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Research article

The Discrete Hyperbolic Sine Distribution: A Comprehensive Survey with New Mathematical Results

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ABSTRACT

The hyperbolic sine (HS) distribution is a discrete probability distribution introduced by Kafi et al. as one of the two component distributions of the discrete hyperbolic mixture (DHM) distribution. Although the mathematical properties established for the DHM distribution, including its moments and index of dispersion, are correct, a careful re-examination reveals that the variance reported for the HS distribution was derived incorrectly. Consequently, the HS distribution was incorrectly classified as overdispersed in the original study. In this paper, we derive the correct variance of the HS distribution and establish that its index of dispersion satisfies $0.5 < \text{IOD} < 1$, demonstrating that the distribution is always underdispersed. In addition, we develop several mathematical properties that were not investigated previously, including the quantile function, characteristic function, factorial moments, skewness, kurtosis, and mode. We also demonstrate that the HS distribution belongs to the exponential family and develop the hyperbolic sine generalized linear model (HS-GLM), enabling its application to regression analysis of underdispersed count data. These results provide a more complete theoretical characterization of the HS distribution and broaden its potential applications in statistical modeling of count data with underdispersion.

1. Introduction

Discrete probability distributions play a fundamental role in modeling count data arising in diverse disciplines. The increasing complexity of modern datasets has motivated the development of flexible discrete distributions capable of accommodating a wide range of dispersion patterns. In particular, the index of dispersion (IOD), defined as the ratio of the variance to the mean, is widely used to characterize count data. A distribution is said to be underdispersed, equidispersed, or overdispersed when its IOD is less than, equal to, or greater than one, respectively. Since many classical count distributions, such as the Poisson distribution, are restricted to equidispersion, numerous alternative models have been proposed to capture non-equidispersed data. However, most of the studies assumed an overdispersed distribution, as in [1, 2, 3, 4, 5, 6, 7, 8].

Recently, Kafi et al. (2025) proposed the discrete hyperbolic mixture (DHM) distribution as a flexible family obtained by mixing two discrete distributions constructed from the hyperbolic cosine and hyperbolic sine functions [9]. This distribution offers an alternative to existing flexible discrete distributions, such as those in [10, 11, 12, 13]. The DHM distribution provides a unified framework in which the hyperbolic cosine (HC) and hyperbolic sine (HS) distributions arise as special cases corresponding to particular values of the mixing parameter. The authors established several important statistical properties of the DHM distribution, including its probability mass function, cumulative distribution function, probability generating function, moment generating function, mean, variance, index of dispersion, and parameter estimation. The proposed model demonstrated competitive performance in fitting several real count datasets.

The HS distribution is of particular interest because it has a relatively simple probability structure and exhibits attractive theoretical properties. Owing to its tractable probability mass function and finite moments of all orders, the distribution could serve as an alternative model for underdispersed count data. However, despite its importance as a component of the DHM family, its mathematical properties have not been investigated as thoroughly as those of many well-established discrete distributions. During our investigation of the HS distribution, we revisited the mathematical derivations presented by Kafi et al. Although the theoretical properties of the DHM distribution were correct, our analysis revealed that the reported variance of the HS distribution was derived incorrectly. The error is confined exclusively to the variance expression of the HS sub-model. It therefore does not invalidate the mathematical properties derived for the DHM distribution, whose moments and variance were established independently from the right first and second moments of the HC and HS component distributions. Nevertheless, the incorrect variance resulted in an erroneous classification of the HS distribution as overdispersed. In this paper, we derive the correct variance and show that the index of dispersion satisfies $0.5 < \text{IOD} < 1$, implying that the HS distribution is always underdispersed.

Beyond correcting the variance, the existing literature provides only a limited theoretical treatment of the HS distribution. Several fundamental distributional properties that are routinely available for classical discrete distributions have not yet been established. These include the quantile function, characteristic function, factorial moments, skewness, kurtosis, and the mode. Such quantities are essential for understanding the distribution's probabilistic behavior, describing its shape, facilitating numerical computation, and supporting statistical inference. Their absence leaves the theoretical development of the HS distribution incomplete and limits its practical applicability. Furthermore, an important aspect of extending discrete distributions beyond theoretical development is integrating them into regression frameworks to model count data with covariate information. The generalized linear model (GLM) provides a well-established frame-

work for relating the distributional characteristics of response variables to explanatory variables through a systematic component and an appropriate link function. However, the application of the HS distribution within the GLM framework has not yet been explored. Establishing that the HS distribution belongs to the exponential family is essential for its incorporation into the GLM framework. In this study, we demonstrate that the HS distribution can be represented as an exponential family distribution and develop the corresponding hyperbolic sine generalized linear model (HSGLM). This extension enables the HS distribution to be used for regression analysis of underdispersed count data while preserving its inherent dispersion characteristics. The proposed HSGLM provides a flexible alternative to existing count regression models, particularly in situations where the assumption of equidispersion or overdispersion is inappropriate.

Motivated by these observations, this paper presents a comprehensive survey of the hyperbolic sine distribution together with several new mathematical results and its extension to regression modeling. First, the variance of the HS distribution is re-derived, and the corrected expression is used to establish its dispersion behavior. Second, several previously unavailable mathematical properties are derived, including the characteristic function, quantile function, factorial moments, skewness, kurtosis, and mode. Third, the membership of the HS distribution in the exponential family is established, allowing the development of the hyperbolic sine generalized linear model (HSGLM). These results complement existing theory and provide a more complete mathematical characterization of the distribution, while expanding its applicability to the regression analysis of underdispersed count data. Collectively, the corrected variance, newly derived properties, and GLM formulation provide additional theoretical foundations and practical tools for studying and applying the HS distribution.

For clarity, the contributions of the present work fall into three categories. First, the variance and dispersion index expression of the HS distribution are corrected. Second, previously available properties are revisited in order to provide a unified mathematical treatment. Third, new results are established concerning the characteristic function, quantiles, factorial moments, skewness, kurtosis, mode, exponential-family representation, and the associated HS regression model.

2. Mass Function and Cumulative Distribution Function

A random variable N is said to have the hyperbolic sine (HS) distribution, denoted by $N \sim \mathrm{HS}(\phi)$, if its probability mass function (PMF) and cumulative distribution function (CDF) are respectively given by

$$\mathbb{P}(N = k) = \frac{\phi^{2k+1}}{(\sinh \phi)\,(2k+1)!}, \qquad k = 0, 1, 2, \ldots, \tag{2.1}$$

and

$$\mathbb{P}(N \le x) = \begin{cases} 0, & x < 0, \\ \displaystyle\sum_{k=0}^{\lfloor x \rfloor} \frac{\phi^{2k+1}}{\sinh \phi\,(2k+1)!} & x \ge 0 \end{cases}, \tag{2.2}$$

where $\phi > 0$. The HS distribution was originally introduced in [9].

3. Generating Functions

This section provides the generating functions of the HS distribution.

Theorem 3.1. *Let $N \sim \mathrm{HS}(\phi)$. Then the moment generating function (MGF) of N is given by:*

$$M(t) = e^{-t/2}\frac{\sinh\left(\phi e^{t/2}\right)}{\sinh\phi}, \qquad \text{for all } t \in \mathbb{R}. \tag{3.1}$$

Proof. The MGF of the HS distribution can be obtained as follows:

$$\begin{aligned} M(t) = \mathbb{E}\left(e^{tN}\right) &= \sum_{k=0}^{\infty} e^{tk}\mathbb{P}(N=k) = \sum_{k=0}^{\infty} e^{tk}\frac{\phi^{2k+1}}{(\sinh\phi)\,(2k+1)!} = \frac{1}{\sinh\phi}\sum_{k=0}^{\infty}\frac{e^{tk}\phi^{2k+1}}{(2k+1)!} \\ &= \frac{1}{\sinh\phi}\sum_{k=0}^{\infty}\frac{\left(e^{t/2}\right)^{2k}\phi^{2k+1}}{(2k+1)!} = \frac{e^{-t/2}}{\sinh\phi}\sum_{k=0}^{\infty}\frac{\left(\phi e^{t/2}\right)^{2k+1}}{(2k+1)!} = \frac{e^{-t/2}\sinh\left(\phi e^{t/2}\right)}{\sinh\phi}. \end{aligned}$$

Since the exponential function and the hyperbolic sine function are defined for every real number, the numerator

$$e^{-t/2}\sinh\left(\phi e^{t/2}\right)$$

is well-defined for all $t \in \mathbb{R}$. Moreover, the denominator $\sinh\phi$ is independent of t. Because $\phi > 0$, we have $\sinh\phi \neq 0$, so no division by zero occurs. Therefore, the domain of $M(t)$ is $\mathbb{R}$. □

