\right)\left(\frac{1}{u}-1\right)^{1/\alpha}\log\left(\frac{1}{u}-1\right)e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{x_i}-1\right)\left(\left(\frac{1}{u}-1\right)^{1/\alpha}+1\right)\right)}}{\alpha\left(e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{x_i}-1\right)\left(\left(\frac{1}{u}-1\right)^{1/\alpha}+1\right)\right)} - 1\right)}$$

$$+ \sum_{i=1}^{n} \frac{-\ln\left(\frac{1}{x_i}-1\right)\left(\frac{1}{u}-1\right)^{1/\alpha}\log\left(\frac{1}{u}-1\right)}{\alpha^2\left(\frac{1}{\mu_i}-1\right)}, \quad (3.5)$$

$$\frac{\partial(logL)}{\partial\mu_i} = (\alpha - 1) \sum_{i=1}^{n} \frac{-\ln\left(\frac{1}{x_i}-1\right)\left(\left(\frac{1}{u}-1\right)^{1/\alpha}+1\right)e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{x_i}-1\right)\left(\left(\frac{1}{u}-1\right)^{1/\alpha}+1\right)\right)}}{\mu_i^2\left(e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{x_i}-1\right)\left(\left(\frac{1}{u}-1\right)^{1/\alpha}+1\right)\right)} - 1\right)}$$

$$- \sum_{i=1}^{n} \frac{-1}{\left(\frac{1}{\mu_i}-1\right)\mu_i^2} - 2\sum_{i=1}^{n} \frac{-\alpha\ln\left(\frac{1}{x_i}-1\right)\left(\left(\frac{1}{u}-1\right)^{1/\alpha}+1\right)e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{x_i}-1\right)\left(\left(\frac{1}{u}-1\right)^{1/\alpha}+1\right)\right)}}{\mu_i^2\left(\left(e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{x_i}-1\right)\left(\left(\frac{1}{u}-1\right)^{1/\alpha}+1\right)\right)} - 1\right)^{\alpha} + 1\right)}$$

$$\times\left(e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{x_i}-1\right)\left(\left(\frac{1}{u}-1\right)^{1/\alpha}+1\right)\right)}-1\right)^{\alpha-1}+\sum_{i=1}^{n}\frac{\ln\left(\frac{1}{x_i}-1\right)\left(\left(\frac{1}{u}-1\right)^{1/\alpha}+1\right)}{\left(\frac{1}{\mu_i}-1\right)^{2}\mu_i^2}, \tag{3.6}$$

where $\frac{\partial\mu_i}{\partial\eta_r}=\mu_i(1-\mu_i)x_{ir}$. The preceding equations may be solved numerally as these equations be made up of non-straight line functions.

3.3. The ULE Quantile Regression

The ULE quantile regression model is embellished based on the re-parameterization of cdf and pdf of the ULE distribution from the quantile function. It is possible to re-parametrize the parameter λ.

$$\mu = x_u = Q(x).$$

Now using x_u.

$$\begin{aligned}
\mu &= [ln((\frac{1}{p}-1)^{\frac{1}{\alpha}}+1)^{\frac{1}{\lambda}}+1]^{-1}\\
\frac{1}{\mu} &= ln((\frac{1}{p}-1)^{\frac{1}{\alpha}}+1)^{\frac{1}{\lambda}}+1\\
\frac{1}{\mu}-1 &= \frac{1}{\lambda}\ ln((\frac{1}{p}-1)^{\frac{1}{\alpha}}+1)\\
\lambda &= (\frac{1}{\mu}-1)^{-1}\ ln((\frac{1}{p}-1)^{\frac{1}{\alpha}}+1).
\end{aligned} \tag{3.7}$$

In such a manner that λ is for the u^{th} quantile of X distribution with fixed known values.

