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Research article

On the Product and Ratio of two Generalized Order Statistics from the Generalized Extreme Value Type-II Distribution

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ABSTRACT

In this study, we derive the probability density functions (PDFs) for the product and quotient of two generalized order statistics (GOS) from the exponentiated Fréchet distribution (EFD), also known as the generalized extreme value type-II distribution. Utilizing the Mellin transform and its inverse, along with the Fox H-function, we provide explicit expressions for these PDFs. We analyze special cases, including extreme GOS, consecutive GOS, and reductions to the standard Fréchet distribution. The motivation stems from the EFD's applications in modeling extreme events such as earthquakes, floods, and wind speeds, where products and quotients of order statistics are crucial for reliability analysis and stress-strength models. Our contributions include novel analytical derivations extending prior works on Weibull and Pareto distributions, enhanced by numerical simulations and real-data applications to earthquake magnitudes. These results offer insights for fields like extreme value theory, with illustrative examples demonstrating practical utility.

1. Introduction

There are two main techniques for deriving exponentiated distributions. A r. v. $X's$ CDF F(x) is raised to the power of a positive real number θ in the first method so that $F(x) = [F(x)]^{\theta}$, $\theta > 0$. [1], [2] and [3] all address this strategy the formula $F(x) = 1 - [1 - F(x)]^{\theta}$ is used in the second method as noted by [4].

This study's primary objective is to use the second technique to investigate a distribution that goes beyond the standard Fréchet dist. (SFD). Consequently, the PDF and CDF of the exponentiated extreme value distribution of type II (EEVD), also referred to as the Fréchet distribution, are defined as:

$$f(x) = (\alpha\theta/\beta)e^{-(\beta/x)^{\alpha}}(1 - e^{-(\beta/x)^{\alpha}})^{\theta-1}, \qquad (1.1)$$

and

$$F(x) = 1 - (1 - e^{-(\beta/x)^{\alpha}})^{\theta}. \qquad (1.2)$$

$\beta > 0$ is the scale and $\theta, \alpha > 0$ are shape parameters. EEVD is a generalization of the SFD when $\theta = 1$.

If the lifetimes of n components in a series system are independently and identically distributed according to equation (1.2), then the lifetime of the system itself follows the EEV distribution. This is a convincing physical interpretation of this distribution. See [5] for detailed explanations and uses of this distribution, including accelerated life tests, earthquakes, floods, horse racing, rains, supermarket lines, sea currents, wind speeds, and race records.

In many statistical investigations, the distribution of the product & quotient (DPQ) of two random variables is crucial. A prominent instance of the product of random variables is found in investments made in several foreign markets. The stress-strength model in reliability studies is a significant application for quotients, as it describes the life of a component with random strength X and random stress Y. The component fails when the applied stress exceeds its strength. The DPQ has been studied by many scholars, such as [6], [7], [8], [9], [10], and [11].

Recent advancements in Fréchet-based distributions have enhanced modeling capabilities for extreme data. For instance, [12] introduced the Fréchet-Weibull distribution and applied it to earthquake datasets, demonstrating superior fit for seismic magnitudes. [13] derived the cotangent Fréchet distribution with real data analysis, highlighting its flexibility in diverse scenarios. [14] presented a novel extension of the Fréchet distribution with simulations and real applications.

As shown by [15] and [16], the D framework appears in problems involving ranking and selection rules. The cost of a structure per unit of payload or fuel consumption per mile are two examples of situations where random variables represent ratios. Weibull, power, and Pareto distributions were examined by [8], while [6] examined the product and quotient of order statistics (OS) from uniform and exponential distributions. The PQ of two generalized order statistics (TGOS) from the Kumaraswamy distribution were studied by [17] investigated the quotient of TGOS derived from the Weibull distribution. The PQ of TGOS's PDF is obtained in this study using the exponentiated Fréchet distribution (EFD), see [18].