Theorem 3.2. *Let $N \sim \mathrm{HS}(\phi)$. Then the probability generating function (PGF) of N is given by:*

$$G(t) = \begin{cases} \dfrac{\sinh(\phi\sqrt{t})}{\sqrt{t}\,\sinh\phi}, & t > 0 \\ \dfrac{\phi}{\sinh\phi}, & t = 0 \\ \dfrac{\sin(\phi\sqrt{-t})}{\sqrt{-t}\,\sinh\phi}, & t < 0 \end{cases}. \tag{3.2}$$

Proof. By the definition of the hyperbolic sine distribution,

$$\mathbb{P}(N=k) = \frac{\phi^{2k+1}}{(2k+1)!\sinh\phi}, \qquad k = 0, 1, 2, \ldots,$$

where $\phi > 0$. Hence, for any $t \in \mathbb{R}$,

$$G(t) = \mathbb{E}(t^N) = \sum_{k=0}^{\infty} t^k\mathbb{P}(N=k) = \sum_{k=0}^{\infty} t^k\frac{\phi^{2k+1}}{\sinh\phi\,(2k+1)!} = \frac{1}{\sinh\phi}\sum_{k=0}^{\infty}\frac{t^k\phi^{2k+1}}{(2k+1)!}.$$

The series converges absolutely for every $t \in \mathbb{R}$. Indeed, if

$$a_k(t) = \frac{t^k\phi^{2k+1}}{(2k+1)!},$$

then

$$\lim_{k\to\infty}\left|\frac{a_{k+1}(t)}{a_k(t)}\right| = \lim_{k\to\infty}\frac{|t|\phi^2}{(2k+3)(2k+2)} = 0.$$

Thus, the PGF is finite for all $t \in \mathbb{R}$.

For $t > 0$, using the Taylor series expansion of $\sinh(x)$, we have

$$G(t) = \frac{1}{\sinh \phi} \sum_{k=0}^{\infty} \frac{\left(t^{1/2}\right)^{2k} \phi^{2k+1}}{(2k+1)!} = \frac{1}{\sqrt{t}\, \sinh \phi} \sum_{k=0}^{\infty} \frac{\left(\phi \sqrt{t}\right)^{2k+1}}{(2k+1)!} = \frac{\sinh\left(\phi \sqrt{t}\right)}{\sqrt{t}\, \sinh \phi}.$$

For $t = 0$, by the definition of the PGF,

$$G(0) = \mathbb{E}(0^N) = \mathbb{P}(N = 0) = \frac{\phi}{\sinh \phi}.$$

Moreover,

$$\lim_{t \to 0} \frac{\sinh\left(\phi \sqrt{t}\right)}{\sqrt{t}\, \sinh \phi} = \frac{\phi}{\sinh \phi} \lim_{t \to 0} \frac{\sinh\left(\phi \sqrt{t}\right)}{\phi \sqrt{t}} = \frac{\phi}{\sinh \phi},$$

because $\lim_{x \to 0} \sinh(x)/x = 1$. Thus, the apparent singularity at $t = 0$ is removable, and the closed-form expression extends continuously to $t = 0$.

For $t < 0$, let $t = -u$, where $u > 0$. Then

$$G(-u) = \frac{1}{\sinh \phi} \sum_{k=0}^{\infty} \frac{(-1)^k \phi^{2k+1} u^k}{(2k+1)!}.$$

Since

$$\phi^{2k+1} u^k = \frac{(\phi \sqrt{u})^{2k+1}}{\sqrt{u}},$$

it follows that

$$G(-u) = \frac{1}{\sqrt{u}\, \sinh \phi} \sum_{k=0}^{\infty} \frac{(-1)^k (\phi \sqrt{u})^{2k+1}}{(2k+1)!}.$$

To verify convergence, define

$$b_k(u) = \frac{(-1)^k (\phi \sqrt{u})^{2k+1}}{(2k+1)!}.$$

Then

$$\lim_{k \to \infty} \left| \frac{b_{k+1}(u)}{b_k(u)} \right| = \lim_{k \to \infty} \frac{u \phi^2}{(2k+2)(2k+3)} = 0.$$

Hence, by the ratio test, the series converges absolutely for every $u > 0$. Moreover, by the Taylor series expansion of the sine function,

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!},$$

we obtain

$$G(-u) = \frac{\sin(\phi \sqrt{u})}{\sqrt{u}\, \sinh \phi}.$$

Therefore, for $t < 0$,

$$G(t) = \frac{\sin\left(\phi \sqrt{-t}\right)}{\sqrt{-t}\, \sinh \phi}.$$

This completes the proof. □

Theorem 3.3. *Let $N \sim \mathrm{HS}(\phi)$. Then the characteristic function of N is given by:*

$$C(t) = e^{-it/2}\frac{\sinh\left(\phi e^{it/2}\right)}{\sinh\phi}, \qquad \text{for all } t \in \mathbb{R}. \tag{3.3}$$

Proof. Following Theorem 3.1, since the MGF exists for all $t \in \mathbb{R}$ which include zero and since $\mathrm{Re}(it) = 0 \in \mathbb{R}$, the replacing t by it is valid in the MGF. Therefore, the characteristic function for the HS distribution is

$$C(t) = \mathbb{E}\left(e^{itN}\right) = \frac{e^{-it/2}\sinh\left(\phi e^{it/2}\right)}{\sinh\phi}, \qquad t \in \mathbb{R}. \qquad \square$$

4. Moments and Index of Dispersion

In this section, we provide the mean, variance, index of dispersion, and some factorial moments of the HS distribution. Although a simple compact general expression for the nth raw moment is not pursued here, raw moments of any nonnegative integer order can be obtained from the moment generating function or probability generating function. These moments facilitate the derivation of several important distributional quantities, including the variance and the coefficient of skewness. Their expressions are given in the following Theorem.

Theorem 4.1. *Let $N \sim \mathrm{HS}(\phi)$. The first four raw moments of N are given by:*

$$\mathbb{E}(N) = \frac{\phi}{2}\frac{\cosh\phi}{\sinh\phi} - \frac{1}{2}. \tag{4.1}$$

$$\mathbb{E}(N^2) = \frac{\phi^2}{4} - \frac{\phi}{4}\frac{\cosh\phi}{\sinh\phi} + \frac{1}{4}. \tag{4.2}$$

$$\mathbb{E}(N^3) = \frac{(\phi^3+\phi)\coth\phi - 1}{8}. \tag{4.3}$$

$$\mathbb{E}(N^4) = \frac{(1+\phi^2+\phi^4) + (2\phi^3-\phi)\coth\phi}{16}. \tag{4.4}$$

Proof. The expected value of N can be obtained as follows:

$$\begin{aligned}
\mathbb{E}(N) = \sum_{k=0}^{\infty} k\mathbb{P}(N=k) &= \sum_{k=0}^{\infty} k\frac{\phi^{2k+1}}{\sinh\phi\,(2k+1)!} = \frac{1}{\sinh\phi}\sum_{k=0}^{\infty}\frac{k\phi^{2k+1}}{(2k+1)!} = \frac{1}{2\sinh\phi}\sum_{k=0}^{\infty}\frac{((2k+1)-1)\phi^{2k+1}}{(2k+1)!} \\
&= \frac{1}{2\sinh\phi}\left[\sum_{k=0}^{\infty}\frac{(2k+1)\phi^{2k+1}}{(2k+1)!} - \sum_{k=0}^{\infty}\frac{\phi^{2k+1}}{(2k+1)!}\right] = \frac{1}{2\sinh\phi}\left[\phi\sum_{k=0}^{\infty}\frac{\phi^{2k}}{(2k)!} - \sinh\phi\right] \\
&= \frac{\phi\cosh\phi - \sinh\phi}{2\sinh\phi} = \frac{\phi}{2}\frac{\cosh\phi}{\sinh\phi} - \frac{1}{2}.
\end{aligned}$$