New cdf of the QULE distribution

$$F_{QULE}(x)=\left((e^{(\frac{1}{\mu}-1)^{-1}\ \ln((\frac{1}{p}-1)^{\frac{1}{\alpha}}+1)\left(1/x-1\right)}-1)^{\alpha}+1\right)^{-1}. \tag{3.8}$$

The new pdf of Quantile ULE distribution

$$f_{QULE}(x)=\frac{\alpha(\frac{1}{\mu}-1)^{-1}\ ln((\frac{1}{p}-1)^{\frac{1}{\alpha}}+1)}{x^2((e^{(\frac{1}{\mu}-1)^{-1}\ \ln((\frac{1}{p}-1)^{\frac{1}{\alpha}}+1)(1/x-1)}-1)^{\alpha}+1)^2}$$

$$\times\ e^{(\frac{1}{\mu}-1)^{-1}\ \ln((\frac{1}{p}-1)^{\frac{1}{\alpha}}+1)(1/x-1)}(e^{(\frac{1}{\mu}-1)^{-1}\ \ln((\frac{1}{p}-1)^{\frac{1}{\alpha}}+1)(x^{-1}-1)}-1)^{\alpha-1}. \tag{3.9}$$

Individually, in which $\alpha \geq 0$, u, $\mu\epsilon$ (0, 1) are recolonized, it is called the quantile ULE (QULE) distribution and it is represented in the following form: QULE(α, u, μ). Assume that n consideration from the pdf of the ULE distribution be $\xi_1, \xi_2, ..., \xi_n$. The QULE regression distribution may be written as:

$$g(\nu_i) = y_i\psi^T,$$

where $\psi = (1, y_{i1}, ..., y_{ip})$ is the familiar regressor vector, $\psi = (\psi_0, \psi_1, ..., \psi_p)^T$ is an unfamiliar parameter vector and $g \in (0, 1)$ is non-increasing and double deviation link function. The greatest link functions are ln – ln, logistic, and probit that are utilized by researchers. The logistic function is applied in the QULE regression distribution for the purpose of link covariates and dependent quantile of the dependent variable. The logistic function may be expressed in the following form:

$$g(\mu_i) = \ln \frac{\mu_i}{1-\mu_i}, \qquad i = 1, 2, ..., n.$$

By finding the parameter values that maximize the likelihood function, the standard MLE method is used to predict the QULE parameters for the data supplied. Suppose that the unbiased sample $\xi = (\xi_1, ..., \xi_n)$ is the actualization of the I.I.D unbiased variables $\Xi = (\Xi_1, ..., \Xi_n)$. Consequently, $\Xi_i \sim$ QULE $(\alpha, \mathrm{v}, \mu)$

$$\mu_i = \frac{\mathrm{e}^{\mathrm{y_i}\beta^{\mathrm{T}}}}{1+\mathrm{e}^{\mathrm{y_i}\beta^{\mathrm{T}}}}.$$

Let $\Omega = (\alpha, \beta)^T$ is unknown parameter vector.

3.4. Log-Likelihood Function (LLF)

The LLF is expressed in the following form:

$$\begin{aligned} \ell_{QULE}(\Omega|\Xi,\mu) = {} & \mathrm{n}\log(\alpha) + \mathrm{n}\log\left(\left(\frac{1}{\mu_{\mathrm{i}}}-1\right)^{-1}\ln\left(\left(\frac{1}{\mathrm{u}}-1\right)^{1/\alpha}+1\right)\right) \\ & + \left(\frac{1}{\mu_i}-1\right)^{-1}\ln\left(\left(\frac{1}{u}-1\right)^{1/\alpha}+1\right)\sum_{i=1}^{n}\log\left(\frac{1}{x_i}-1\right) \\ & - 2\sum_{i=1}^{n}\log\left(\left(\mathrm{e}^{\left(\frac{1}{\mu_{\mathrm{i}}}-1\right)\ln\left(\frac{1}{\mathrm{x_i}}-1\right)\left(\left(\frac{1}{\mathrm{u}}-1\right)^{1/\alpha}+1\right)}-1\right)^{\alpha}+1\right) \\ & - 2\sum_{i=1}^{n}\log(x_i) + (\alpha-1)\sum_{i=1}^{n}\log\left(\mathrm{e}^{\left(\frac{1}{\mu_{\mathrm{i}}}-1\right)\ln\left(\frac{1}{\xi_{\mathrm{i}}}-1\right)\left(\left(\frac{1}{\mathrm{v}}-1\right)^{1/\alpha}+1\right)}-1\right). \end{aligned} \tag{3.10}$$