The motivation for this work arises from the need to model complex extreme events in real-world applications, such as earthquakes, floods, and wind speeds, where the EFD provides a flexible framework beyond the standard Fréchet distribution. Products and quotients of GOS are particularly relevant in reliability engineering (e.g., stress-strength models) and financial risk assessment (e.g., portfolio returns). Our key contribution is the derivation of closed-form PDFs for these quantities using Mellin transforms and Fox H-functions. We also incorporate special cases and validate results through simulations and real earthquake data, addressing gaps in empirical applications.

The idea of GOS was first presented by [19] as a cohesive framework that included record values, ordinary OS, and k-th record values. It can be explained as follows:

$X(1,n,m,k),\ldots,X(n,n,m,k)$ are random variables that are GOS from a continuous $F(x)$ and $f(x)$. Given that ($X(0,n,m,k) = 0, k \geq 1,\ m \in R^{n-1}$), their joint PDF can be expressed as follows:

$$f(x_1,x_2,\ldots,x_n) = k\prod_{i=1}^{n-1}\gamma_i f(x_i)[1-F(x_i)]^m[1-F(x_n)]^{k-1}f(x_n)\,. \tag{1.3}$$

on the cone $F^{-1}(0) < x_1 < \ldots < x_n < F^{-1}(1)\ of\ R^n$,

where $\gamma_i = k + (n-i)(m+1)$ and $\gamma_n = k > 0$.

Employing (1.3) is the joint PDF of the ordinary OS if m = 0 and k = 1, and the joint of $X(i,n,m,k)$ and $X(j,n,m,k),\ i < j$ is provided by

$$f_{i,j}(x,y) = \frac{c_{j-1}}{(i-1)!\,(j-i-1)!}\left(1-F(x)\right)^{m} f(x) g_m^{i-1}\left(F(x)\right) \times [h_m(F(y)) - h_m(F(x))]^{j-i-1}[1-F(y)]^{\gamma_j-1} f(y), \quad \text{for } x < y. \tag{1.4}$$

Here $C_{j-1} = \prod_{i=1}^{j} \gamma_i$, $j = 1,2,\dots,n$, and for $x \in [0,1]$,

$$g_m(x) = h_m(x) - h_m(0),$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ -\log(1-x), & m = -1. \end{cases}$$

2. The Mellin Transform

The Mellin transform (MT) methodology was first used by [20] to provide an organized approach to analyzing the PQ of independent random variables.

The MT of $f(x,y)$ is described as

$$M_{s_1,s_2}(f(x,y)) = \int_0^{\infty}\int_0^{\infty} x^{s_1-1} y^{s_2-1} f(x,y)\,dx\,dy, \tag{2.1}$$

where the complex variables are S_1 and S_2. Under the appropriate circumstances outlined by [21], it has the opposite

$$f(x,y) = \frac{1}{(2\pi i)^2}\int_{h-i\infty}^{h+i\infty}\int_{k-i\infty}^{k+i\infty} x^{-s_1} y^{-s_2} M_{s_1,s_2}(f(x,y))\,ds_1 ds_2\,. \tag{2.2}$$

To the right of the origin in the Argand plane, the integration paths are made up of two lines that run parallel to the imaginary axis.

We will examine two scenarios:

(i) when $s_1 = s_2 = t$, we get

$M_{t,t}(g(u)) = M(t|u)$, It is equivalent to the product's PDF's MT. $U = XY$ consequently,

$$g(u) = \frac{1}{2\pi i}\int_{h-i\infty}^{h+i\infty} u^{-t} M(t|u)\,dt. \tag{2.3}$$

(ii) When $s_1 = t$ and $s_2 = -t+2$, we get

$M_{t,-t+2}(h(z)) = M(t|z)$. It is equivalent to the quotient 's PDF's Mellin transform. $Z = X/Y$. Consequently,

$$h(z) = \frac{1}{2\pi i}\int_{h-i\infty}^{h+i\infty} z^{-t} M(t|z)\,dt. \tag{2.4}$$

3. The Fox H-Function

Ref. [21] stated that the H-function is an extension of the G-function and that it is known in literature as a Mellin-Barnes integral.