The second moment of N can be obtained as follows:

$$\mathbb{E}(N^2) = \sum_{k=0}^{\infty} k^2\mathbb{P}(N=k) = \sum_{k=0}^{\infty} k^2\frac{\phi^{2k+1}}{\sinh\phi\,(2k+1)!} = \frac{1}{\sinh\phi}\sum_{k=0}^{\infty}\frac{k^2\phi^{2k+1}}{(2k+1)!} = \frac{1}{4\sinh\phi}\sum_{k=0}^{\infty}\frac{((2k+1)-1)^2\phi^{2k+1}}{(2k+1)!}$$

$$
\begin{aligned}
&= \frac{1}{4\sinh\phi}\sum_{k=0}^{\infty}\frac{((2k+1)^2-2(2k+1)+1)\phi^{2k+1}}{(2k+1)!}\\
&= \frac{1}{4\sinh\phi}\left[\sum_{k=0}^{\infty}\frac{(2k+1)^2\phi^{2k+1}}{(2k+1)!}-2\sum_{k=0}^{\infty}\frac{(2k+1)\phi^{2k+1}}{(2k+1)!}+\sum_{k=0}^{\infty}\frac{\phi^{2k+1}}{(2k+1)!}\right]\\
&= \frac{1}{4\sinh\phi}\left[\sum_{k=0}^{\infty}\frac{((2k+1)(2k)+(2k+1))\phi^{2k+1}}{(2k+1)!}-2\phi\cosh\phi+\sinh\phi\right]\\
&= \frac{1}{4\sinh\phi}\left[\sum_{k=0}^{\infty}\frac{(2k+1)(2k)\phi^{2k+1}}{(2k+1)!}+\sum_{k=0}^{\infty}\frac{(2k+1)\phi^{2k+1}}{(2k+1)!}-2\phi\cosh\phi+\sinh\phi\right]\\
&= \frac{1}{4\sinh\phi}\left[\phi\sum_{k=1}^{\infty}\frac{\phi^{2k}}{(2k-1)!}+\phi\sum_{k=0}^{\infty}\frac{\phi^{2k}}{(2k)!}-2\phi\cosh\phi+\sinh\phi\right]\\
&= \frac{1}{4\sinh\phi}\left[\phi\sum_{j=0}^{\infty}\frac{\phi^{2j+2}}{(2j+1)!}+\phi\cosh\phi-2\phi\cosh\phi+\sinh\phi\right]\\
&= \frac{\phi^2\sinh\phi-\phi\cosh\phi+\sinh\phi}{4\sinh\phi}=\frac{\phi^2}{4}-\frac{\phi}{4}\frac{\cosh\phi}{\sinh\phi}+\frac{1}{4}.
\end{aligned}
$$

Besides the direct approach, the third and fourth moments of N can also be obtained from the third and fourth derivatives of the MGF of the HS distribution, respectively, as follows:

$$
M^{(3)}(t)=\frac{\left(\phi^3e^t+\phi\right)\cosh\left(\phi e^{t/2}\right)-e^{-t/2}\sinh\left(\phi e^{t/2}\right)}{8\sinh\phi}.
$$

$$
\begin{aligned}
M^{(4)}(t) &= \frac{e^{-t/2}\sinh\left(\phi e^{t/2}\right)-\phi\cosh\left(\phi e^{t/2}\right)+2\phi^3e^t\cosh\left(\phi e^{t/2}\right)+\phi^2e^{t/2}\sinh\left(\phi e^{t/2}\right)+\phi^4e^{3t/2}\sinh\left(\phi e^{t/2}\right)}{16\sinh\phi}\\
&= \frac{\left(e^{-t/2}+\phi^2e^{t/2}+\phi^4e^{3t/2}\right)\sinh\left(\phi e^{t/2}\right)+\left(2\phi^3e^t-\phi\right)\cosh\left(\phi e^{t/2}\right)}{16\sinh\phi}.
\end{aligned}
$$

Recall that the k-th raw moment equals the k-th derivative of the MGF at point $t=0$. Therefore,

$$
\begin{aligned}
\mathbb{E}(N^3) &= M^{(3)}(0)=\frac{(\phi^3+\phi)\coth\phi-1}{8},\\
\mathbb{E}(N^4) &= M^{(4)}(0)=\frac{(1+\phi^2+\phi^4)+(2\phi^3-\phi)\coth\phi}{16}.
\end{aligned}
$$

This completes the proof. □

Theorem 4.2. *Let $N\sim\mathrm{HS}(\phi)$. Then the variance of N is given by:*

$$
\mathrm{Var}(N)=\frac{-\phi^2+\frac{1}{2}\phi\sinh(2\phi)}{4\sinh^2\phi}. \tag{4.5}
$$

Proof. The variance of N can be obtained as follows:

$$\begin{aligned}
\mathrm{Var}(N) &= \mathbb{E}(N^2) - [\mathbb{E}(N)]^2 \\
&= \frac{\phi^2}{4} - \frac{\phi}{4}\frac{\cosh\phi}{\sinh\phi} + \frac{1}{4} - \left[\frac{\phi}{2}\frac{\cosh\phi}{\sinh\phi} - \frac{1}{2}\right]^2 \\
&= -\frac{\phi}{4}\frac{\cosh\phi}{\sinh\phi} + \frac{\phi^2}{4} + \frac{1}{4} - \frac{\phi^2}{4}\frac{\cosh^2\phi}{\sinh^2\phi} + \frac{\phi}{2}\frac{\cosh\phi}{\sinh\phi} - \frac{1}{4} \\
&= -\frac{\phi^2}{4}\frac{\cosh^2\phi - \sinh^2\phi}{\sinh^2\phi} + \frac{\phi}{4}\frac{\cosh\phi}{\sinh\phi} \\
&= -\frac{\phi^2}{4\sinh^2\phi} + \frac{\phi}{4}\frac{\cosh\phi}{\sinh\phi} \\
&= \frac{-\phi^2 + \phi\cosh\phi\sinh\phi}{4\sinh^2\phi} \\
&= \frac{-\phi^2 + \frac{1}{2}\phi\sinh(2\phi)}{4\sinh^2\phi}. \qquad \square
\end{aligned}$$

Remark 4.3. *In [9], the variance of the HS distribution is reported as*

$$\frac{\phi^2 + \phi\sinh(2\phi)}{4\sinh^2\phi}.$$

This expression resulted from an inaccuracy in the mathematical derivation and affects the resulting dispersion classification of the HS distribution. The correct variance of the HS distribution is given in Theorem 4.2.

Theorem 4.4. *Let $N \sim \mathrm{HS}(\phi)$. Then the index of dispersion (IOD) of N is given by:*

$$\mathrm{IOD} = \frac{-\phi^2 + \frac{1}{2}\phi\sinh(2\phi)}{\phi\sinh(2\phi) - 2\sinh^2\phi}. \tag{4.6}$$

The HS distribution is underdispersed.

Proof. The IOD of N can be obtained as follows:

$$\begin{aligned}
\mathrm{IOD} = \frac{\mathrm{Var}(N)}{\mathbb{E}(N)} &= \frac{\frac{-\phi^2 + \frac{1}{2}\phi\sinh(2\phi)}{4\sinh^2\phi}}{\frac{\phi}{2}\frac{\cosh\phi}{\sinh\phi} - \frac{1}{2}} = \frac{-\phi^2 + \frac{1}{2}\phi\sinh(2\phi)}{4\sinh^2\phi} \cdot \frac{1}{\frac{\phi\cosh\phi - \sinh\phi}{2\sinh\phi}} = \frac{-\phi^2 + \frac{1}{2}\phi\sinh(2\phi)}{4\sinh^2\phi} \cdot \frac{2\sinh\phi}{\phi\cosh\phi - \sinh\phi} \\
&= \frac{-\phi^2 + \frac{1}{2}\phi\sinh(2\phi)}{2\sinh\phi(\phi\cosh\phi - \sinh\phi)} = \frac{-\phi^2 + \frac{1}{2}\phi\sinh(2\phi)}{\phi\sinh(2\phi) - 2\sinh^2\phi}.
\end{aligned}$$

Note that the obtained expression can be rewritten as follows:

$$\mathrm{IOD} = \frac{-\phi(\phi - \sinh\phi\cosh\phi)}{2\sinh\phi(\phi\cosh\phi - \sinh\phi)} = \frac{1}{2}\left(-\frac{\phi(\phi\,\mathrm{csch}\,\phi - \cosh\phi)}{\phi\cosh\phi - \sinh\phi}\right)$$

Let

$$f(\phi) = \frac{1}{2}\left(-\frac{\phi(\phi\ \mathrm{csch}\,\phi - \cosh\phi)}{\phi\cosh\phi - \sinh\phi}\right), \qquad \phi > 0.$$

Since $\operatorname{csch}\phi = 1/\sinh\phi$, we can write

$$f(\phi) = \frac{\phi\left(\cosh\phi - \dfrac{\phi}{\sinh\phi}\right)}{2(\phi\cosh\phi - \sinh\phi)} = \frac{\phi(\sinh\phi\cosh\phi - \phi)}{2\sinh\phi\,(\phi\cosh\phi - \sinh\phi)}.$$

First observe that

$$\phi\cosh\phi - \sinh\phi > 0 \qquad \text{for all } \phi > 0.$$

Indeed, define

$$g(\phi) = \phi\cosh\phi - \sinh\phi.$$

Then $g(0) = 0$, and

$$g'(\phi) = \cosh\phi + \phi\sinh\phi - \cosh\phi = \phi\sinh\phi > 0 \qquad (\phi > 0).$$

Thus $g(\phi) > 0$ for every $\phi > 0$.