3.5. Score Function

The QULE's score function distribution may be written as:

$$U(\Omega|\xi, u) = [U_{\beta_r}, U_\alpha],$$

where $U_\beta = \frac{\partial(logL)}{\partial\mu_i}\frac{\partial\mu_i}{\partial\beta_r}$ and $U_\alpha = \frac{\partial(logL)}{\partial\alpha}$ for $i = 1, ..., n$ and $r = 1, ..., p$. Consequently, the score vector with coordinates may be written in the following form:

$$\begin{aligned} \frac{\partial(logL)}{\partial\alpha} = {} & \frac{n}{\alpha} - \frac{n\left(\frac{1}{v}-1\right)^{1/\alpha}\log\left(\frac{1}{v}-1\right)}{\alpha^2\left(\left(\frac{1}{v}-1\right)^{1/\alpha}+1\right)} + \sum_{i=1}^{n}\log\left(\mathrm{e}^{\left(\ln\left(\frac{1}{\mu_{\mathrm{i}}}-1\right)\left(\frac{1}{\xi_{\mathrm{i}}}-1\right)\left(\left(\frac{1}{\mathrm{v}}-1\right)^{1/\alpha}+1\right)\right)}-1\right) \\ & + (\alpha-1)\sum_{i=1}^{n}\frac{-\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{\xi_i}-1\right)\left(\frac{1}{v}-1\right)^{1/\alpha}\log\left(\frac{1}{v}-1\right)\mathrm{e}^{\left(\ln\left(\frac{1}{\mu_{\mathrm{i}}}-1\right)\left(\frac{1}{\xi_{\mathrm{i}}}-1\right)\left(\left(\frac{1}{\mathrm{v}}-1\right)^{1/\alpha}+1\right)\right)}}{\alpha^2\left(\mathrm{e}^{\left(\ln\left(\frac{1}{\mu_{\mathrm{i}}}-1\right)\left(\frac{1}{\xi_{\mathrm{i}}}-1\right)\left(\left(\frac{1}{\mathrm{v}}-1\right)^{1/\alpha}+1\right)\right)}-1\right)} \end{aligned}$$

$$
-2\sum_{i=1}^{n}\frac{\left(e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{\xi_i}-1\right)\left(\left(\frac{1}{v}-1\right)^{1/\alpha}+1\right)\right)}-1\right)^{\alpha}\log\left(e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{\xi_i}-1\right)\left(\left(\frac{1}{v}-1\right)^{1/\alpha}+1\right)\right)}-1\right)}{\left(e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{\xi_i}-1\right)\left(\left(\frac{1}{v}-1\right)^{1/\alpha}+1\right)\right)}-1\right)^{\alpha}+1}
$$

$$
-\frac{\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{\xi_i}-1\right)\left(\frac{1}{v}-1\right)^{1/\alpha}\log\left(\frac{1}{v}-1\right)e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{\xi_i}-1\right)\left(\left(\frac{1}{v}-1\right)^{1/\alpha}+1\right)\right)}}{\alpha\left(e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{\xi_i}-1\right)\left(\left(\frac{1}{v}-1\right)^{1/\alpha}+1\right)\right)}-1\right)}
$$

$$
+\sum_{i=1}^{n}\frac{-\ln\left(\frac{1}{\xi_i}-1\right)\left(\frac{1}{v}-1\right)^{1/\alpha}\log\left(\frac{1}{v}-1\right)}{\alpha^2\left(\frac{1}{\mu_i}-1\right)}, \tag{3.11}
$$

$$
\frac{\partial(logL)}{\partial\mu_i}=(\alpha-1)\sum_{i=1}^{n}\frac{-\ln\left(\frac{1}{\xi_i}-1\right)\left(\left(\frac{1}{v}-1\right)^{1/\alpha}+1\right)e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{\xi_i}-1\right)\left(\left(\frac{1}{v}-1\right)^{1/\alpha}+1\right)\right)}}{\mu_i^2\left(e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{\mathrm{x_i}}-1\right)\left(\left(\frac{1}{\mathrm{u}}-1\right)^{1/\alpha}+1\right)\right)}-1\right)}
$$