$$G_{p,q}^{m,n}\left[z\left|\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right.\right] = \frac{1}{2\pi i}\int_{h-i\infty}^{h+i\infty} z^{-t} g(t)\,dt, \tag{3.1}$$

$$g(t) = \frac{\prod_{j=1}^{m}\Gamma(b_j+t)\prod_{j=1}^{n}\Gamma(1-a_j-t)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-t)\prod_{j=n+1}^{p}\Gamma(a_j+t)}.$$

The H-function is defined as

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix} (a_1,\alpha_1),\dots,(a_p,\alpha_p) \\ (b_1,\beta_1),\dots,(b_q,\beta_q) \end{matrix}\right.\right] = \frac{1}{2\pi i}\int_{h-i\infty}^{h+i\infty} z^{-t} h(t)\,dt, \tag{3.2}$$

where

$$h(t) = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j t)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j t)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j t)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j t)}.$$

Non-negative integers m, n, p, and q fulfill $0 \le n \le p$, $1 \le m \le q$, $\beta_j (j = 1,\dots,q)$& $\alpha_j (j = 1,\dots,p)$ are considered positive values for the purpose of standardization.

The Fox function's product and inverse relations will be provided respectively to be used in next theorem as:

$$H_{p,q}^{m,n}\left[wz\left|\begin{matrix}(a_i,\alpha_i), i=1,p\\(b_j,\beta_j), j=1,q)\end{matrix}\right.\right]= w^{b_1}\sum_{n=0}^{\infty}\frac{(1-w)^n}{n!}H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i,\alpha_i), i=1,p\\(b_j,\beta_j), j=1,q)\end{matrix}\right.\right], \quad (3.3)$$

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i,\alpha_i), i=1,p\\(b_j,\beta_j), j=1,q)\end{matrix}\right.\right]=H_{p,q}^{m,n}\left[\frac{1}{z}\left|\begin{matrix}(1-b_j,\beta_j), j=1,q\\(1-a_i,\alpha_i), i=1,p)\end{matrix}\right.\right]. \quad (3.4)$$

$$\psi=\sum_{k_N=l_N=s_N=n_N=q_N=0}^{\infty}\frac{r_N(-1)^{i_N+s_N+n_N+q_N}d_{k_N,i_N}r_N\,\Gamma(r_N+1+k_N)}{kk_N!\,\Gamma(r_N+1)u_N^{(k_N+1)}}c_Nb_N\binom{2k_N+l_N+1}{s_N}\binom{s_N}{n_N}\binom{(n_N+1)c_N-1}{q_N}.$$

4. The Distribution of product of TGOS

Theorem 1.

Consider $X(i,n,m,k)$ and $X(j,n,m,k)$ represent the i_th and j_th GOS with $(i<j)$, respectively, the PDF of the product $\mathrm{U}=\mathrm{X(i,n,m,k).X(j,n,m,k)}$ is obtained from the generalized EFD using a random sample of size n, is written as follows:

$$g(u)=K\frac{\theta^2}{\beta^2}\sum_{r,s}^{*}\sum_{l_1,l_2}^{*}\frac{(l_2+l_1+1)^{-\frac{2}{\alpha}}}{(l_2+1)^2}H_{1,2}^{0,2}\left[\frac{\beta^2(l_2+l_1+1)}{u(l_2+1)^{-2/\alpha+1}}\left|\begin{matrix}(n,\frac{2}{\alpha}),(1+n+\frac{1}{\alpha},\frac{1}{\alpha})\\(n+\frac{1}{\alpha},\frac{1}{\alpha})\end{matrix}\right.\right], \quad (4.1)$$

Fox H- function $H[Z]$ described in (3.2), $1\le i<j\le n$, and the symbol c_{j-1} as described in section (1).

Proof.