Differentiating f and simplifying yields

$$f'(\phi) = -\frac{(\sinh\phi - \phi\cosh\phi)^2 + \phi^2\sinh^2\phi}{2\sinh^2\phi\,(\phi\cosh\phi - \sinh\phi)^2}.$$

For $\phi > 0$, the denominator is positive, since

$$\sinh^2\phi > 0 \qquad \text{and} \qquad (\phi\cosh\phi - \sinh\phi)^2 > 0.$$

Moreover, the numerator satisfies.

$$(\sinh\phi - \phi\cosh\phi)^2 + \phi^2\sinh^2\phi > 0.$$

Therefore, $f'(\phi) < 0$ for all $\phi > 0$. Hence, IOD $= f(\phi)$ is strictly decreasing on $(0, \infty)$.

Now, consider the following behavior of $\frac{-\phi^2 + \frac{1}{2}\phi\sinh(2\phi)}{\phi\sinh(2\phi) - 2\sinh^2\phi}$ as $\phi \to \infty$,

$$\lim_{\phi\to\infty} \frac{-\phi^2 + \frac{1}{2}\phi\sinh(2\phi)}{\phi\sinh(2\phi) - 2\sinh^2\phi} = \lim_{\phi\to\infty} \frac{-2\phi + \frac{1}{2}\sinh(2\phi) + \phi\cosh(2\phi)}{2\phi\cosh(2\phi) - \sinh(2\phi)} = \lim_{\phi\to\infty} \frac{-2 + 2\cosh(2\phi) + 2\phi\sinh(2\phi)}{4\phi\sinh(2\phi)}$$
$$= \lim_{\phi\to\infty} \frac{-\frac{1}{\phi\sinh(2\phi)} + \frac{\cosh(2\phi)}{\phi\sinh(2\phi)} + 1}{2} = \frac{0 + 0 + 1}{2} = \frac{1}{2}.$$

Consider the following behavior of $\frac{-\phi^2 + \frac{1}{2}\phi\sinh(2\phi)}{\phi\sinh(2\phi) - 2\sinh^2\phi}$ as $\phi \to 0^+$. Using L'Hopital, we can evaluate as follows:

$$\lim_{\phi\to 0^+} \frac{-\phi^2 + \frac{1}{2}\phi\sinh(2\phi)}{\phi\sinh(2\phi) - 2\sinh^2\phi} = \lim_{\phi\to 0^+} \frac{-2\phi + \frac{1}{2}\sinh(2\phi) + \phi\cosh(2\phi)}{\sinh(2\phi) + 2\phi\cosh(2\phi) - 4\sinh\phi\cosh\phi}$$
$$= \lim_{\phi\to 0^+} \frac{-2\phi + \frac{1}{2}\sinh(2\phi) + \phi\cosh(2\phi)}{\sinh(2\phi) + 2\phi\cosh(2\phi) - 2\sinh(2\phi)}$$
$$= \lim_{\phi\to 0^+} \frac{-2\phi + \frac{1}{2}\sinh(2\phi) + \phi\cosh(2\phi)}{-\sinh(2\phi) + 2\phi\cosh(2\phi)}$$

$$
\begin{aligned}
&= \lim_{\phi\to 0^+} \frac{-2+\cosh(2\phi)+\cosh(2\phi)+2\phi\sinh(2\phi)}{-2\cosh(2\phi)+2\cosh(2\phi)+4\phi\sinh(2\phi)} \\
&= \lim_{\phi\to 0^+} \frac{-2+2\cosh(2\phi)+2\phi\sinh(2\phi)}{4\phi\sinh(2\phi)} \\
&= \lim_{\phi\to 0^+} \frac{4\sinh(2\phi)+2\sinh(2\phi)+4\phi\cosh(2\phi)}{4\sinh(2\phi)+8\phi\cosh(2\phi)} \\
&= \lim_{\phi\to 0^+} \frac{8\cosh(2\phi)+4\cosh(2\phi)+4\cosh(2\phi)+8\phi\sinh(2\phi)}{8\cosh(2\phi)+8\cosh(2\phi)+16\phi\sinh(2\phi)} \\
&= \frac{8+4+4+0}{8+8+0} = 1.
\end{aligned}
$$

Since the IOD of the HS distribution is a strictly decreasing function over $\phi > 0$ such that

$$\lim_{\phi\to 0^+} \text{IOD} = 1 \quad \text{and} \lim_{\phi\to\infty} \text{IOD} = \frac{1}{2},$$

then IOD given in Equation 4.6 belongs to the open interval $(1/2, 1)$. This implies that HS distribution is underdispersed. □

The following Figure 1 supports Theorem 4.4.

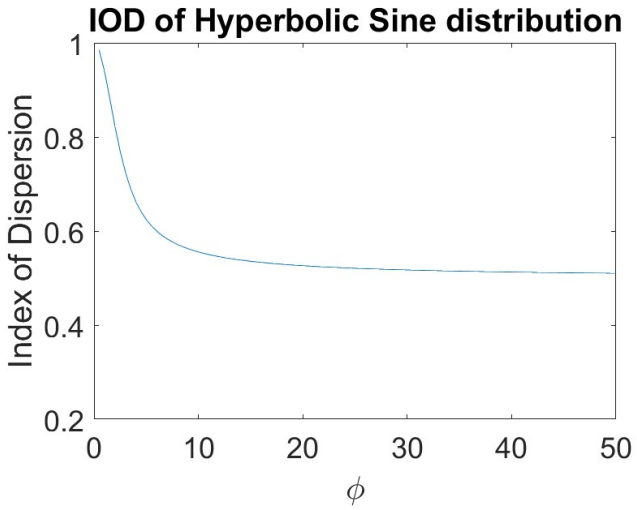

Figure 1. Index of dispersion of the HS distribution.

Recall that the k-th factorial moment of N is defined by

$$\mathbb{E}[(N)_k] = \mathbb{E}\left[N(N-1)\cdots(N-k+1)\right].$$

Theorem 4.5. *Let $N \sim \text{HS}(\phi)$. The first three factorial moments of N are given by:*

$$\mathbb{E}[(N)_1] = \frac{\phi}{2}\coth\phi - \frac{1}{2}. \tag{4.7}$$

$$\mathbb{E}[(N)_2] = \frac{3}{4} + \frac{\phi^2}{4} - \frac{3\phi}{4}\coth\phi. \tag{4.8}$$

$$\mathbb{E}[(N)_3] = -\frac{15}{8} + \frac{\phi^3}{8}\coth\phi - \frac{6\phi^2}{8} + \frac{15\phi}{8}\coth\phi. \tag{4.9}$$

Proof. The first three derivatives of the PGF in Equation 3.2 consecutively are

$$G'(t) = -\frac{\sinh\left(\phi\sqrt{t}\right) - \phi\sqrt{t}\cosh\left(\phi\sqrt{t}\right)}{2t^{3/2}\sinh\phi},$$

$$G''(t) = \frac{3\sinh\left(\phi\sqrt{t}\right) - 3\phi\sqrt{t}\cosh\left(\phi\sqrt{t}\right) + \phi^2 t\sinh\left(\phi\sqrt{t}\right)}{4t^{5/2}\sinh\phi},$$

$$G^{(3)}(t) = -\frac{15\sinh\left(\phi\sqrt{t}\right) - \phi^3 t^{3/2}\cosh\left(\phi\sqrt{t}\right) - 15\phi\sqrt{t}\cosh\left(\phi\sqrt{t}\right) + 6\phi^2 t\sinh\left(\phi\sqrt{t}\right)}{8t^{7/2}\sinh\phi}.$$

Recall that the k-th factorial moment equals the k-th derivative of the PGF at point $t = 1$. Therefore,

$$\mathbb{E}[(N)_1] = G'(1) = -\frac{\sinh\phi - \phi\cosh\phi}{2\sinh\phi} = \frac{\phi}{2}\coth\phi - \frac{1}{2},$$

$$\mathbb{E}[(N)_2] = G''(1) = \frac{3\sinh\phi + \phi^2\sinh\phi - 3\phi\cosh\phi}{4\sinh\phi} = \frac{3}{4} + \frac{\phi^2}{4} - \frac{3\phi}{4}\coth\phi,$$

$$\begin{aligned}\mathbb{E}[(N)_3] = G^{(3)}(1) &= -\frac{15\sinh\phi - \phi^3\cosh\phi + 6\phi^2\sinh\phi - 15\phi\cosh\phi}{8\sinh\phi} \\ &= -\frac{15}{8} + \frac{\phi^3}{8}\coth\phi - \frac{6\phi^2}{8} + \frac{15\phi}{8}\coth\phi. \quad \square\end{aligned}$$

Remark 4.6. *In general, the k-th factorial moment exists for every positive integer k because the PGF*

$$G(t) = \frac{\sinh\left(\phi\sqrt{t}\right)}{\sqrt{t}\,\sinh\phi}$$

is k times differentiable at $t = 1$.