$$
-\sum_{i=1}^{n}\frac{-1}{\left(\frac{1}{\mu_i}-1\right)\mu_i^2}-2\sum_{i=1}^{n}\frac{-\alpha\ln\left(\frac{1}{\xi_i}-1\right)\left(\left(\frac{1}{v}-1\right)^{1/\alpha}+1\right)e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{\mathrm{x_i}}-1\right)\left(\left(\frac{1}{\mathrm{u}}-1\right)^{1/\alpha}+1\right)\right)}}{\mu_i^2\left(\left(e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{\xi_i}-1\right)\left(\left(\frac{1}{v}-1\right)^{1/\alpha}+1\right)\right)}-1\right)^{\alpha}+1\right)}
$$

$$
\times\left(e^{\left(\ln\left(\frac{1}{\mu_i}-1\right)\left(\frac{1}{\xi_i}-1\right)\left(\left(\frac{1}{v}-1\right)^{1/\alpha}+1\right)\right)}-1\right)^{\alpha-1}+\sum_{i=1}^{n}\frac{\ln\left(\frac{1}{\xi_i}-1\right)\left(\left(\frac{1}{v}-1\right)^{1/\alpha}+1\right)}{\left(\frac{1}{\mu_i}-1\right)^2\mu_i^2}, \tag{3.12}
$$

where $\frac{\partial\mu_i}{\partial\beta_r}=\mu_i(1-\mu_i)\xi_{ir}$.

4. Monte Carlo Simulation (MCS) Study

We use these results to examine the consistency of the MLE for the ULE parameters through a MCS study. All calculations were performed using the software $\Re$. The samples of sizes $m = 50, ..., 100$ have been generated for $S = 500$ from the ULE distribution with fixed parameter values 1: $\alpha = 1.10$, $\vartheta = 0.50$, 2: $\alpha = 1.20$, $\vartheta = 0.50$, 3: $\alpha = 0.80$, $\vartheta = 0.60$, 4: $\alpha = 0.80$, $\vartheta = 0.70$. The mean squared error (MSE) of bias and MLE was also empirically calculated. Lets take the empirical MSE correlating with h for $\eta\epsilon(\alpha,\vartheta)$ defined by:

$$
MSE_\eta=\frac{1}{S}\sum_{r=1}^{s}(\hat{\upsilon}-\upsilon)^2\,, \tag{4.1}
$$

where $\hat{\upsilon}_r$ is the MLE of υ discover at the r^{th} iteration of the simulation.

MSE stands for mean squared error. It is a common metric used to measure the accuracy of a model or estimator, particularly in regression analysis or forecasting. The MSE quantifies the average squared difference between the observed (true) values and the values predicted by a model or estimator. BIAS refers to the systematic error or deviation of an estimator's expected value from the true value of the parameter being estimated.

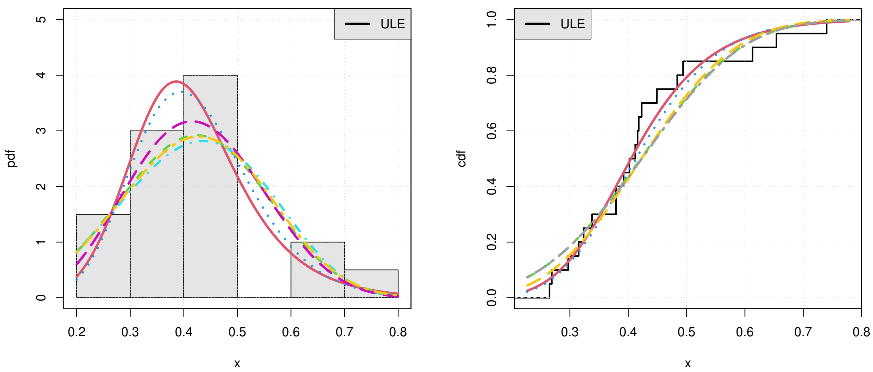

Figure 3. The comparison of ULE distribution with other distributions for Data Set I: ULE (red), PBH (green), BX (blue), Kw (pink), beta (purple) and bXII (yellow)