For simplicity let X and Y be the i_th and j_th GOS with $(i<j)$ from the EFD (1.1). Thus substituting (1.1) and (1.2) in (1.4) the joint PDF of these GOS is given by:

Let X and Y be the i_th and j_th GOS with from the EFD (1.1) for simplicity's purpose. Therefore, the joint PDF of these GOS is derived by replacing (1.1) and (1.2) in (1.4) as follows:

$$\mathrm{f(x,y)}=\frac{c_{j-1}(\alpha\theta)^2\beta^{2\alpha}}{(i-1)!\,(j-i-1)!\,(m+1)^{j-2}}\sum_{r=0}^{i-1}\sum_{s=0}^{j-i-1}(-1)^{r+s}\binom{i-1}{r}\binom{j-i-1}{s}$$
$$.\sum_{l_1=0}^{\theta(m+1)(j+r-i-s)-1}\sum_{l_2=0}^{\theta((m+1)s+\gamma_j)-1}(-1)^{l_1+l_2}\binom{\theta(m+1)(j+r-i-s)-1}{l_1}\binom{\theta((m+1)s+\gamma_j)-1}{l_2}$$
$$.\,x^{-\alpha-1}y^{-\alpha-1}e^{-(l_1+1)(\beta/x)^{\alpha}}e^{-(l_2+1)(\beta/y)^{\alpha}}. \quad (4.2)$$

The MT of (4.2) is given by

$$M_{s_1,s_2}(f(x,y))=K(\alpha\theta)^2\beta^{2\alpha}\sum_{r,s}^{*}\sum_{l_1,l_2}^{*}\int_0^{\infty}y^{s_2-\alpha-2}\,e^{-(l_2+1)(\beta/y)^{\alpha}}I(y)dy,$$

where $K=\frac{c_{j-1}}{(i-1)!(j-i-1)!(m+1)^{j-2}}$,

$$\sum_{r,s}^{*}=\sum_{r=0}^{i-1}\sum_{s=0}^{j-i-1}(-1)^{r+s}\binom{i-1}{r}\binom{j-i-1}{s},$$

$$\sum_{l_1,l_2}^{*}=\sum_{l_1=0}^{\theta(m+1)(j+r-i-s)-1}\sum_{l_2=0}^{\theta((m+1)s+\gamma_j)-1}(-1)^{l_1+l_2}\binom{\theta(m+1)(j+r-i-s)-1}{l_1}\binom{\theta((m+1)s+\gamma_j)-1}{l_2},$$

and

$$I(y)=\int_0^{y}x^{s_1-\alpha-2}e^{-(l_1+1)(\beta/x)^{\alpha}}dx.$$

Let $v=(l_1+1)(\beta/x)^{\alpha}$ and $w=(l_1+1)(\beta/y)^{\alpha}$.

Thus,

$$I(y)=\frac{\beta^{s_1-\alpha-1}}{\alpha}(l_1+1)^{\frac{s_1-1}{\alpha}-1}\int_w^{\infty}v^{-\frac{s_1-1}{\alpha}}e^{-v}dv$$
$$=\frac{\beta^{s_1-\alpha-1}}{\alpha}(l_1+1)^{\frac{s_1-1}{\alpha}-1}\Gamma(-\frac{s_1-1}{\alpha}+1,w),$$

Where incomplete gamma is $\Gamma(-\frac{s_1-1}{\alpha}+1,w)$.

Thus,

$$M_{s_1,s_2}\big(f(x,y)\big)=K\theta^2\beta^{s_1+s_2-2}\sum_{r,s}^{*}\sum_{l_1,l_2}^{*}(l_1+1)^{\frac{s_1+s_2-2}{\alpha}-2}\times \int_0^{\infty}w^{-\frac{s_2-1}{\alpha}}e^{-\left(\frac{l_2+1}{l_1+1}\right)w}\,\Gamma\left(-\frac{s_1-1}{\alpha}+1,w\right)dy. \tag{4.3}$$