5. Skewness and Kurtosis

This section presents the coefficients of skewness and kurtosis for the HS distribution, as given in Theorems 5.1 and 5.2.

Theorem 5.1. *The coefficient of skewness of the HS distribution is given by:*

$$\gamma_1 = \frac{1}{\left(\frac{-\phi^2 + \frac{1}{2}\phi\sinh(2\phi)}{4\sinh^2\phi}\right)^{3/2}}\left[\frac{\phi\coth(\phi) + 2\phi^3\coth(\phi)\,\mathrm{csch}^2(\phi) - 3\phi^2\,\mathrm{csch}^2(\phi)}{8}\right]. \tag{5.1}$$

Furthermore, γ_1 *is positive and monotonically decreasing on* $0 < \phi < \infty$.

Proof. The coefficient of skewness of $N \sim \mathrm{HS}(\phi)$ can be obtained as follows:

$$\gamma_1 = \frac{1}{[\mathrm{Var}(N)]^{3/2}}\left(\mathbb{E}(N^3) - 3\mathbb{E}(N)\mathbb{E}(N^2) + 2(\mathbb{E}(N))^3\right)$$

$$= \frac{1}{\left(\frac{-\phi^2+\frac{1}{2}\phi\sinh(2\phi)}{4\sinh^2\phi}\right)^{3/2}}\left[\frac{(\phi^3+\phi)\coth\phi-1}{8} - 3\left(\frac{\phi\coth\phi-1}{2}\right)\left(\frac{\phi^2-\phi\coth\phi+1}{4}\right) + 2\left(\frac{\phi\coth\phi-1}{2}\right)^3\right]$$

$$= \frac{1}{\left(\frac{-\phi^2+\frac{1}{2}\phi\sinh(2\phi)}{4\sinh^2\phi}\right)^{3/2}}\left[\frac{-1+\phi\coth(\phi)+\phi^3\coth(\phi)}{8} - \frac{3}{8}(\phi\coth(\phi)-1)(\phi^2+1-\phi\coth(\phi))\right.$$
$$\left. + 2\left(\frac{\phi\coth(\phi)-1}{2}\right)^3\right]$$

$$= \frac{1}{\left(\frac{-\phi^2+\frac{1}{2}\phi\sinh(2\phi)}{4\sinh^2\phi}\right)^{3/2}}\left[\frac{-5\phi\coth(\phi)-2\phi^3\coth(\phi)+3\phi^2\coth^2(\phi)+3\phi^2+2}{8}\right.$$
$$\left. + \frac{2\phi^3\coth^3(\phi)-6\phi^2\coth^2(\phi)+6\phi\coth(\phi)-2}{8}\right]$$

$$= \frac{1}{\left(\frac{-\phi^2+\frac{1}{2}\phi\sinh(2\phi)}{4\sinh^2\phi}\right)^{3/2}}\left[\frac{\phi\coth(\phi)+2\phi^3\coth^3(\phi)-2\phi^3\coth(\phi)-3\phi^2\coth^2(\phi)+3\phi^2}{8}\right]$$

$$= \frac{1}{\left(\frac{-\phi^2+\frac{1}{2}\phi\sinh(2\phi)}{4\sinh^2\phi}\right)^{3/2}}\left[\frac{\phi\coth(\phi)+2\phi^3\coth(\phi)(\coth^2(\phi)-1)-3\phi^2(\coth^2(\phi)-1)}{8}\right]$$

$$= \frac{1}{\left(\frac{-\phi^2+\frac{1}{2}\phi\sinh(2\phi)}{4\sinh^2\phi}\right)^{3/2}}\left[\frac{\phi\coth(\phi)+2\phi^3\coth(\phi)\operatorname{csch}^2(\phi)-3\phi^2\operatorname{csch}^2(\phi)}{8}\right].$$

Let

$$A(\phi) = \phi\coth\phi - \phi^2\operatorname{csch}^2\phi, \qquad \phi > 0.$$

Using

$$\sinh(2\phi) = 2\sinh\phi\cosh\phi,$$

we obtain

$$\frac{-\phi^2+\frac{1}{2}\phi\sinh(2\phi)}{4\sinh^2\phi} = \frac{1}{4}\left(\phi\coth\phi-\phi^2\operatorname{csch}^2\phi\right) = \frac{1}{4}A(\phi).$$

Hence, since $(\frac{1}{4})^{3/2} = \frac{1}{8}$, the expression for γ_1 simplifies to

$$\gamma_1(\phi) = \frac{\phi\coth\phi + 2\phi^3\coth\phi\ \operatorname{csch}^2\phi - 3\phi^2\operatorname{csch}^2\phi}{A(\phi)^{3/2}}.$$

Differentiating A, we find

$$A'(\phi) = \frac{d}{d\phi}\left(\phi\coth\phi-\phi^2\operatorname{csch}^2\phi\right) = \coth\phi - \phi\operatorname{csch}^2\phi - 2\phi\operatorname{csch}^2\phi + 2\phi^2\coth\phi\ \operatorname{csch}^2\phi$$
$$= \coth\phi + 2\phi^2\coth\phi\ \operatorname{csch}^2\phi - 3\phi\operatorname{csch}^2\phi.$$

Therefore,

$$\phi A'(\phi) = \phi\coth\phi + 2\phi^3\coth\phi\ \operatorname{csch}^2\phi - 3\phi^2\operatorname{csch}^2\phi,$$

and consequently the skewness coefficient γ_1 can be rewritten as follows

$$\gamma_1(\phi) = \frac{\phi A'(\phi)}{A(\phi)^{3/2}}.$$

We first show the positivity of $\gamma_1(\phi)$. Since $\phi > 0$ and $A(\phi)^{3/2} > 0$, we only need to show $A'(\phi) > 0$. Using

$$\coth \phi = \frac{\cosh \phi}{\sinh \phi} \qquad \text{and} \qquad \operatorname{csch}^2 \phi = \frac{1}{\sinh^2 \phi},$$

we obtain

$$A'(\phi) = \frac{\cosh \phi \, \sinh^2 \phi + 2\phi^2 \cosh \phi - 3\phi \sinh \phi}{\sinh^3 \phi}.$$

Since $\sinh \phi > 0$ for $\phi > 0$, it suffices to show that the numerator of $A'(\phi)$ is positive. Let

$$h(\phi) = \cosh \phi \, \sinh^2 \phi + 2\phi^2 \cosh \phi - 3\phi \sinh \phi.$$

Since $\sinh \phi > \phi$ for $\phi > 0$, we have

$$\cosh \phi \, \sinh^2 \phi > \phi^2 \cosh \phi.$$

Hence,

$$h(\phi) > \phi^2 \cosh \phi + 2\phi^2 \cosh \phi - 3\phi \sinh \phi = 3\phi(\phi \cosh \phi - \sinh \phi).$$

Now, define

$$q(\phi) = \phi \cosh \phi - \sinh \phi.$$

Consider that

$$q'(\phi) = \cosh \phi + \phi \sinh \phi - \cosh \phi = \phi \sinh \phi > 0$$

for all $\phi > 0$. This implies that $q(\phi)$ is an increasing function over $\phi > 0$. However, $q(0) = 0$ and therefore, $q(\phi) > 0$ for every $\phi > 0$.

Consequently,

$$h(\phi) > 3\phi q(\phi) > 0.$$

Since $\sinh^3 \phi > 0$ for $\phi > 0$, it follows that

$$A'(\phi) = \frac{h(\phi)}{\sinh^3 \phi} > \frac{3\phi q(\phi)}{\sinh^3 \phi} > 0, \qquad \phi > 0.$$

Therefore,

$$\gamma_1(\phi) = \frac{\phi A'(\phi)}{A(\phi)^{3/2}} > 0$$

Now, we show that $\gamma_1(\phi)$ is monotonically decreasing on $0 < \phi < \infty$. The first derivative of γ_1 with respect to ϕ gives

$$\gamma_1'(\phi) = \frac{d}{d\phi}\left[\phi A'(\phi) A(\phi)^{-3/2}\right] = \frac{(A'(\phi) + \phi A''(\phi))A(\phi) - \frac{3}{2}\phi (A'(\phi))^2}{A(\phi)^{5/2}}.$$

For $\phi > 0$, we have $A(\phi) > 0$. Indeed,

$$A(\phi) = \frac{\phi(\sinh\phi\cosh\phi - \phi)}{\sinh^2\phi},$$

and

$$\sinh\phi\cosh\phi - \phi = \frac{1}{2}\sinh(2\phi) - \phi > 0, \qquad \phi > 0,$$

because $\sinh x > x$ for $x > 0$.