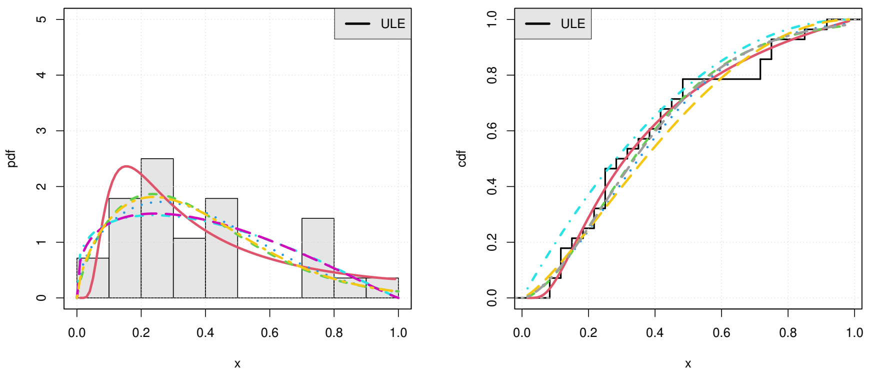

Figure 4. The comparison of ULE distribution with other distributions for Data Set II: : ULE (red), PBH (green), BX (blue), Kw (pink), beta (purple) and bXII (yellow)

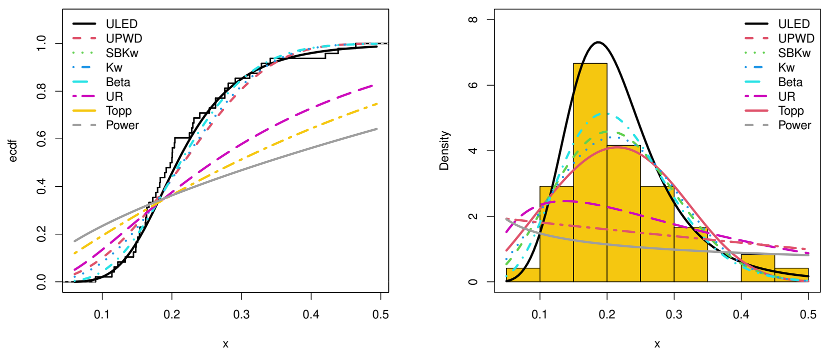

Figure 5. Estimated cdf and pdf plots for Data Set III.

Table 2. Numerical values of simulation.

α	ϑ	N	$MSE(\alpha)$	BIASE(α)	$MSE(\vartheta)$	$BIASE(\vartheta)$
1.10	0.50	50	0.667	0.301	1.380	0.890
-	-	100	0.6201	0.260	1.269	0.859
-	-	150	0.609	0.249	1.250	0.849
-	-	200	0.601	0.250	1.261	0.851
-	-	250	0.579	0.218	1.201	0.819
-	-	300	0.557	0.210	1.200	0.810
-	-	350	0.550	0.201	1.190	0.801
-	-	400	0.520	0.180	1.179	0.779
-	-	500	0.509	0.190	1.100	-0.239
1.20	0.50	50	1.710	0.820	3.350	1.520
-	-	100	1.701	0.748	3.259	1.450
-	-	150	1.600	0.720	2.729	1.419
-	-	200	1.550	0.639	2.729	1.340
-	-	250	1.401	0.560	2.669	1.259
-	-	300	1.350	0.549	2.609	1.249
-	-	350	1.290	0.519	2.549	1.240
-	-	400	1.281	0.510	2.448	1.240
-	-	500	1.240	0.500	2.401	1.210
0.80	0.60	50	0.22	0.13	0.32	0.33
-	-	100	0.220	0.109	0.301	0.309
-	-	150	0.219	0.101	0.289	0.290
-	-	200	0.221	0.091	0.280	0.289
-	-	250	0.210	0.079	0.279	0.279
-	-	300	0.210	0.080	0.270	0.280
-	-	350	0.210	0.080	0.270	0.280
-	-	400	0.201	0.070	0.260	0.270
-	-	500	0.201	0.061	0.251	0.270
0.80	0.70	50	1.110	0.690	1.140	0.849
-	-	100	1.101	0.681	1.130	0.830
-	-	150	1.101	0.671	1.130	0.820
-	-	200	1.090	0.650	1.121	0.811
-	-	250	1.081	0.641	1.091	0.810
-	-	300	1.070	0.641	1.080	0.780
-	-	350	1.061	0.631	1.061	0.780
-	-	400	1.051	0.621	1.051	0.762
-	-	500	1.031	0.610	1.021	0.731