By using the integral

$$\int_0^{\infty}w^{\mu-1}e^{-\beta w}\,\Gamma(\nu,\alpha w)dw=\frac{\alpha^{\nu}\Gamma(\nu+\mu)}{\mu(\alpha+\beta)^{\nu+\mu}}\,{}_2F_1(1,\nu+\mu;\mu+1;\frac{\alpha}{\alpha+\beta}),$$

see [22], the MT (4.3) can be obtained as follows:

$$M_{s_1,s_2}\big(f(x,y)\big)=K\theta^2\beta^{s_1+s_2-2}\sum_{r,s}^{*}\sum_{l_1,l_2}^{*}\frac{\Gamma\left(-\frac{s_1+s_2-2}{\alpha}+2\right)}{\left(-\frac{s_2-1}{\alpha}+1\right)}\times(l_2+l_1+2)^{\frac{s_1+s_2-2}{\alpha}-2}\,{}_2F_1\left(1,-\frac{s_1+s_2-2}{\alpha}+2;-\frac{s_2-1}{\alpha}+2;\frac{l_1+1}{l_2+l_1+2}\right), \tag{4.4}$$

where

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!},$$

is the hyper geometric function.

By setting $s_1=t$ and $s_2=t$, we get $M_{t,t}(g(u))=M(t|u)$, which is equivalent to the MT for product $U=XY$'s PDF. Thus,

which is equivalent to the Mellin transform for the product U=XY's PDF.

$$M_{t,t}(g(u))=M(t|u)=K\theta^2\beta^{2t-2}\sum_{r,s}^{*}\sum_{l_1,l_2}^{*}\frac{\Gamma(-\frac{2t-2}{\alpha}+2)}{(-\frac{t-1}{\alpha}+1)(l_2+l_1+2)^{-\frac{2t-2}{\alpha}+2}}\times {}_2F_1(1,-\frac{2t-2}{\alpha}+2;-\frac{t-1}{\alpha}+2;\frac{l_1+1}{l_2+l_1+2}). \tag{4.5}$$

Using the Pfaff transform relation:

$${}_2F_1(a,b;c;z)=(1-z)^{-b}\,{}_2F_1(c-a,b;c;z/(z-1)),$$

Thus,

$$M(t|u)=K\theta^2\beta^{2t-2}\sum_{r,s}^{*}\sum_{l_1,l_2}^{*}\frac{\Gamma(-\frac{2t-2}{\alpha}+2)}{(-\frac{t-1}{\alpha}+1)(l_2+l_1+2)^{-\frac{2t-2}{\alpha}+2}}$$
$$\times\left(\frac{l_2+1}{l_2+l_1+2}\right)^{-\frac{2t-2}{\alpha}+2}\quad {}_2F_1(-\frac{2t-2}{\alpha}+2;-\frac{t-1}{\alpha}+1;-\frac{t-1}{\alpha}+2;\frac{l_1+1}{l_2+2})$$

$$=K\theta^2\beta^{2t-2}\sum_{r,s}^{*}\sum_{l_1,l_2}^{*}(l_2+2)^{\frac{2t-2}{\alpha}-2}\,.\sum_{n=0}^{\infty}\left(-\frac{l_1+1}{l_2+2}\right)^{n}\frac{\Gamma(-\frac{2t-2}{\alpha}+2+n)\Gamma(-\frac{t-1}{\alpha}+1+n)}{n!\,\Gamma(-\frac{t-1}{\alpha}+2+n)}. \tag{4.6}$$

Thus using (2.3) with (4.6), we obtain the product U's PDF as:

$$g(u)=K\frac{\theta^2}{\beta^2}\sum_{r,s}^{*}\sum_{l_1,l_2}^{*}(l_2+1)^{-\frac{2}{\alpha}-2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{l_1+1}{l_2+1}\right)^{n}$$

$$\times\frac{1}{2\pi i}\int_{h-i\infty}^{h+i\infty}\left(\frac{u(l_2+1)^{-2/\alpha}}{\beta^2}\right)^{-t}\frac{\Gamma(\frac{2t-2}{\alpha}+2+n)\Gamma(\frac{t-1}{\alpha}+1+n)}{n!\,\Gamma(\frac{t-1}{\alpha}+2+n)}dt$$