Let

$$N(\phi) = (A'(\phi) + \phi A''(\phi))A(\phi) - \frac{3}{2}\phi(A'(\phi))^2.$$

A direct simplification yields

$$2\sinh^6\phi\, N(\phi) = -\phi^3(\sinh(2\phi) - 2\phi)^2 - 2\phi^4\sinh^2\phi(\cosh(2\phi) - 1).$$

For every $\phi > 0$,

$$\sinh(2\phi) - 2\phi > 0, \qquad \cosh(2\phi) - 1 > 0, \qquad \sinh\phi > 0.$$

Thus,

$$2\sinh^6\phi\, N(\phi) < 0,$$

and hence

$$N(\phi) < 0.$$

Since $A(\phi)^{5/2} > 0$, it follows that $\gamma_1'(\phi) < 0$ for $\phi > 0$. Therefore, $\gamma_1(\phi)$ is monotonically decreasing on $(0, \infty)$. □

By Theorem 5.1, the HS distribution is right-skewed. Figure 2 is illustrated in support of Theorem 5.1, that is, the coefficient of skewness of the HS distribution as a function of the parameter ϕ. The coefficient of skewness is positive for all $\phi > 0$. The skewness decreases monotonically as ϕ increases, implying that the distribution becomes progressively less asymmetric. For small values of ϕ, the distribution exhibits substantial positive skewness, whereas for large values of ϕ, the coefficient of skewness approaches zero, suggesting that the distribution becomes approximately symmetric.

Theorem 5.2. *The coefficient of kurtosis of the HS distribution is given by:*

$$\gamma_2 = \frac{\sinh^4\phi\left[(7\phi + \phi^3) + (-6\phi^2 + 1)\coth\phi + (2\phi^3 - 4\phi)\coth^2\phi + 6\phi^2\coth^3\phi - 3\phi^3\coth^4\phi\right]}{\phi\left(-\phi + \frac{1}{2}\sinh(2\phi)\right)^2}. \tag{5.2}$$

Furthermore, γ_2 is monotonically decreasing on $0 < \phi < \infty$ and is bounded below by 3.

Proof. The coefficient of kurtosis of $N \sim \mathrm{HS}(\phi)$ can be obtained as follows:

$$\begin{aligned}\gamma_2 &= \frac{1}{[\mathrm{Var}(N)]^2}\left(E(N^4) - 4E(N)E(N^3) + 6E^2(N)E(N^2) - 3(E(N))^4\right)\\ &= \frac{1}{\left(\frac{-\phi^2 + \frac{1}{2}\phi\sinh(2\phi)}{4\sinh^2\phi}\right)^2}\left[\frac{(1 + \phi^2 + \phi^4) + (2\phi^3 - \phi)\coth\phi}{16}\right.\end{aligned}$$

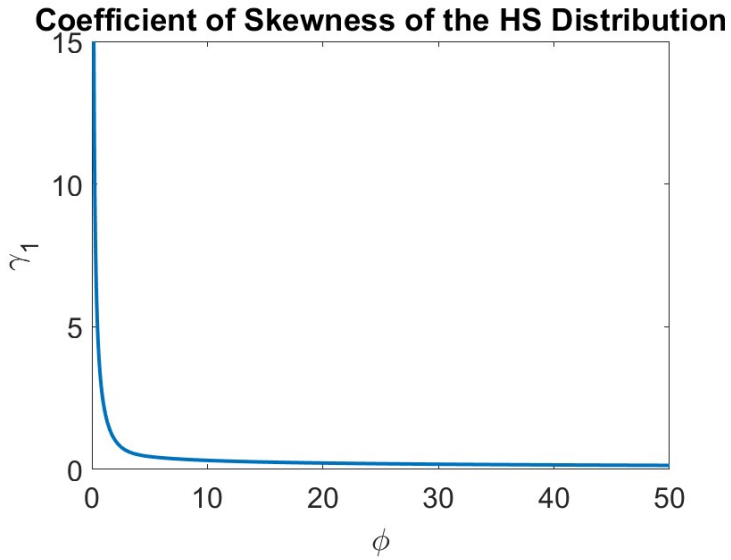

Figure 2. Skewness coefficient values of the HS distribution.

$$
\begin{aligned}
&\quad - 4\left(\frac{\phi\coth\phi - 1}{2}\right)\left(\frac{(\phi^3+\phi)\coth\phi - 1}{8}\right) + 6\left(\frac{\phi\coth\phi - 1}{2}\right)^2\left(\frac{\phi^2 - \phi\coth\phi + 1}{4}\right) \\
&\quad - 3\left(\frac{\phi\coth\phi - 1}{2}\right)^4\Bigg] \\
&= \frac{\frac{1}{16}\left[(7\phi^2+\phi^4) + (-6\phi^3+\phi)\coth\phi + (2\phi^4 - 4\phi^2)\coth^2\phi + 6\phi^3\coth^3\phi - 3\phi^4\coth^4\phi\right]}{\left(\frac{-\phi^2 + \frac{1}{2}\phi\sinh(2\phi)}{4\sinh^2\phi}\right)^2} \\
&= \frac{\sinh^4\phi\left[(7\phi^2+\phi^4) + (-6\phi^3+\phi)\coth\phi + (2\phi^4 - 4\phi^2)\coth^2\phi + 6\phi^3\coth^3\phi - 3\phi^4\coth^4\phi\right]}{\left(-\phi^2 + \frac{1}{2}\phi\sinh(2\phi)\right)^2} \\
&= \frac{\sinh^4\phi\left[(7\phi+\phi^3) + (-6\phi^2+1)\coth\phi + (2\phi^3 - 4\phi)\coth^2\phi + 6\phi^2\coth^3\phi - 3\phi^3\coth^4\phi\right]}{\phi\left(-\phi + \frac{1}{2}\sinh(2\phi)\right)^2}.
\end{aligned}
$$

For $\phi > 0$, define

$$
\gamma_2(\phi) = \frac{\sinh^4\phi\left[(7\phi+\phi^3) + (-6\phi^2+1)\coth\phi + (2\phi^3 - 4\phi)\coth^2\phi + 6\phi^2\coth^3\phi - 3\phi^3\coth^4\phi\right]}{\phi\left(-\phi + \frac{1}{2}\sinh(2\phi)\right)^2}.
$$

First, use

$$
\sinh(2\phi) = 2\sinh\phi\cosh\phi.
$$

Moreover, multiply the expression in square brackets by $\sinh^4\phi$. After collecting powers of $\sinh\phi$ and $\cosh\phi$, we obtain

$$
\gamma_2(\phi) = \frac{(7\phi+\phi^3)\sinh^4\phi + (-6\phi^2+1)\sinh^3\phi\cosh\phi + (2\phi^3 - 4\phi)\sinh^2\phi\cosh^2\phi}{\phi\left(\sinh\phi\cosh\phi - \phi\right)^2}
$$

$$+\frac{6\phi^2\sinh\phi\cosh^3\phi-3\phi^3\cosh^4\phi}{\phi\,(\sinh\phi\cosh\phi-\phi)^2}.$$

Let

$$D(\phi)=\sinh\phi\cosh\phi-\phi=\frac{1}{2}\sinh(2\phi)-\phi.$$

Since $\sinh(2\phi)>2\phi$ for $\phi>0$, we have $D(\phi)>0$ for $\phi>0$. Hence the denominator $\phi D(\phi)^2$ is strictly positive for $\phi>0$. Differentiating $\gamma_2(\phi)$ and simplifying gives

$$\gamma_2'(\phi)=-\frac{2\sinh^2\phi}{\phi^2D(\phi)^3}\,Q(\phi),$$

where

$$Q(\phi)=\phi^2\sinh^2\phi(\cosh(2\phi)-1)+\phi\sinh\phi\cosh\phi(\sinh(2\phi)-2\phi)^2+(\sinh(2\phi)-2\phi)^3.$$

For $\phi>0$, each factor appearing in $Q(\phi)$ is nonnegative, and in fact, strictly positive:

$$\phi>0,\qquad \sinh\phi>0,\qquad \cosh\phi>0,$$

$$\cosh(2\phi)-1>0,\qquad \sinh(2\phi)-2\phi>0.$$

Therefore, $Q(\phi)>0$ for $\phi>0$. Moreover,

$$\frac{2\sinh^2\phi}{\phi^2D(\phi)^3}>0,\quad \text{for}\quad \phi>0.$$

It follows that $\gamma_2'(\phi)<0$ for $\phi>0$. Therefore, $\gamma_2(\phi)$ is strictly monotonically decreasing on $(0,\infty)$.