5. Applications

The proposed distribution is compared with the some well known distributions i.e. the PBH [16], the BX [22], the Kw [15], the beta [12] and the bXII [28], the UPW [9], the power, the TL [27], the UR [8] and the SBKw [25] distributions. For this some well-known goodness-of-fit test statistics and test criterion are used.

5.1. Data set I

The twenty samples of flood levels obtained from a river in pennsylvania [8].

5.2. Data set II

The twenty eight samples of kidney patients [8].

Table 3. The MLEs and standard errors (in brackets) for data set I

Model	MLEs and standard errors	
ULE	2.6489646	0.4713822
(α, ϑ)	(0.51543859)	(0.04896027)
PBH	3.590317	7.793646
(α, β)	(0.5552271)	(3.1335253)
BX	-3.445231	4.908819
(λ, θ)	(0.391762)	(2.130048)
KU	3.362684	11.783406
(a,b)	(0.6031874)	(5.3558189)
Beta	6.754817	9.108713
(α, β)	(2.093899)	(2.850841)
BX12	14.39778	3.51854
(α, β)	(5.946767)	(0.564428)

Table 4. Test statistics values for data set I.

Distribution	ℓ	W	A	KS	p-value
ULE	-15.644169	0.05907509	0.3783913	0.13224	0.8755
PBH	-13.387829	0.1400184	0.8373746	0.19555	0.4288
BX	-15.241529	0.07516161	0.4694526	0.16481	0.649
Ku	-12.86620	0.1435952	0.8511986	0.97343	0.498
Beta	-14.062229	0.1257911	0.7524871	0.19878	0.4081
BX12	-13.263910	0.1456184	0.8669111	0.19782	0.4143

Table 5. The goodness-of-fit statistics for data set I.

Distribution	AIC	CAIC	BIC	HQIC
ULE	-27.28834	-26.58246	-25.29688	-26.89959
PBH	22.77566	-22.06977	-20.78419	-22.3869
BX	-26.48306	-25.77718	-24.4916	-25.776
Kw	-21.73238	-21.0265	-19.74092	-21.34363
Beta	-24.12447	-23.41859	-22.133	-23.73571
bxII	-22.52781	-21.82193	-20.53635	-22.13906

Table 6. The MLEs and standard errors (in brackets) for data set II.

Model	MLEs and standard errors	
ULE	0.9864255	0.3129172
(α,ϑ)	(0.17141538)	(0.06580135)
PBH	1.709434	2.327284
(α,β)	(0.2417161)	(0.5422318)
BX	-2.0201806	0.7483259
(λ, θ)	(0.2722531)	(0.1773432)
kw	2.079743	1.265078
(a,b)	(0.5713896)	(0.2544237)
Beta	1.356751	2.105821
(α,β)	(0.3332307)	(0.5496509)
bxII	4.077114	1.638028
(α,β)	(0.9617489)	(0.2405367)

Table 7. The goodness-of-fit statistics for data set II.

Distribution	ℓ	W	A	KS	p-value
ULE	-6.3319149	0.03402816	0.2600505	0.095061	0.962
PBH	-3.8086359	0.05741463	0.4374684	0.1199	0.8156
BX	-3.5547230	0.07569003	0.5363167	0.13663	0.6727
Kw	-3.662399	0.1154183	0.7143394	0.16553	0.4268
Beta	-3.7776190	0.1101202	0.6859319	0.14119	0.6321
bxII	-3.7774879	0.0625623	0.4645672	0.12075	0.8089

Table 8. The goodness-of-fit statistics for data set II.