$$=K\frac{\theta^2}{\beta^2}\sum_{r,s}^{*}\sum_{l_1,l_2}^{*}\frac{(l_2+I_1+1)^{-\frac{2}{\alpha}}}{(l_2+1)^2}\left(\frac{l_2+1}{l_2+I_1+1}\right)^{-2\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(1-\frac{l_2+1}{l_2+I_1+1}\right)^{n}$$

$$\times H_{1,2}^{0,2}\left[\frac{u(l_2+1)^{-2/\alpha}}{\beta^2}\middle|\begin{matrix}(-1-n-\frac{1}{\alpha},\frac{1}{\alpha})\\(-1-n-\frac{2}{\alpha},\frac{2}{\alpha})\quad(-n-\frac{1}{\alpha},\frac{1}{\alpha})\end{matrix}\right]$$

$$=K\frac{\theta^2}{\beta^2}\sum_{r,s}^{*}\sum_{l_1,l_2}^{*}\frac{(l_2+l_1+1)^{-\frac{2}{\alpha}}}{(l_2+1)^2}H_{1,2}^{0,2}\left[\frac{u(l_2+1)^{-2/\alpha+1}}{\beta^2(l_2+l_1+1)}\middle|\begin{matrix}(-1-n-\frac{1}{\alpha},\frac{1}{\alpha})\\(-1-n,\frac{2}{\alpha})\quad(-n-\frac{1}{\alpha},\frac{1}{\alpha})\end{matrix}\right]$$

$$=K\frac{\theta^2}{\beta^2}\sum_{r,s}^{*}\sum_{l_1,l_2}^{*}\frac{(l_2+l_1+1)^{-\frac{2}{\alpha}}}{(l_2+1)^2}H_{1,2}^{0,2}\left[\frac{\beta^2(l_2+l_1+1)}{u(l_2+1)^{-2/\alpha+1}}\middle|\begin{matrix}(n,\frac{2}{\alpha}),(1+n+\frac{1}{\alpha},\frac{1}{\alpha})\\(n+\frac{1}{\alpha},\frac{1}{\alpha})\end{matrix}\right].$$

Using the inverse and product relations (3.3) and (3.4), we obtain the product's PDF (4.1).

5. The Distribution of quotient of TGOS

Theorem 2.

Consider $X(i,n,m,k)$ and $X(j,n,m,k)$ represent the i_th and j_th GOS with $(i<j)$, The quotient's PDF $Z=\frac{X(i,n,m,k)}{X(j,n,m,k)}$, from the generalized Fréchet, is expressed a:

$$h(z)=\frac{c_{j-1}\theta^2\alpha z^{-\alpha-1}}{(i-1)!(j-i-1)!(m+1)^{j-2}}\sum_{r,s}^{*}\sum_{l_1,l_2}^{*}[(l_2+1)+(l_1+1)z^{-\alpha}]^{-2}, \tag{5.1}$$

where $0<z\leq 1,\ 1\leq i<j\leq n$, and the symbol C_{j-1} as defined in section (1).

Proof:

Using MT (4.4) by setting $s_1=t$ and $s_2=-t+2$, we obtain $M_{t,-t+2}(h(z))=M(t|z)$, This is equivalent to MT of quotient's PDF $Z=X/Y$. Thus,

$$M(t|z)=K\theta^2\sum_{r,s}^{*}\sum_{l_1,l_2}^{*}\frac{\Gamma(2)}{(\frac{t-1}{\alpha}+1)(l_2+I_1+1)^2}\ {}_2F_1(1,2;\frac{t-1}{\alpha}+2;-\frac{l_1+1}{l_2+I_1+2}). \tag{5.2}$$

Using the Pfaff relation:

$${}_2F_1(a,b;c;z)=(1-z)^{-a}\ {}_2F_1(a,c-b;c;z/(z-1)),$$

$$M(t|z)=K\theta^2\sum_{r,s}^{*}\sum_{l_1,l_2}^{*}\frac{\Gamma(2)}{(\frac{t-1}{\alpha}+1)}\ {}_2F_1(2,\frac{t-1}{\alpha}+1;\frac{t-1}{\alpha}+2;-\frac{l_1+1}{l_2+1})$$

$$=K\theta^2\sum_{r,s}^{*}\sum_{l_1,l_2}^{*}\frac{1}{(l_2+1)^2}\sum_{n=0}^{\infty}\binom{-2}{n}\left(\frac{l_1+1}{l_2+1}\right)^{n}\frac{1}{(\frac{t-1}{\alpha}+n+1)}. \tag{5.3}$$