Next, given $\gamma_2(\phi)$ is monotonically decreasing for $\phi>0$. We show that γ_2 as a function of ϕ is bounded below by 3.

Recall that

$$\gamma_2(\phi)=\frac{\sinh^4\phi\left[(7\phi+\phi^3)+(-6\phi^2+1)\coth\phi+(2\phi^3-4\phi)\coth^2\phi+6\phi^2\coth^3\phi-3\phi^3\coth^4\phi\right]}{\phi\left(-\phi+\frac{1}{2}\sinh(2\phi)\right)^2}.$$

As $\phi\to\infty$,

$$\coth\phi\to1,\qquad \sinh\phi\sim\frac{e^\phi}{2},\qquad \frac{1}{2}\sinh(2\phi)=\sinh\phi\cosh\phi\sim\frac{e^{2\phi}}{4}.$$

Consequently,

$$\frac{\sinh^4\phi}{\phi\left(-\phi+\frac{1}{2}\sinh(2\phi)\right)^2}\sim\frac{e^{4\phi}/16}{\phi(e^{2\phi}/4)^2}=\frac{1}{\phi}.$$

Moreover,

$$\begin{aligned}&(7\phi+\phi^3)+(-6\phi^2+1)\coth\phi+(2\phi^3-4\phi)\coth^2\phi+6\phi^2\coth^3\phi-3\phi^3\coth^4\phi\\&\sim(7\phi+\phi^3)+(-6\phi^2+1)+(2\phi^3-4\phi)+6\phi^2-3\phi^3\\&=3\phi+1.\end{aligned}$$

Thus,

$$\gamma_2(\phi) \sim \frac{1}{\phi}(3\phi + 1),$$

and hence

$$\lim_{\phi\to\infty} \gamma_2(\phi) = 3.$$

Since γ_2 is decreasing on $(0, \infty)$, for every $\phi > 0$,

$$\gamma_2(\phi) \geq \lim_{t\to\infty} \gamma_2(t) = 3.$$

Therefore, $\gamma_2(\phi)$ is bounded below by 3. □

Figure 3 illustrates the coefficient of kurtosis of the HS distribution as a function of the parameter ϕ. It shows that the coefficient of kurtosis of the HS distribution is always greater than three for all $\phi > 0$, indicating that the distribution is leptokurtic. The kurtosis is extremely large when ϕ is close to zero, reflecting a highly peaked distribution with relatively heavy tails. As ϕ increases, the kurtosis decreases monotonically and approaches the limiting value of 3, implying that the HS distribution gradually becomes approximately mesokurtic for large values of the parameter ϕ.

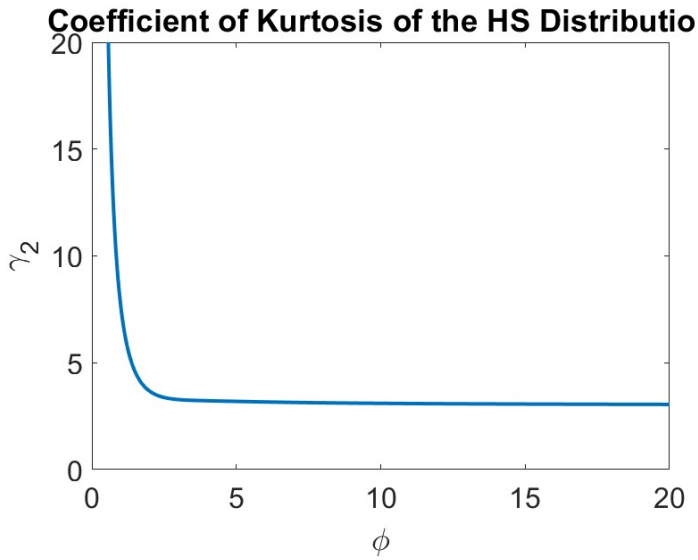

Figure 3. Kurtosis coefficient values of the HS distribution.

6. Quantiles and Mode

This section presents the quantiles and mode for the HS distribution, as given in Theorems 6.1 and 6.2.

Theorem 6.1. *Let* $0 \leq q < 1$ *and* $N \sim \text{HS}(\phi)$. *The q-quantile (or* $100q$*-th percentile) of* N *is the non-negative integer* x_q *such that the following inequality holds:*

$$q \sinh \phi \leq \sum_{k=0}^{x_q} \frac{\phi^{2k+1}}{(2k+1)!} \leq \frac{\phi^{2x_q+1}}{(2x_q+1)!} + q \sinh \phi. \tag{6.1}$$

Proof. The q-quantile(s) of an integer-valued discrete distribution are the non-negative integers x_q satisfying

$$\mathbb{P}(N < x_q) \leq q \leq \mathbb{P}(N \leq x_q).$$

We note that $\mathbb{P}(N \leq x_q) \geq q$ can be written as

$$\sum_{k=0}^{x_q} \frac{1}{\sinh\phi} \frac{\phi^{2k+1}}{(2k+1)!} \geq q \quad \text{or} \quad \sum_{k=0}^{x_q} \frac{\phi^{2k+1}}{(2k+1)!} \geq q \sinh\phi.$$

Also, $\mathbb{P}(N < x_q) \leq q$ can be written as

$$\sum_{k=0}^{x_q-1} \frac{1}{\sinh\phi} \frac{\phi^{2k+1}}{(2k+1)!} \leq q \quad \text{or} \quad \sum_{k=0}^{x_q-1} \frac{\phi^{2k+1}}{(2k+1)!} \leq q \sinh\phi.$$

Therefore, we obtain that x_q is any non-negative integer satisfying

$$q \sinh\phi \leq \sum_{k=0}^{x_q} \frac{\phi^{2k+1}}{(2k+1)!} \leq \frac{\phi^{2x_q+1}}{(2x_q+1)!} + q \sinh\phi. \quad \square$$

Theorem 6.2. *The mode of $N \sim \mathrm{HS}(\phi)$ is*

$$\text{Mode} = \begin{cases} \left\lfloor \frac{-1+\sqrt{4\phi^2+1}}{4} \right\rfloor, & \phi^2 \neq 2m(2m+1) \\ \{m-1, m\}, & \phi^2 = 2m(2m+1) \end{cases}. \tag{6.2}$$

where $m \in \{1, 2, \ldots\}$.

Proof. Recall that the mode of a discrete probability distribution is defined as any point at which its PMF attains a local maximum. Consider the ratio

$$\frac{\mathbb{P}(N = k+1)}{\mathbb{P}(N = k)}, \qquad k \geq 0,$$

that is,

$$\frac{\mathbb{P}(N = k+1)}{\mathbb{P}(N = k)} = \frac{\frac{1}{\sinh\phi}\frac{\phi^{2k+3}}{(2k+3)!}}{\frac{1}{\sinh\phi}\frac{\phi^{2k+1}}{(2k+1)!}} = \frac{\phi^2}{(2k+3)(2k+2)}.$$

Note that the PMF is increasing as long as

$$\frac{\phi^2}{(2k+3)(2k+2)} > 1,$$

that is,

$$\phi^2 > (2k+3)(2k+2).$$

Since the mode must be a nonnegative integer, the mode k^* is the integer satisfying

$$(2k^*)(2k^* + 1) \leq \phi^2 \leq (2k^* + 3)(2k^* + 2).$$

This guarantees that

$$\mathbb{P}(N = k^*) \geq \mathbb{P}(N = k^* - 1) \quad \text{and} \quad \mathbb{P}(N = k^*) \geq \mathbb{P}(N = k^* + 1).$$

The largest k for which the PMF is still increasing is the solution of

$$(2k^*)(2k^* + 1) = \phi^2.$$

Solving this quadratic equation yields

$$k_{1,2} = \frac{-2 \pm \sqrt{4 - 4(4)(-\phi^2)}}{2(4)} = \frac{-1 \pm \sqrt{4\phi^2 + 1}}{4}.$$

Since k must be a nonnegative integer, we take the positive root. If $\frac{-1+\sqrt{4\phi^2+1}}{4}$ is not an integer, the unique mode is

$$k^* = \left\lfloor \frac{-1 + \sqrt{4\phi^2 + 1}}{4} \right\rfloor.$$

Now suppose that $\frac{-1+\sqrt{4\phi^2+1}}{4} = m$ for some positive integer m. Then

$$m = \frac{-1 + \sqrt{4\phi^2 + 1}}{4},$$

which is equivalent to

$$\phi^2 = 2m(2m + 1).$$

Consequently,

$$\frac{p_m}{p_{m-1}} = \frac{\phi^2}{(2m + 1)(2m)} = \frac{2m(2m + 1)}{(2m + 1)(2m)} = 1.$$