Distribution	AIC	CAIC	BIC	HQIC
ULE	-8.66383	-8.18383	-5.999421	-7.849293
PBH	-3.617273	-3.137273	-0.9528635	-2.802736
BX	-3.109446	-2.629446	-0.4450373	-2.29491
Kw	-3.32496	-2.84496	-0.6605506	-2.510423
Beta	-3.555241	-3.075241	-0.8908323	-2.740705
bxII	-3.554977	-3.074977	-0.8905675	-2.74044

5.3. *Data set III*

The data set III consists of samples of petroleum rock. [20].

Table 9. The MLEs and SEs (in brackets) for data set III.

Distribution	MLEs and SEs (in brackets)		
ULE	2.7286	0.1813	
(α, ϑ)	(0.3432)	(0.0124)	
SBKw	2.1929	32.8035	
(α_1, α_2)	(0.2945)	(12.235)	
Kw	2.718	44.6081	
(λ, θ)	(0.2932)	(17.546)	
Beta	5.9399	21.200	
(α_1, α_2)	(1.1810)	(4.3457)	
UPW	169.6064	2.179	0.3159
(α, β, ϑ)	(338.6027)	(0.2232)	(0.2520)
UR	0.3782		
(τ)	(0.0546)		
Topp	0.9893		
(ϑ)	(0.1429)		
Power	0.6302		
(ϑ)	(0.0910)		

Figures 3, 4 and 5 represent The comparison of ULE distribution with other distributions for Data Set I, Data Set II and Data Set III. **Comments on Tables 2 to 11**, the ULE distribution is compared with several other rival distributions for different test criterion and goodness-of-fit statistics. The p-value associated with K-S test is also calculated. From Tables 2 to 11 it appears that the values of all test criterion and goodness-of-fit statistics of the ULE distribution are lowest among all several other rival probability distributions. Thus, the proposed ULE distribution is the best fit for all the three data sets.

Table 10. The goodness-of-fit statistics values for data set III.

Distribution	ℓ	W	A	KS	p-value
ULE	-66.56863	0.0453	0.2633	0.12241	0.4682
UPWD	-50.7835	0.2501	1.5434	0.1542	0.2038
SBKw	-53.6087	0.1420	0.8417	182.22	2.2e-17
Kw	-52.4915	0.2083	1.2799	0.1532	0.2096
Beta	-55.6002	0.1281	0.7782	0.1427	0.2820
UR	-35.6046	0.0804	0.4823	0.2920	0.00054
Topp	-21.1659	0.1190	0.7218	0.368	4.517e-05
Power	-6.0118	0.1139	0.6904	0.4295	4.067e-07

Table 11. The goodness-of-fit statistics values for data set III.

Distribution	AIC	BIC	CAIC	HQIC
ULE	-129.1369	-125.3948	-128.8705	-127.7230
UPW	-95.5672	-89.9535	-95.0216	-93.4457
SBKw	-103.2176	-99.4752	-102.9508	-101.8032
Kw	-100.9833	-97.2407	-100.7164	-99.5688
Beta	-107.2014	-103.459	-106.9338	-105.7862
UR	-69.2091	-67.3381	-69.1222	-68.5020
Topp	-40.3320	-38.4608	-40.2449	-39.6248
Power	-10.0238	-8.1526	-9.9367	-9.3166

6. Concluding Remarks

In this paper, we have proposed and studied a novel unit logistic exponential distribution. Several of its structural properties including explicit expressions for the quantile, survival, odd, hazard rate, reversed hazard rate, cumulative hazard rate, residual lifetime, reversed residual lifetime functions have been discussed. Moreover order statitics, reversed order and upper record statistics and moments have also been derived. The quantile regression analysis is also conducted. The method of MLE is employed to estimate the distribution parameters. To exhibit the efficacy of the proposed ULE distribution, three real data sets have been re-analyzed by fitting the ULE distribution along with several existing probability distributions. From the goodness of fit summary measures, it appears that the ULE distribution could be a viable alternative to several of those well-known probability distributions defined on the unit interval. As a natural extension, one can most certainly envision of constructing bivariate and multivariate versions of the ULE distribution that are not developed in this paper. However, from a practitioner's point of view, such distributions must find a way to fit various types of observed phenomena before it can be considered further. All such ideas will be taken up in a future article.

Conflict of Interest

The authors declare there is no existing conflict of interest.

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