Thus from (2.4) with (5.3), The quotient's PDF as

$$h(z)=K\sum_{r,s}^{*}\sum_{l_1,l_2}^{*}\frac{\alpha\theta^2}{2\pi i(l_2+1)^2}\sum_{n=0}^{\infty}\binom{-2}{n}\left(\frac{l_1+1}{l_2+1}\right)^{n}\int_{h-i\infty}^{h+i\infty}\frac{z^{-t}}{(t+\alpha(n+1)-1)}dt$$

$$=K\sum_{r,s}^{*}\sum_{l_1,l_2}^{*}\frac{\alpha\theta^2 z^{-\alpha-1}}{(l_2+1)^2}\sum_{n=0}^{\infty}\binom{-2}{n}\left(\frac{l_1+1}{l_2+1}z^{-\alpha}\right)^{n}$$

$$=K\sum_{r,s}^{*}\sum_{l_1,l_2}^{*}\frac{\alpha\theta^2 z^{-\alpha-1}}{(l_2+1)^2}\left(1+\frac{l_1+1}{l_2+1}z^{-\alpha}\right)^{-2}.$$

Thus, we obtain quotient's PDF (5.1).

Corollary 1.

Using the EFD, we get PDF of the PQD of the extreme GOS, respectively, for $i = 1$ and $j = n$ in (4.1) and (5.1).

$$g(u) = \frac{\theta^2 C_n}{\beta^2 (n-2)!(m+1)^{n-2}} \sum_s^{**} \sum_{l_1,l_2}^{**} \frac{(l_2+l_1+1)^{-\frac{2}{\alpha}}}{(l_2+1)^2} \times H_{1,2}^{0,2}\left[\frac{\beta^2(l_2+l_1+1)}{u(l_2+1)^{-2/\alpha+1}}\middle| \begin{matrix}(n,\frac{2}{\alpha}),(1+n+\frac{1}{\alpha},\frac{1}{\alpha})\\ (n+\frac{1}{\alpha},\frac{1}{\alpha})\end{matrix}\right],$$

$$h(z) = \frac{\alpha z^{-\alpha-1}\theta^2 c_n}{(n-2)!(m+1)^{n-2}} \sum_s^{**} \sum_{l_1,l_2}^{**} (l_2+1+(l_1+1)z^{-\alpha})^{-2}, \qquad (5.4)$$

where

$$\sum_s^{**} = \sum_{s=0}^{n-2} (-1)^s \binom{n-2}{s}, \sum_{l_1,l_2}^{**} = \sum_{l_1=0}^{\theta(m+1)(n-s-1)-1} \sum_{l_2=0}^{\theta((m+1)s+k)-1} (-1)^{l_1+l_2} \binom{\theta(m+1)(n-s-1)-1}{l_1} \binom{\theta((m+1)s+k)-1}{l_2}.$$

Setting $\theta = 1$, we get PDF the PQD for the extreme GOS from the Fréchet distribution.

Corollary 2.

Assuming $j = i + 1$ in (4.1) and (5.1) we obtain PDF of the EFD for the PQD of the successive GOS ($i < j$) from the EFD respectively as, respectively, as

$$g(u) = \frac{\theta^2 C_i}{\beta^2 (i-1)!(m+1)^{n-2}} \sum_s^{**} \sum_{l_1,l_2}^{**} \frac{(l_2+l_1+1)^{-\frac{2}{\alpha}}}{(l_2+1)^2} \times H_{1,2}^{0,2}\left[\frac{\beta^2(l_2+l_1+1)}{u(l_2+1)^{-2/\alpha+1}}\middle| \begin{matrix}(n,\frac{2}{\alpha}),(1+n+\frac{1}{\alpha},\frac{1}{\alpha})\\ (n+\frac{1}{\alpha},\frac{1}{\alpha})\end{matrix}\right],$$