Hence,

$$p_{m-1} = p_m.$$

Moreover, because the successive ratios are strictly decreasing in k,

$$\frac{p_k}{p_{k-1}} > 1 \qquad \text{for } k < m,$$

and

$$\frac{p_k}{p_{k-1}} < 1 \qquad \text{for } k > m.$$

Therefore,

$$p_0 < p_1 < \cdots < p_{m-1} = p_m > p_{m+1} > p_{m+2} > \cdots.$$

Thus, the distribution has exactly two modes, that is, $m - 1$ and m. □

7. Hyperbolic Sine Regression Model

This section presents a regression extension of the Hyperbolic Sine (HS) distribution. Specifically, we incorporate the HS distribution into the generalized linear model (GLM) framework by allowing its parameter to depend on a vector of covariates through a suitable link function. This formulation yields a new count regression model, hereafter referred to as the Hyperbolic Sine Generalized Linear Model (HSGLM).
To incorporate the HS distribution into the generalized linear model (GLM) framework, it is necessary to establish that the HS distribution belongs to the exponential family of distributions, as stated in Theorem 7.1.

Theorem 7.1. *The Hyperbolic Sine (HS) distribution belongs to the exponential family of distributions.*

Proof. Let N be a random variable following the HS distribution with PMF

$$\mathbb{P}(N = k) = \frac{\phi^{2k+1}}{\sinh(\phi)(2k+1)!}, \qquad k = 0, 1, 2, \ldots, \quad \phi > 0.$$

Taking the natural logarithm of the PMF gives

$$\begin{aligned} \ln \mathbb{P}(N = k) &= (2k+1)\ln\phi - \ln(\sinh\phi) - \ln((2k+1)!) \\ &= 2k\ln\phi + \ln\phi - \ln(\sinh\phi) - \ln((2k+1)!) \\ &= k(2\ln\phi) - [\ln(\sinh\phi) - \ln\phi] - \ln((2k+1)!). \end{aligned} \tag{7.1}$$

Define the canonical parameter,

$$\eta = 2\ln\phi,$$

so that

$$\phi = e^{\eta/2}, \qquad -\infty < \eta < \infty.$$

Substituting this expression into the PMF yields

$$\mathbb{P}(N = k) = \exp\left\{k\eta - \left[\ln\left(\sinh(e^{\eta/2})\right) - \frac{\eta}{2}\right] - \ln((2k+1)!)\right\}.$$

Therefore, the PMF can be written in the standard exponential family form,

$$\mathbb{P}(N = k) = \exp\{k\eta - A(\eta) + C(k)\},$$

where

$$A(\eta) = \ln\left(\sinh(e^{\eta/2})\right) - \frac{\eta}{2},$$

and

$$C(k) = -\ln((2k+1)!).$$

Since the support of N, namely $\{0, 1, 2, \ldots\}$, does not depend on the parameter and the natural parameter space

$$\Omega = \{\eta : -\infty < \eta < \infty\}$$

is an open interval, the Hyperbolic Sine distribution is a member of the exponential family. □

Theorem 7.1 establishes that the Hyperbolic Sine (HS) distribution belongs to the regular exponential family, thereby providing the theoretical foundation for embedding it within the generalized linear model (GLM) framework. In particular, the canonical parameter of the HS distribution is

$$\eta = 2\ln\phi.$$

Therefore, the canonical GLM is obtained by relating the conditional mean μ to this canonical parameter, that is,

$$2\ln\phi_i = g(\mu_i) = 2\ln\left\{m^{-1}(\mu_i)\right\} = x_i^\top\beta,$$

where

$$\mu = m(\phi) = \frac{\phi\coth(\phi) - 1}{2}$$

and

$$x_i = (1, x_{i1}, \ldots, x_{ip})^\top$$

is the vector of covariates corresponding to the i-th observation, and

$$\beta = (\beta_0, \beta_1, \ldots, \beta_p)^\top$$

is the vector of regression coefficients.

Although the canonical link is formally a function of the conditional mean μ_i, the equivalent representation in terms of ϕ_i is particularly convenient for likelihood estimation. Specifically,

$$2\ln(\phi_i) = x_i^\top\beta$$

directly yields

$$\phi_i = \exp\left(\frac{x_i^\top\beta}{2}\right),$$

so numerical evaluation of $m^{-1}(\mu_i)$ is not required.

Now, by substituting the expression

$$\phi_i = \exp\left(\frac{x_i^\top\beta}{2}\right)$$

into Equation 2.1, the PMF of HSGLM is

$$\mathbb{P}(N_i = k_i; x_i) = \frac{\left[\exp\left(\frac{x_i^\top\beta}{2}\right)\right]^{2k_i+1}}{\sinh\left(\exp\left(\frac{x_i^\top\beta}{2}\right)\right)(2k_i+1)!}, \qquad k_i = 0, 1, 2, \ldots$$

For a random sample (k_i, x_i), $i = 1, \ldots, n$, the likelihood function is

$$L(\beta) = \prod_{i=1}^{n} \frac{\phi_i^{(2k_i+1)}}{\sinh(\phi_i)(2k_i+1)!},$$

where

$$\phi_i = \exp\left(\frac{x_i^\top\beta}{2}\right).$$

Hence, the log-likelihood is

$$\ell(\beta) = \sum_{i=1}^{n} \left[(2k_i + 1) \ln(\phi_i) - \ln(\sinh(\phi_i)) - \ln((2k_i + 1)!) \right]. \tag{7.2}$$

Substituting the log-link gives

$$\ell(\beta) = \sum_{i=1}^{n} \left[(2k_i + 1) \frac{x_i^\top \beta}{2} - \ln \left\{ \sinh \left(\exp \left(\frac{x_i^\top \beta}{2} \right) \right) \right\} - \ln((2k_i + 1)!) \right]. \tag{7.3}$$

We shall use numerical methods to maximize $\ell(\beta)$ and obtain the maximum likelihood estimates of β.

Remark 7.2. *This regression model differs from the classical Poisson regression because the conditional mean is*

$$\mathbb{E}(N_i) = \frac{\phi_i \coth(\phi_i) - 1}{2},$$

rather than simply ϕ_i. Since the HS distribution is intrinsically underdispersed,

$$0.5 < \text{IOD} < 1,$$

the proposed regression model is particularly suitable for count data exhibiting underdispersion, where Poisson regression may provide an inadequate fit.

8. Conclusion

In this paper, we have revisited and comprehensively investigated the hyperbolic sine (HS) distribution, which was originally introduced as a component distribution of the discrete hyperbolic mixture (DHM) distribution. Our investigation identified an error in the previously reported variance expression of the HS distribution, which consequently led to an incorrect classification of its dispersion behavior. By deriving the correct variance, we established that the index of dispersion of the HS distribution satisfies ($0.5 < \text{IOD} < 1$), confirming that the HS distribution is inherently underdispersed rather than overdispersed. This correction provides a more accurate understanding of the distribution's fundamental characteristics.

Beyond correcting its variance, we developed several previously unexplored theoretical properties of the HS distribution, including the quantile function, characteristic function, factorial moments, skewness coefficient, kurtosis coefficient, and mode. These results provide deeper insights into the distribution's probabilistic structure and shape characteristics. In particular, the mode analysis confirms that the HS distribution is unimodal, while the skewness coefficient indicates positive skewness, indicating a longer right tail. Furthermore, the kurtosis coefficient is greater than three, indicating positive excess kurtosis and a greater degree of peakedness and/or tail weight relative to the normal distribution in terms of the fourth standardized moment.

To extend the applicability of the HS distribution in statistical modeling, we established that it belongs to the exponential family and developed the hyperbolic sine generalized linear model (HSGLM). This extension enables the HS distribution to be incorporated into the generalized linear modeling framework for analyzing count data with underdispersion and for investigating the effects of explanatory variables. Therefore, the HSGLM provides a flexible alternative to existing count regression models, particularly in

applications where the equidispersion assumption of the Poisson model or the overdispersion structure of many alternative models is inadequate.

Overall, the results presented in this study provide a corrected, complete, and unified theoretical foundation for the HS distribution. The newly established properties and regression framework enhance its potential as a flexible model for underdispersed count data. Future research may focus on developing more advanced inferential procedures, investigating Bayesian extensions, and exploring additional applications of the HS distribution in diverse fields involving underdispersed count responses.

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