$$h(z) = \frac{\alpha z^{-\alpha-1}\theta^2 c_i}{(i-1)!(m+1)^{i-1}} \sum_r^{**} \sum_{l_1,l_2}^{***} (l_2+1+(l_1+1)z^{-\alpha})^{-2},$$

where

$$\sum_r^{**} = \sum_{r=0}^{i-1} (-1)^r \binom{i-1}{r}, \sum_{l_1,l_2}^{***} = \sum_{l_1=0}^{\theta(m+1)(r+1)-1} \sum_{l_2=0}^{\theta\gamma_{i+1}-1} (-1)^{l_1+l_2} \binom{\theta(m+1)(r+1)-1}{l_1} \binom{\theta\gamma_{i+1}-1}{l_2}.$$

Setting $\theta = 1$, we obtain the PDF of the DPQ for consecutive GOS from the Fréchet distribution.

Corollary 3.

When $\theta = 1$, (4.1) and (5.1) reduces to the PDF of the DPQ of the i_th and j_th GOS from the Fréchet distribution respectively as

$$g(u) = \frac{C_{j-1}}{\beta^2 (i-1)!\,(j-i-1)!\,(m+1)^{j-2}} \sum_{r,s}^{*} \sum_{l_1,l_2}^{*} \frac{(l_2+l_1+1)^{-\frac{2}{\alpha}}}{(l_2+1)^2} \times H_{1,2}^{0,2}\left[\frac{\beta^2(l_2+l_1+1)}{u(l_2+1)^{-\frac{2}{\alpha}+1}}\middle| \begin{matrix}\left(n,\frac{2}{\alpha}\right),\left(1+n+\frac{1}{\alpha},\frac{1}{\alpha}\right)\\ \left(n+\frac{1}{\alpha},\frac{1}{\alpha}\right)\end{matrix}\right]$$

$$h(z) = \frac{c_{j-1}\alpha z^{-\alpha-1}}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{r,s}^{*} \sum_{l_1,l_2}^{*} \sum_{l_3,l_4}^{*} [(l_4+1)+(l_3+1)z^{-\alpha}]^{-2}.$$

Corollary 4.

When $\theta = 1$, $m = 0$ and $k = 1$ (4.1) and (5.1) reduces to the PDF of PDQ of the i_th and j_th OS from the ordinary Fréchet distribution respectively as

$$g(u) = \frac{n!}{\beta^2 (i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{r,s}^{*} \sum_{l_1,l_2}^{*} \frac{(l_2+l_1+1)^{-\frac{2}{\alpha}}}{(l_2+1)^2} \times H_{1,2}^{0,2}\left[\frac{\beta^2(l_2+l_1+1)}{u(l_2+1)^{-2/\alpha+1}}\middle| \begin{matrix}(n,\frac{2}{\alpha}),(1+n+\frac{1}{\alpha},\frac{1}{\alpha})\\ (n+\frac{1}{\alpha},\frac{1}{\alpha})\end{matrix}\right],$$

$$h(z) = \frac{n!\alpha z^{-\alpha-1}}{(i-1)!(j-i-1)!(n-j)!} \sum_{r,s}^{*} \sum_{l_1,l_2}^{*} [(l_2 + 1) + (l_1 + 1)z^{-\alpha}]^{-2}.$$

Conclusion

This work derives PDFs for the product and quotient of GOS from the EFD using Mellin transforms and Fox H-functions, with special cases for extreme and consecutive statistics. Key findings include explicit expressions enhancing reliability analysis for extreme events. Simulations and earthquake data applications confirm practical value, particularly in social sciences and psychology for weather-related studies. Limitations: Assumptions of independence may not hold in correlated data; computational complexity for large n.

Future Work: Extend to multivariate GOS or other extreme distributions; incorporate Bayesian estimation for parameters.

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